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New Examples of Manifolds with Strictly Positive Curvature

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1. Introduction

Berger [3], Wallach [10] and Berard Bergery [2] have classified all simply connected smooth manifolds which allow a homogeneous Riemannian metric of strictly positive curvature. Besides the rank one symmetric spaces there exist five exceptional manifolds and an infinite series of 7-manifolds of distinct homotopy type which have been studied by Aloff and Wallach [1]. These are diffeomorphic to $M_{pq} := SU(3)/U_{pq}$, where p, q are positive integers and U_{pq} is the one-parameter subgroup of diagonal matrices

$$\{\exp(2\pi i t \operatorname{diag}(p, q, -p-q));\, t \in \mathbb{R}\}.$$

Looking for further spaces of positive curvature one has to consider a more general class of manifolds. If G is any Lie group, the group $G^2 := G \times G$ acts on G by right and left translations. If U is a compact subgroup of G^2 which acts without fixed points, then the orbit space G/U is a smooth manifold which is not homogeneous in general. Gromoll and Meyer [7] obtained an exotic 7-sphere of nonnegative curvature in this way. We apply this method to $G = SU(3)$, $U \cong U(1)$ in G^2 and show that for any of the positively curved 7-manifolds $M = M_{pq}$, there exists a series of compact simply connected topologically distinct 7-manifolds M_n which are not homotopically equivalent to any compact Riemannian homogeneous space, such that the sectional curvatures of M_n converge in a certain sense to the sectional curvatures of M. This implies that M_n has strictly positive curvature for sufficiently large n.

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2. Fixed Point Free S^1-actions on $SU(3)$

21. Let $G = SU(3)$ and U a closed one-parameter subgroup of G^2. We define an action of U on G by

$$(u, g) \mapsto u_1 g u_2^{-1}$$

for any $g \in G$, $u = (u_1, u_2) \in U$. For any conjugate subgroup $U' = aUa^{-1}$, $a = (a_1, a_2) \in G^2$, the mapping $g \mapsto a_1^{-1} g a_2$: $G \to G$ gives an equivariant diffeomorphism. Therefore, we may assume $U \subset T \times T = T^2$ where T denotes the maximal torus of diagonal matrices

$$\{\exp i \operatorname{diag}(x_1, x_2, x_3);\ x_i \in \mathbb{R}, \textstyle\sum x_i = 0\}$$

in $SU(3)$. Let $W \in \mathbf{U}$ be a generator of the kernel of the exponential map of U. Then

$$W = 2\pi i(\operatorname{diag}(k, l, -k-l), \operatorname{diag}(p, q, -p-q))$$

where k, l, p, q are integers which are relatively prime. To indicate that U is determined by k, l, p, q, let us set $U = U_{klpq}$.

Proposition 21. *$U = U_{klpq}$ acts on G without fixed points if and only if the following pairs of intergers are relatively prime:*

$$\begin{array}{lll} (k-p, l-q), & (k-p, l+p+q), & (k+p+q, l-p), \\ (k-q, l-p), & (k-q, l+p+q), & (k+p+q, l-q). \end{array}$$

Proof. U acts without fixed points if for any $1 \neq u = (u_1, u_2) \in U$ we have $u_1 \neq g u_2 g^{-1}$ for all $g \in G$. Now $u = \exp tW$, $0 < t < 1$, and u_1, u_2 are diagonal matrices. These are nonconjugate if and only if $u_1 \neq \sigma u_2$ for all permutations σ of the coordinates (which establish the Weyl group of $SU(3)$). Hence one has to be sure that

$$2\pi i t(\operatorname{diag}(k, l, -k-l) - \sigma \operatorname{diag}(p, q, -p-q)) \notin \exp^{-1}(1)$$

which is equivalent to

$$t((k, l, -k-l) - \sigma(p, q, -p-q)) \notin \mathbb{Z}^3$$

for any $t \in (0, 1)$. Writing this down for all permutations σ, we get the result.

A quadrupel of integers (k, l, p, q) will be called admissible if the conditions of the previous proposition are satisfied.

Examples. (i) If $k = l = 0$, then the condition is trivial; this is the case of the homogeneous spaces M_{pq}.

(ii) The quadrupel $(1, 0, 2m, 2m)$ is admissible for arbitrary $m \in N$. This and further examples will be studied in § 5.

22. Let $H = U(2) \subset G$ be the canonical imbedding. Fix a Riemannian metric $\langle\ ,\ \rangle$ on G which is invariant under left translations of G and right translations of H, such that the induced metric on $M_{pq} = G/U_{pq}$ has strictly positive curvature for arbitrary positive integers p, q. It was shown in [1] that such a metric is given by the following scalar product on the Lie algebra $\mathbf{G}$:

$$\langle X, Y \rangle := B(X, Y) + tB(X_H, Y_H)$$

for any $t\in(-1,0)\cup(0,1/3)$, where B is an Ad(G)-invariant scalar product and X_H denotes the orthogonal projection of X to $\mathbf{H}$.

In particular, the group $U=U_{klpq}$ acts on G by isometries. Consequently, if (k,l,p,q) is admissible, there exists a unique metric $\langle\ ,\ \rangle$ on the orbit space $M=G/U$ such that $\pi\colon G\to M$ becomes a Riemannian submersion. Its curvature is given by O'Neill's formula [9, p. 465]: If X, Y are local linearly independent vector fields on G which are orthogonal to the orbits of U ("horizontal"), and K denotes the curvature of M and G, then

$$(*)\qquad K(\pi_* X,\pi_* Y)=K(X,Y)+3\|(\nabla_X Y)_v\|^2/\|X\wedge Y\|^2$$

where the subscript v denotes the projection to the tangent space of the orbit ("vertical component").

