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AN ELEMENTARY PROOF OF THE CHEEGER-GROMOLL SPLITTING THEOREM

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We give a short proof of the Cheeger-Gromoll Splitting Theorem which says that a line in a complete manifold of nonnegative Ricci curvature splits off isometrically. Our proof avoids the existence and regularity theory of elliptic PDE's.

1. Introduction

The purpose of this note is to give an elementary proof of the following

Splitting Theorem: Let M be a complete connected Riemannian manifold of nonnegative Ricci curvature. If M contains a line (i.e. a complete geodesic which realizes the distance between any two of its points) then M splits isometrically as $M = M' \times \mathbb{R}$.

The theorem is of fundamental importance in Riemannian geometry and has many applications. It is due to Cohn-Vossen [4, Satz 5] in the 2-dimensional case, to Toponogov [8] under the assumption of nonnegative sectional curvature and finally to Cheeger and Gromoll in its above generality. As a principal tool Cheeger and Gromoll use the existence and regularity theory of elliptic equations. Actually, they prove a stronger result, namely the subharmonicity of any Busemann function and deduce the Splitting Theorem as an easy corollary. For a simplified proof of the subharmonicity see Wu [9]. It is our purpose to point out that the Splitting Theorem is more elementary in nature and can be proved quickly by a direct application of the maximum principle (Lemma 3) together with a closer look at the geometry of the Busemann function associated to a line (Lemma 2). For convenience we give full proofs of all the details.

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2. Busemann functions

Let M be a complete connected Riemannian manifold and γ a ray, i.e. a geodesic defined on $[0,\infty)$ which realizes the distance between any two of its points. The functions $b_r(x) = r - d(x,\gamma(r))$ for $r \geq 0$ are increasing with r, bounded by $d(x,\gamma(0))$ and equicontinuous. Therefore, the Busemann function (associated to γ) $b := \lim_{r\to\infty} b_r$ exists and is continuous. Its level sets are called horospheres.

For each $p \in M$ and any positive sequence $r_n \to \infty$, a subsequence of the unit tangent vectors of minimal geodesics from p to $\gamma(r_n)$ converges. The geodesic in such a limit direction is a ray, called an <u>asymptote</u> of γ. Asymptotes are not necessarily unique.
If $v \in T_pM$ is the direction of an asymptote, we put

$$b_{p,r}(x) := b(p) - r + d(x, \exp rv)$$

for each $r > o$. It follows easily from the triangle inequality that $b_{p,r}$ is a <u>support function</u> of b at p, i.e. a continuous function with $b_{p,r}(p) = b(p)$ and $b_{p,r} \le b$ [5]. Furthermore, $b_{p,r}$ is C^∞ in a neighborhood of p (depending on r), since p is not in the cut locus of exp rv.

The splitting of M as stated in the theorem will be achieved by the level sets and gradient curves of the Busemann function associated to the given line.

3. Nonnegative Ricci curvature

The assumption of nonnegative Ricci curvature will enter only via the following well known lemma.

<u>Lemma 1</u>: Let M be a Riemannian manifold and $f \in C^\infty(M)$ with $\|\text{grad } f\| = 1$. If c is an integral curve of grad f, then c is a geodesic realizing the distance between any two of its points and

$$-\mathrm{Ric}(\dot{c},\dot{c}) = (\Delta f \circ c)' + \|\mathrm{Hess}\, f \circ c\|^2 \geq (\Delta f \circ c)' + \frac{1}{n-1}(\Delta f \circ c)^2$$

where $\Delta f = + \mathrm{tr}\, \mathrm{Hess}\, f$.

Remark 1: The equality is a special case of the Bochner-Lichnerowicz formula (cf. [1], p. 131), the inequality was already known to Calabi [2].

Proof: Since f is a Riemannian submersion, it does not increase the length of any curve while the length of integral curves of grad f (horizontal curves) remains unchanged. Thus, if c is an integral curve it is a minimizing geodesic. Now choose a basis $E_1,\ldots,E_n$ of orthonormal vector fields in a neighborhood U of $c(t_o)$ for $t_o \in \mathbb{R}$, such that $E_n = \mathrm{grad}\, f$ and E_i are parallel along c for $i = 1,\ldots,n$. As in [3], one has along c in U

$$\begin{aligned}\mathrm{Ric}(E_n,E_n) &= \sum_{i=1}^{n} \langle R(E_i,E_n)E_n,E_i\rangle \\ &= \sum_{i=1}^{n}(-\langle \nabla_{E_n}\nabla_{E_i}E_n,E_i\rangle - \langle \nabla_{\nabla_{E_i}E_n}E_n,E_i\rangle) \\ &= -E_n(\sum_{i=1}^{n}\langle \nabla_{E_i}E_n,E_i\rangle) - \sum_{i,j=1}^{n}\langle \nabla_{E_i}E_n,E_j\rangle\langle \nabla_{E_j}E_n,E_i\rangle \\ &= -E_n(\Delta f) - \|\mathrm{Hess}\, f\|^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{Ric}(E_n,E_n) &\le -E_n(\Delta f) - \sum_{i=1}^{n-1} \langle \mathrm{Hess}\, f(E_i),E_i\rangle^2 \\ &\le -E_n(\Delta f) - \frac{1}{n-1}(\Delta f)^2\end{aligned}$$

by Schwarz' inequality which finishes the proof.