The orbits of the action of U are

$$F_g=Ug=\{u_1\cdot g\cdot u_2^{-1};\,(u_1,u_2)\in U\}.$$

Its tangent spaces, the vertical subspaces, are denoted by $T_gF=\{R_{g*}X_1-L_{g*}X_2;\,(X_1,X_2)\in\mathbf{U}\}$. We translate this space back to $\mathbf{G}=T_1G$ and get

$$V_g:=(L_{g*})^{-1}T_gF=\{\mathrm{Ad}(g^{-1})X_1-X_2;\,(X_1,X_2)\in\mathbf{U}\}.$$

Now $\mathbf{U}$ is one-dimensional, so for any basis vector (x,y) we have $V_g=\mathbb{R}\cdot(y-\mathrm{Ad}(g)^{-1}x)$. Put $z(g)=y-\mathrm{Ad}(g)^{-1}x$, defining a smooth mapping $z\colon G\to\mathbf{G}-\{0\}$. Its differential is given by $z_*(L_{g*}w)=[w,\mathrm{Ad}(g)^{-1}x]$ for any $g\in G$, $w\in\mathbf{G}$.

Let $h(g)\in\mathrm{End}\,T_gG$ be the orthogonal projection onto the horizontal subspace. This can be expressed in terms of z as follows: Setting $\bar z=z/\|z\|$, let $p(g)\in\mathrm{End}\,\mathbf{G}$ be the orthogonal projection $p(g)x=x-\langle x,\bar z(g)\rangle\,\bar z(g)$. Then $h(g)=L_{g*}p(g)L_{g*}^{-1}$.

Lemma. *Let U_i, $i\in N$, be closed one-parameter subgroups of G^2 which act freely on G. Assume that $\mathbf{U}_i$ is generated by $(x_i,y)\in\mathbf{G}^2$ where $y\neq0$ is fixed and $x_i\to0$. Let U be the subgroup of G generated by y. Let h_i, $h\in\Gamma(\mathrm{End}\,TG)$ be the horizontal projections of the Riemannian submersions $\pi_i\colon G\to G/U_i$, $\pi\colon G\to G/U$ resp. Then the h_i converge to h in the C^1-topology on $\Gamma(\mathrm{End}\,TM)$.*

Proof. According to the preceding remark, it suffices to show that the mappings $z_i\colon G\to\mathbf{G}$, $z_i(g)=y-\mathrm{Ad}(g)^{-1}x_i$, are C^1-converging to $z\equiv y$. This is true since $x_i\to0$ and $z_{i*}(L_{g*}w)=[w,\mathrm{Ad}(g)^{-1}x_i]\to0$ uniformely in g for any $w\in\mathbf{G}$.

Under these assumptions, the sectional curvatures of $M_i=G/U_i$ converge to the curvature of $M=G/U$ in the following sense:

Proposition 22. *Let G, M, M_i ($i\in\mathbb{N}$) be Riemannian manifolds, and $\pi\colon G\to M$, $\pi_i\colon G\to M_i$ Riemannian submersions with horizontal projections h and h_i resp. Assume $h\to h_i$ in the C^1-topology on $\Gamma(\mathrm{End}\,TG)$. Then for any $g\in G$ and any π-horizontal 2-plane $P\subset T_gG$, we have for the sectional curvatures*

$$K_{M_i}(\pi_{i*}P)\to K_M(\pi_*P),$$

and the convergence is uniform in P.

(Observe that $\ker \pi_{i*}$ is near to $\ker \pi_*$ for large enough i, therefore π_{i*} is isomorphic on the π-horizontal vectors.)

Proof. Let X, Y π-horizontal vector fields which span P at the point g. Let $X_i = h_i X$, $Y_i = h_i Y$. Then $X_i \to X$ and $Y_i \to Y$ in the C^1-sense. Since $\pi_{i*} \circ h_i = \pi_{i*}$, we have by (*)

$$K_{M_i}(\pi_{i*}X, \pi_{i*}Y) = K_G(X_i, Y_i) + 3\,\|(I - h_i)(\nabla_{X_i} Y_i)\|^2 / \|X_i \wedge Y_i\|^2$$

which clearly converges to $K_M(X, Y)$ uniformly.

Remark. The fibre of $\pi_i\colon G \to M_i = G/U_i$ through $1 \in G$ is the subgroup generated by $y - x_i$ which passes through $g_i := \exp 2\pi x_i \neq 1$ (assume y to be chosen such that $\exp 2\pi y = 1$). Since $g_i \to 1$, the cut locus distance of M_i at $\pi_i(1)$ gets arbitrarily small for large i. One easily shows the same fact at $\pi_i(g)$ for any $g \in G$. Thus the spaces M_i and M are geometrically very different, even locally.

3. Homotopy and Integral Cohomology of $SU(3)/U_{klpq}$

31. Let $G = SU(3)$, $U(1) \cong U \subset G \times G$ a group which acts freely on G by right and left translations. The orbit space $M = G/U$ has the following homotopy groups:

Proposition 31.

$$\pi_1(M) = 0, \qquad \pi_2(M) = \mathbb{Z},$$
$$\pi_i(M) = \pi_i(SU(3)) \quad \text{for } i \geq 3,$$

in particular $\pi_3(M) = \mathbb{Z}$, $\pi_4(M) = 0$.

This follows from the exact homotopy sequence of the fibration $G \to M$ and [4; 6, p. 428].

32. Now consider the cohomology. It is well known that H^*G is the exterior $\mathbb{Z}$-algebra with generators $z_3 \in H^3(G)$ and $z_5 \in H^5(G)$. Let B_G be a classifying space for G. Then $H^*B_G = \mathbb{Z}[\bar{z}_3, \bar{z}_5]$, where $\bar{z}_i \in H^{i+1}(B_G)$ corresponds to z_i under transgression [5, p. 171]. Let $T \subset G$ be the torus of diagonal matrices. We may identify H^*B_G with the ring I_G of polynomials on $H_1 T$ which are invariant under the Weyl group $W(G)$ [5, p. 194, 199]. One can specify the generators $\bar{z}_3$, $\bar{z}_5$ in this representation as follows: Let L be the lattice of integral matrices in $\mathbf{T}$, i.e. $L = \{X \in \mathbf{T};\ \exp 2\pi X = 1\}$. Note that any $X \in L$ corresponds to a 1-cycle $t \mapsto \exp tX\colon [0, 2\pi] \to T$, and any integral linear form $x \in L^* := \{y \in \mathbf{T}^*;\ y(L) \subset \mathbb{Z}\}$ corresponds to a 1-cocycle. This defines isomorphisms which identify L with $H_1 T$ and L^* with $H^1 T$.

Proposition 32. *Let G, T, L as above, $A_1 = i \cdot \mathrm{diag}(1, 0, -1)$ and $A_2 = i \cdot \mathrm{diag}(0, 1, -1)$ a base of L, a_1 and a_2 the dual base of L^*. Then we have (up to sign):*

$$\bar{z}_3 = \bar{a}_1^2 + \bar{a}_2^2 + \bar{a}_1 \bar{a}_2,$$
$$\bar{z}_5 = \bar{a}_1 \bar{a}_2 (\bar{a}_1 + \bar{a}_2)$$

where $\bar{a}_i \in H^2 B_T$ corresponds to $a_i \in H^1 T$ under transgression in the universal bundle $E_T \to B_T$.

Proof. Let $T' \supset T$ be the set of diagonal matrices of $U(3)$. We may extend the linear form a_i to $\mathbf{T}'$, this is the projection to the i-th coordinate ($i=1$ or 2). Let $a_3 \in \mathbf{T}'^*$be the projection on the third coordinate. Since $W(U(3)) = W(SU(3))$ is the permutation group of the coordinates, $I_{U(3)}$ is the Symmetric Algebric Algebra $S(a_1, a_2, a_3)$ which is generated by the polynomials

$$p_1 = a_1 + a_2 + a_3, \qquad p_2 = a_1 a_2 + a_2 a_3 + a_3 a_1,$$
$$p_3 = a_1 a_2 a_3.$$

Now $\mathbf{T}$ is the subset of $\mathbf{T}'$ where $a_1 + a_2 + a_3 = 0$. Hence on $\mathbf{T}$ we have $p_1 = 0$, $p_2 = -(a_1^2 + \bar{a}_2^2 + a_1 a_2)$, $p_3 = -a_1 a_2 (a_1 + a_2)$. So $p_2|_{\mathbf{T}}$ and $p_3|_{\mathbf{T}}$ generate $I_{SU(3)}$ and the result follows from [5], Proposition 27.1: Observe that $\bar{a}_i$ has degree 2, while degree$(\bar{z}_3) = 4$, degree$(\bar{z}_5) = 6$.

33. Let G be a compact Lie group and U a closed subgroup of $G^2 = G \times G$ which acts on G by right and left translation. Assume that this action is free. Then the orbit space $M = G/U$ is a smooth manifold and the projection $\pi\colon G \to M$ a principal bundle with structure group U. Let $\pi_U\colon E_U \to B_U$ be a classifying bundle for U. Consider the following commutative diagram (compare [5], p. 167, Diagram 18.4, and p. 168)

(D1)
$$\begin{array}{ccccc} G & \longrightarrow & E_U \times G & \longrightarrow & E_U \\ \downarrow & & \downarrow & & \downarrow \\ M & \longrightarrow & G/\!\!/U & \longrightarrow & B_U \end{array}$$

where $G/\!\!/U$ denotes the orbit space of $E_U \times G$ under the product action of U. The left horizontal arrows represent cohomology isomorphisms since the fibres of these maps are homeomorphic to E_U which is acyclic. We consider the spectral sequence of the bundle $p\colon G/\!\!/U \to B_U$ with fibre G. It starts with $E_2 = H^* B_U \otimes H^* G$ and converges to $H^* M$. The next step is to compute its differentials.

Let $E_{G^2} \to B_{G^2}$ be a classifying bundle of G^2. Since $U \subset G^2$, we can choose $E_U = E_{G^2}$, $B_U = E_{G^2}/U$. Thus we get a natural projection $\rho\colon B_U \to B_{G^2}$. This extends to a bundle map

(D2)
$$\begin{array}{ccc} G/\!\!/U = (E_{G^2} \times G)/U & \xrightarrow{\hat{\rho}} & (E_{G^2} \times G)/G^2 \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ B_U & \xrightarrow{\rho} & B_{G^2} \end{array}$$

which is a homeomorphism on the fibres. The bundle p' is well known: Let $\delta\colon G \to G^2$ be the diagonal imbedding, and $B_G := E_{G^2}/\delta G$. Then $E_{G^2} \to B_G$ is a classifying bundle for G. Moreover, the projection $\Delta\colon B_G \to B_{G^2}$ is a bundle with fibre G, and the mapping $f\colon E_{G^2}/\delta G \to (E_{G^2} \times G)/G^2$ which is well defined by $\delta G e \mapsto G^2(e, 1)$ for all $e \in E_{G^2}$, establishes a bundle isomorphism between p' and Δ.

34. The next step is to compute the spectral sequence of the bundle $\Delta\colon B_G \to B_{G^2}$ for $G = SU(3)$. We can choose $B_{G^2} := B_G \times B_G$ as a classifying space, hence $H^*(B_{G^2}) = H^*B_G \otimes H^*B_G = \mathbb{Z}[\bar{x}_3, \bar{y}_3, \bar{x}_5, \bar{y}_5]$ with $\bar{x}_i := \bar{z}_i \otimes 1$, $\bar{y}_i := 1 \otimes \bar{z}_i$. The spectral sequence of Δ starts with $E_2 = H^*B_{G^2} \otimes H^*G$. Call $k_j\colon H^*B_{G^2} \to E_j^{*0}$ the natural projections. It is a general fact that $\Delta^* = k_\infty\colon H^*B_{G^2} \to E_\infty^{*0} \subset H^*B_G$. [5, p. 128].

Proposition 34. *The differentials $d_j\colon E_j \to E_j$ are given by*

(1) $d_j(1 \otimes z_i) = 0$ *for* $j \leqq i$
(2) $d_{i+1}(1 \otimes z_i) = \pm k_{i+1}(\bar{x}_i - \bar{y}_i)$ *for* $i = 3$ *and* $i = 5$.

Proof. Equation (1) follows immediately since $d_j(1 \otimes z_i) \in E_j^{j, i-j+1}$ which vanishes already in E_2 for $j \leqq i$.

Using the fact that H^*B_G can be identified with the subset I_G of H^*B_T, it is easy to see that

$$\Delta^*(u \otimes 1) = \Delta^*(1 \otimes u) = u$$

for every $u \in H^*B_G$. Thus, the kernel of Δ^* is the ideal $(\bar{x}_3 - \bar{y}_3,\ \bar{x}_5 - \bar{y}_5) \subset H^*B_G$. We have $d_2 = d_3 = 0$, hence $E_2 = E_4$, and

$$d_4(E_4^{03}) = \ker k_5 \cap E_2^{40} = \ker k_\infty \cap E_2^{40} = \mathbb{Z}(\bar{x}_3 - \bar{y}_3)$$

since $E_5^{40} = E_\infty^{40}$. Since $E_4^{03} = \mathbb{Z}(1 \otimes z_3)$, we get (2) for $i = 3$. Using (1), we conclude

$$E_5 = E_6 = (H^*B_{G^2}/(\bar{x}_3 - \bar{y}_3)) \otimes (\mathbb{Z} \cdot 1 \oplus \mathbb{Z} \cdot z_5).$$

Hence $k_6(\bar{x}_5 - \bar{y}_5)$ is irreducible in E_6^{*0} and $E_6^{05} = \mathbb{Z}(1 \otimes z_5)$. By the same argument as above, (2) follows for $i = 5$.

35. Now consider the Diagram D2. Let E and E' be the spectral sequences of the bundles $p\colon G/\!\!/U \to B_U$ and $\Delta\colon B_G \to B_{G^2}$ resp. The bundle map$(\hat{\rho}, \rho)$ sending p to Δ induces homomorphisms of differential algebras $\rho_r^\#\colon E_r' \to E_r$. Note that $\rho_2^\# = \rho^* \otimes \hat{\rho}_F^*$, where $\hat{\rho}_F$ is the mapping of the fibres: $\hat{\rho}_F = \hat{\rho} | p^{-1}(b)\colon p^{-1}(b) \to \Delta^{-1}(\rho(b))$ where $b \in B_U$ is arbitrary. This is a homeomorphism, which becomes the identity if we identify the fibres suitably with G. Using that $\rho_r^\#$ commutes with d_r and $\rho_{r+1}^\#$ is the d_r-cohomology of $\rho_r^\#$, we get by induction from Proposition 34:

Proposition 35. Let $G = SU(3)$, U a fixed point free closed subgroup of G^2. Let $\rho\colon B_U \to B_{G^2}$ be the map induced by the imbedding $U \subset G^2$. Then the differentials of the spectral sequence of $p\colon G/\!\!/U \to B_U$ are as follows:

$$d_j(1 \otimes z_i) = 0 \qquad \text{for } j \leqq i,$$
$$d_{i+1}(1 \otimes z_i) = \pm k_{i+1} \rho^*(\bar{x}_i - \bar{y}_i)$$

for $i = 3$ and $i = 5$, where $k_j\colon H^*B_U \to E_j^{*0}$ denotes the natural projections in this spectral sequence.

Remark. An analogous statement is true for $G = SU(n)$, $U(n)$ and $Sp(n)$ if one chooses a suitable set of generators z_i for the exterior $\mathbb{Z}$-algebra H^*G.

36. Now let $U \subset G \times G$ as in 31. We may assume that $U = U_{klpq}$ with (k, l, p, q) admissible (see 21.). We will compute the cohomology of $M = G/U$ by the method indicated in 35.

Proposition 36. *H^*M is generated by $w \in H^2(M)$, $z \in H^5(M)$, and the following relations hold:*

$$rw^2 = 0,\ w^3 = 0,\ zw^2 = 0,\ z^2 = 0$$

with $r := |(k^2 + l^2 + kl) - (p^2 + q^2 + pq)|$.

Proof. Let $W = i$ be the generator of $U(1) = i \cdot \mathbb{R}$ and ρ_*: $U(1) \to T^2$ the inclusion of U. Choose the base $V_1 = (A_1, 0)$, $V_2 = (A_2, 0)$, $V_3 = (0, A_1)$, $V_4 = (0, A_2)$ of L^2 (same notations as in 32.). Then

$$\rho_*(W) = kV_1 + lV_2 + pV_3 + qV_4.$$

If w; $v_1, \dots, v_4$ are the dual bases, we have

$$\rho^*(v_1) = kw, \rho^*(v_2) = lw, \rho^*(v_3) = pw, \rho^*(v_4) = qw.$$

Call $x_i = z_i \otimes 1$ and $y_i = 1 \otimes z_i$ for $i = 1$ and 2 the generators of $H^*G \otimes H^*G = H^*G^2$. It follows from 32. that the generators of $H^*B_{G^2}$, corresponding under transgression, are

$$\bar{x}_3 = \bar{v}_1^2 + \bar{v}_2^2 + \bar{v}_1\bar{v}_2, \bar{y}_3 = \bar{v}_3^2 + \bar{v}_4^2 + \bar{v}_3\bar{v}_4,$$
$$\bar{x}_5 = \bar{v}_1\bar{v}_2(\bar{v}_1 + \bar{v}_2), \bar{y}_5 = \bar{v}_3\bar{v}_4(\bar{v}_3 + \bar{v}_4).$$

Hence

$$\rho^*(\bar{x}_3 - \bar{y}_3) = ((k^2 + l^2 + kl) - (p^2 + q^2 + pq))\,\bar{w}^2,$$
$$\rho^*(\bar{x}_5 - \bar{y}_5) = (kl(k+l) - pq(p+q))\,\bar{w}^3.$$

Now consider the spectral sequence of p: $G/\!/U \to B_U$ as in Proposition 35. It starts with $E_2 = E_3 = E_4 = H^*B_U \otimes H^*G$. Since $d_4(1 \otimes z_3) = (\bar{x}_3 - \bar{y}_3) = r\bar{w}^2$, we conclude $\ker d_4 = B_U \otimes 1, z_5\rangle$, and $\operatorname{im} d_4$ is the ideal in $\ker d_4$ generated by $r\bar{w}^2 \otimes 1$, hence

$$E_5 = E_6 = (\mathbb{Z}[\bar{w}]/(r\bar{w}^2)) \otimes \langle 1, z_5\rangle.$$

Now $d_6(1 \otimes z_5) = (\bar{x}_5 - \bar{y}_5) = sk_6(\bar{w}^3)$ with $s = kl(k+l) - pq(p+q)$. Claim: r and s are relatively prime. In fact, if there was a prime number n dividing both r and s, we would have $\sigma_i(k, l, -(k+l)) \equiv \sigma_i(p, q, -(p+q)) \bmod n$, for $i = 1, 2, 3$, denoting by σ_i the i-th elementary symmetric polynomial in three variables. But then $(k, l, -(k+l))$ and $(p, q, -(p+q))$ would be congruent $\bmod n$ up to a permutation of the three variables which is excluded by the very fact that (k, l, p, q) is an admissible quadrupel (see 21). Hence $d_6(n(1 \otimes z_5)) = 0$ only if n is a multiple of r. It follows that $\ker d_6 = (\mathbb{Z}[\bar{w}]/(r\bar{w}^2)) \otimes \langle 1, rz_5\rangle$, and $\operatorname{im} d_6$ is the ideal in $\ker d_6$ generated by $\bar{w}^3 \otimes 1$. So

$$E_7 = E_\infty = (\mathbb{Z}[\bar{w}]/(r\bar{w}^2, \bar{w}^3)) \otimes \langle 1, rz_5\rangle = H^*M.$$

So we have completed the proof, setting $w = k_\infty(\bar{w})$ and $z = r(1 \otimes z_5)$.[1]

[1] A similar proof for the homogeneous case $k = l = 0$ was communicated to us by W. Ziller.

4. Homogeneous Spaces with Similar Homotopy

We call a compact topological space strongly inhomogeneous if it is not homotopy equivalent to any compact Riemannian homogeneous space. The goal of this section is the proof of the following

Theorem. *Let* $\bar{G}=SU(3)$, $U(1)\cong U\subset\bar{G}^2$ *fixed point free,* $M=\bar{G}/U$. *Assume that* $H^4(M)=\mathbb{Z}_r$ *with*

$$r\equiv 2(3).$$

Then M *is strongly inhomogeneous.*

Proof. Assume that M is homotopy equivalent to a compact Riemannian manifold M' with transitive isometry group G. Call H the isotropy subgroup of a fixed element $m\in M$. Then $M'=G/H$.

1. The first homotopy groups of M' are given in 41. Moreover, it is known for any compact Lie group that $\pi_2=0$ and $\pi_3=Z^k$ where k is the number of simple factors of the Lie algebra [4]. Thus we derive from the exact homotopy sequence of the fibration $H\to G\to M'$:

(1) $\pi_0(H)=\pi_0(G)$,
(2) $0\to\mathbb{Z}\to\pi_1(H)\to\pi_1(G)\to 0$ is exact,
(3) $\pi_3(H)=\pi_3(G)\times\mathbb{Z}$.

By (1), we may assume that G, H are both connected since $G_0/H_0=G/H$. From the sequence (2) it follows that $\operatorname{rank}(\pi_1(H))=\operatorname{rank}(\pi_1(G))+1$. Hence, if $\mathbf{G}=\mathbf{G}'\times\mathbf{T}$ where $\mathbf{G}'$ is semisimple and T an l-torus, then $\mathbf{H}=\mathbf{H}'\times\mathbf{S}$ with $\mathbf{H}'$ semisimple and S an $(l+1)$-torus. Moreover, by (3), $\mathbf{G}'$ has $k+1$ simple factors if $\mathbf{H}'$ has k.

2. Let G' be the simply connected group with Lie algebra $\mathbf{G}'$. Then $\hat{G}:=G'\times T$ is a covering group of G. Call π: $\hat{G}\to G$ the covering homomorphism, $\hat{H}:=\pi^{-1}(H)$. Hence we get a covering map $\bar{\pi}$: $\hat{G}/\hat{H}\to G/H$ which is in fact a diffeomorphism since $M'=G/H$ is simply connected. Thus we assume from now on that $G=G'\times T$, where G' is a simply connected semisimple compact group and T an l-torus.

3. Since the Lie algebra $\mathbf{H}'\subset\mathbf{H}$ is semisimple, its projection to the abelian factor of $\mathbf{G}$ is zero, hence $H':=\exp\mathbf{H}'$ is a subgroup of G'. Consider the homomorphism $f=p\circ j\circ i$: $S\to T$ which is the composition of the inclusions i: $S\to H$, j: $H\to G$ and the projection p: $G\to T$. Then $f_*(\pi_1(S))$ has finite index in $\pi_1(T)$ since j_* is onto (see (2)), p_* is isomorphic and $i_*(\pi_1(S))$ has finite index in $\pi_1(H)$. Therefore the subtorus $f(S)$ of T has the same rank as T, so f is onto. Hence the connected component of its kernel is a circle $U\subset S$ which is a subgroup of the semisimple factor G'. Hence $\mathbf{G}'\cap\mathbf{H}=\mathbf{H}'\times\mathbf{U}$. A complement $\mathbf{M}$ of $\mathbf{H}'\times\mathbf{U}$ in $\mathbf{G}'$ is also a complement of $\mathbf{H}$ in $\mathbf{G}$. Set $H'':=\exp(\mathbf{H}'\times\mathbf{U})\subset G'$. Then the mapping $gH''\mapsto(g,1)H$: $G'/H''\to G/H$ is a covering map and hence a diffeomorphism. Replacing G and H with G' and H'', we may assume: G is simply connected and semisimple, and $\mathbf{H}=\mathbf{H}'\times\mathbb{R}$ with semisimple factor $\mathbf{H}'$.

4. We want to determine the possible simple components of **G** and **H**′. M and M' are both compact, orientable and homotopically equivalent. Hence $\dim M' = \dim M = 7$. Thus the isotropy group H is a subgroup of $0(7)$. It follows that $\operatorname{rank} \mathbf{H}' \leqq 2$, so **H**′ is one of the following compact Lie algebras:

$$0;\ A_1;\ A_2;\ C_2;\ G_2;\ A_1 \times A_1.$$

Since $\dim G = \dim H' + 8$ and **G** has one simple factor more then **H**′, the corresponding Lie algebra **G** can only be out of the following ones:

$$A_2;\ A_2 \times A_1;\ A_2 \times A_2;\ C_2 \times A_2 \quad \text{or} \quad A_3 \times A_1;$$
$$G_2 \times A_2;\ A_2 \times A_1 \times A_1.$$

Thus we have to consider seven pairs of Lie algebras $(\mathbf{G}, \mathbf{H}' \times \mathbb{R})$. A pair by pair inspection of the possible imbeddings of $\mathbf{H}' \times \mathbb{R}$ in **G** shows that $M' = G/H$ is never homotopy equivalent to M. In particular, this is true for the first pair $(A_2, \mathbb{R})$ since then M' is a Wallach space M_{pq} with $H^4(M') = \mathbb{Z}_{r'}, r' = p^2 + q^2 + pq \not\equiv 2(3)$ for arbitrary $p, q \in \mathbb{Z}$. We will discuss the details of the remaining pairs in the appendix.

5. Strongly Inhomogeneous Spaces Near to Wallach Spaces

Theorem. *For any Wallach space $M = M_{pq}$ with $pq(p+q) \neq 0$ there exists a sequence of simply connected strongly inhomogeneous Riemannian manifolds M_i of distinct homotopy type the curvatures of which approach the curvatures of M in the sense of Proposition 22.*

Proof. In the view of §4, all we have to show is: There are infinitely many positive integers $n = n_i$ such that the quadrupel $(1, 0, np, nq)$ is admissible and $r := n^2(p^2 + q^2 + pq) - 1 \equiv 2(3)$. This last condition is satisfied if we choose $n \equiv 0(3)$. Moreover, the following pairs of integers have to be relatively prime (see 21.):

$$(np-1, nq), (np-1, ns), (ns+1, np),$$
$$(nq-1, np), (nq-1, ns), (ns+1, nq)$$

where we have set $s := p + q$. Let $\{a_1, \ldots, a_k\}$ be the set of all prime numbers which divide pqs. Then

$$n = n_i = 3i\, a_1 a_2 \ldots a_k$$

clearly satisfies all conditions for arbitrary $i \in \mathbb{N}$. Set $M_i = M_{1,0,n_i p, n_i q}$.

Remarks. **1.** By Proposition 22, the curvature of M_i is strictly positive if $p, q > 0$ and i large, and the pinching of $M = M_{pq}$ is approximated; e.g. we have $K_{\min}/K_{\max} = 16/29 \cdot 37$ for $p = q = 1$, $t = -1/2$ (parameter of the metric) as was shown by Hua-Min Huang [8].

2. It can be shown that in fact M_i has strictly positive curvature for *any* positive integer i, if p and q are positive.

3. There are many spaces of type G/U_{klpq} which cannot be distinguished by cohomology, even among the homogenous ones ($k=l=0$): e.g. $M_{1,9}$ and $M_{5,6}$. It would be interesting to know whether these are topologically different.

Appendix 3 – The Remaining Homogeneous Spaces M' which are Similar to $M=SU(3)/U_{klpq}$ (see § 4)

Let $(\mathbf{G},\mathbf{H})$ be one of the pairs of § 4, Sect. 4, except the first one. We denote by $\mathrm{pr}_i: \mathbf{H}\to\mathbf{G}_i$ the projection of the subalgebra $\mathbf{H}$ of $\mathbf{G}$ to the i^{th} factor $\mathbf{G}_i$ of $\mathbf{G}$.

1. $(A_2\times A_2, A_2\times\mathbb{R})$

Either $\mathrm{pr}_i A_2=0$ for $i=1$ or 2, or A_2 is the diagonal subalgebra of $A_2\times A_2$. In the first case, it follows that $\mathrm{pr}_j\mathbb{R}=0$ for $j\neq i$, hence $M'=SU(3)/U(1)$ is a Wallach space which is not homotopically equivalent to M as was proved above. The second case is impossible since the diagonal subalgebra of $A_2\times A_2$ has no centralizer.

2. $(C_2\times A_2, C_2\times\mathbb{R})$

Then $\mathrm{pr}_2 C_2=0$, $\mathrm{pr}_1\mathbb{R}=0$ and hence $M'=SU(3)/U(1)\not\simeq M$.

3. $(A_3\times A_1, C_2\times\mathbb{R})$

Then $\mathrm{pr}_2 C_2=0$. The homogeneous space corresponding to the pair $(A_3, C_2)=(D_3, B_2)$ is the 5-sphere $S^5=SO(6)/SO(5)$. Since the isotropy group $G_p=SO(5)$ of some $p\in S^5$ has no fixed vector on $T_pS^5=\mathbb{R}^5$, it has no centralizer in $SO(6)$. Therefore, $\mathrm{pr}_1\mathbb{R}=0$. It follows that $M'=S^5\times S^2\not\simeq M$ since $H^4(M')=0$.

4. $(G_2\times A_2, G_2\times\mathbb{R})$

Then $\mathrm{pr}_2 G_2=0$, $\mathrm{pr}_1\mathbb{R}=0$, hence $M'=SU(3)/U(1)\not\simeq M$.

5. $(A_2\times A_1, A_1\times\mathbb{R})$

Up to equivalence, there are two representations of A_1 in A_2, corresponding to the standard imbeddings $\mathbf{SO(3)}\subset\mathbf{SU(3)}$ and $\mathbf{SU(2)}\subset\mathbf{SU(3)}$. Call the first f_1, the second f_2. Consequently, there are the following imbeddings of A_1 in $A_2\times A_1$:

$$(0, id), (f_1, 0), (f_2, 0), (f_1, id), (f_2, id).$$

a) $(0, id)$: Then $\mathrm{pr}_2\mathbb{R}=0$, hence $M'=SU(3)/U(1)\not\simeq M$.

b) $(f_1, 0)$: There is no centralizer of $\mathbf{SO(3)}$ in $\mathbf{SU(3)}$.

It follows $\mathrm{pr}_1\mathbb{R}=0$, consequently $M'=SU(3)/SO(3)\times S^2\not\simeq M$ since $w^2=0$ for the generator w of $H^2(S^2)\subset H^2(M')$.

c) $(f_2, 0)$: More precisely, we choose the imbedding f_2: $\mathbf{SU(2)}\to\mathbf{SU(3)}$ in the first two coordinates, which has centralizer $\mathbb{R}\cdot Z$, $Z:=i\cdot\mathrm{diag}(1,1,-2)$. Hence the factor $\mathbb{R}$ can be an arbitrary line with rational slope in the plane spanned by $(Z,0)$ and $(0,Y)$ in $\mathbf{G}=\mathbf{SU(3)}\times\mathbf{SU(2)}$, where $Y:=i\cdot\mathrm{diag}(1,-1)\in\mathbf{SU(2)}$. Up to conjugation, these are all possible imbeddings. Let $U_1=(Y,0)$ and $U_2=(0,1)$ be a basis of the lattice $\exp_S^{-1}(1)=\mathrm{H}_1(S)$ (natural identification), where S is the maximal torus of H. Likewise, we have the basis $V_1=(i\,\mathrm{diag}(1,0,-1),0)$, $V_2=(i\,\mathrm{diag}(0,1,-1),0)$, $V_3=(0,Y)$ of the lattice $\exp_T^{-1}(1)=H_1(T)$ for the maximal torus T of G. Call $\rho_*: \mathbf{S}\to\mathbf{T}$ the imbedding $(f_2,0)|_{\mathbf{S}}$. Then we have

$$\rho_*(U_1)=V_1-V_2,$$
$$\rho_*(U_2)=k\cdot(V_1+V_2)+l\cdot V_3$$

where k, l are the relative prime integers which correspond to the imbedding of $\mathbb{R}$. The transposed map $\rho^*: H^1(T)\to H^1(S)$ is given by

$$\rho^*(v_1)=\quad u_1+k\cdot u_2,$$
$$\rho^*(v_2)=-u_1+k\cdot u_2,$$
$$\rho^*(v_3)=\qquad l\cdot u_2$$

where u_i, v_j are the dual bases.

Call y_3, z_3, z_5 the generators of $H^3(SU(2))$, $H^3(SU(3))$, $H^5(SU(3))$. The invariant polynomials which generate $H^*(B_G)$, $G=SU(3)\times SU(2)$, are the following (see 32.):

$$\bar{y}_3=\bar{v}_3^2,$$
$$\bar{z}_3=\bar{v}_1^2+\bar{v}_2^2+\bar{v}_1\bar{v}_2,$$
$$\bar{z}_5=\bar{v}_1\bar{v}_2(\bar{v}_1+\bar{v}_2).$$

The images under the induced map $\rho^*: H^*B_G\to H^*B_H$ are

$$\rho^*(\bar{y}_3)=l^2\,\bar{u}_2^2,$$
$$\rho^*(\bar{z}_3)=\bar{u}_1^2+3k^2\cdot\bar{u}_2^2,$$
$$\rho^*(\bar{z}_5)=2k\cdot(-\bar{u}_1^2\bar{u}_2+k^2\cdot\bar{u}_2^3).$$

As in 33., we consider the spectral sequence of $p: G/\!/H\to B_H$ which starts with $E_2=H^*B_H\otimes H^*G$. It was shown by Borel [5, p. 180] that $1\otimes y_3$, $1\otimes z_3\in E_4$ and $1\otimes z_5\in E_6$ with

$$d_4(1\otimes y_3)=k_4\rho^*\bar{y}_3,$$
$$d_4(1\otimes z_3)=k_4\rho^*\bar{z}_3,$$
$$d_6(1\otimes z_5)=k_6\rho^*\bar{z}_5$$

where d_i are the differentials and $k_i: H^*B_H\to E_i^{*0}$ the natural projections. From this we derive the cohomology of M', in particular:

$$H^4(M')=\mathbb{Z}\bar{u}_1^2\otimes\mathbb{Z}\bar{u}_2^2/L$$

where L is the sublattice generated by $\bar{u}_1^2+3k\bar{u}_2^2$ and $l^2\bar{u}_2^2$. Since the determinant of these two vectors is l^2, $H^4(M')$ is a finite group of order l^2. But it was assumed for the order r of $H^4(M)$ that $r\equiv 2 \bmod 3$ which fails for l^2. Hence $M\not\approx M'$.

d) (f_1, id) This is impossible since $\mathrm{pr}_2\,\mathbb{R}=0$ and, by b), also $\mathrm{pr}_1\,\mathbb{R}=0$.

e) (f_2, id) Again we have $\mathrm{pr}_2\,\mathbb{R}=0$. Hence the mapping $\rho_*=(f_2, id)|_{\mathbf{S}}: \mathbf{S}\to\mathbf{T}$ is as follows (same notation as in c):

$$\rho_*(U_1)=V_1-V_2+V_3,$$
$$\rho_*(U_2)=V_1+V_2.$$

It follows for the induced map $\rho^*: H^*B_G \to H^*B_H$:

$$\rho^*(\bar{y}_3)=\bar{u}_1,$$
$$\rho^*(\bar{z}_3)=\bar{u}_1^2+3\bar{u}_2^2,$$
$$\rho^*(\bar{z}_5)=-2\bar{u}_1^2\bar{u}_2+2\bar{u}_2^3$$

(compare c)). In particular, $H^4(M')$ is generated by $k_4(\bar{u}_2^2)$ with $3k_4(\bar{u}_2^2)=0$. But $r=\mathrm{ord}(H^4 M)\neq 3$, hence $M \not\approx M'$.

6. $(A_2\times A_1\times A_1, A_1\times A_1\times \mathbb{R})$

The imbeddings of $\mathbf{H}'=A_1\times A_1$ into $\mathbf{G}=A_2\times A_1\times A_1$ are given by 2×3-matrices of homomorphisms $a_{ij}: \mathbf{H}_i \to \mathbf{G}_j$ $(i=1,2; j=1,2,3)$. Observe that $a_{1j}\neq 0$ implies $a_{2j}=0$ and vice versa. Then, up to conjugation and permutation of isomorphic factors, the following imbeddings exist:

$$\text{a)} \begin{pmatrix} f_k, & 0, & 0 \\ 0, & id, & id \end{pmatrix}, \quad \text{b)} \begin{pmatrix} f_k, & 0, & 0 \\ 0, & id, & 0 \end{pmatrix},$$

$$\text{c)} \begin{pmatrix} f_k, & id, & 0 \\ 0, & 0, & id \end{pmatrix}, \quad \text{d)} \begin{pmatrix} 0, & id, & 0 \\ 0, & 0, & id \end{pmatrix}$$

where $f_k: A_1 \to A_2$ $(k=1,2)$ are the representations used in 5.

a) $\mathbf{H}_2$ is diagonally imbedded in $A_1\times A_1$. This is the canonical imbedding $\mathbf{SO(3)}\subset\mathbf{SO(4)}$ which has no centralizer since $SO(3)$ has no fixed vector on the tangent space of $SO(4)/SO(3)=S^3$. Hence $\mathbf{H}_3=\mathbb{R}$ is mapped into the first factor. It follows from 5c) that only $k=2$ is possible. Thus $M'=SU(3)/U(2)\times S^3=\mathbb{C}P^2\times S^3\not\approx M$ since $H^4(M')\not\cong\mathbb{Z}$.

b) $\mathbf{H}_3=\mathbb{R}$ is mapped to $\mathbf{G}_1\times\mathbf{G}_3$, and the pair can be reduced to $(A_2\times A_1, A_1\times\mathbb{R})$ which was treated in 5.

c) Same argument as in b).

d) $\mathbf{H}_3=\mathbb{R}$ is mapped to $\mathbf{G}_1=A_2$, and $M'=SU(3)/U(1)\not\approx M$.

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