Remark 2: From the differential inequality above for $\varphi = \Delta f \circ c$, one can easily derive: If M is complete with $\mathrm{Ric} \ge 0$ and $f : M \to \mathbb{R}$ is C^∞ (in fact C^2 suffices) with $\|\mathrm{grad}\, f\| = 1$, then M splits isometrically as $M = M' \times \mathbb{R}$. In fact, if ϕ is a primitive of $\frac{\varphi}{n-1}$, the function e^ϕ is concave, hence constant, and thus φ and Hess f have to vanish. This can be used for another proof of the Splitting Theorem (see §5).

For a connected Riemannian manifold M and $p \in M$, we denote by ρ_p the distance to p, i.e. $\rho_p(x) = d(x,p)$. Note that ρ_p is C^∞ outside p and its cut locus $C(p)$, with $\|\mathrm{grad}\, \rho_p\| = 1$.

Corollary: (cf. [2], [3]) If M is a complete connected Riemannian manifold of nonnegative Ricci curvature and dimension n, then

$$\Delta\rho_p \le \frac{n-1}{\rho_p}$$

in $M \setminus (\{p\} \cup C(p))$.

Proof: Let c be an integral curve of $\operatorname{grad} \rho_p$ with $c(0) = p$. For $\varphi := \Delta\rho_p \circ c$ we have $\lim_{t\to 0} \varphi(t) = \infty$, as in the euclidean case. The corollary now is proved by looking at the derivative of $1/\varphi$.

Lemma 2: (cf. [6], Lemma 6) Let M be a complete connected Riemannian manifold of nonnegative Ricci curvature. If c is a line in M and $b_r^{\pm} := r - \rho_{c(\pm r)}$, then $\lim_{r\to\infty} \operatorname{Hess} b_r^{\pm}(c(t)) = 0$ for all $t \in \mathbb{R}$.

Remark: Geometrically this means that horospheres associated to a line are totally geodesic at the points of intersection with the line.

Proof: As follows easily from the triangle inequality, the functions $b_r^{\pm}$ are monotonously increasing with r, and $b_r^{+} \le - b_{r'}^{-}$, for $r,r' \ge 0$ with equality at $c(s)$ for $s \in [-r',r]$. Hence $L_r^{\pm}(s) := \operatorname{Hess} b_r^{\pm}(c(s))$ converge monotonously from below to some limit $L^{\pm}(s)$ as $r \to \infty$, and $L^{+}(s) \le - L^{-}(s)$. The preceding corollary implies $\operatorname{tr} L_r^{\pm} = - \Delta\rho_{c(\pm r)} \ge 0$ so that $\operatorname{tr}(L^{+} + L^{-}) \ge 0$. Thus $L^{+} = - L^{-}$ with $\operatorname{tr} L^{\pm} = 0$. Since the convergence of L_r^{+} and $- L_r^{-}$ to L^{+} is monotonous, it is also locally uniform. Therefore, $L^{+}(t) \ne 0$ for some $t \in \mathbb{R}$ would imply $\|L_r^{+}\|^2 > \varepsilon$ in an interval around t, for some $\varepsilon > 0$ and sufficiently large r. But this contradicts $\|L_r^{+}\|^2 \le -(\operatorname{tr} L_r^{+})'$ from Lemma 1, since $\operatorname{tr} L_r^{+}$ is nonpositive and pointwise convergent to zero as $r \to \infty$.

4. Proof of the Splitting Theorem

From now on, let M be a complete connected Riemannian manifold of nonnegative Ricci curvature and $\gamma : \mathbb{R} \to M$ a line. Let $b^{\pm}(x) := \lim_{r\to\infty} (r - d(x,\gamma(\pm r)))$ be the Busemann function associated to the rays $\gamma^{\pm}(t) := \gamma(\pm t)$, $t \geq 0$. As follows from the triangle inequality, $b^{+} + b^{-} \leq 0$. On the other hand, $b^{+} + b^{-} = 0$ on γ. Using a slight generalization of E. Hopf's maximum principle due to Calabi (see § 6), we get:

Lemma 3: $b^{+} + b^{-} \equiv 0$, and b^{+} is once differentiable with $\|\text{grad } b^{+}\| = 1$. The asymptotes of $\gamma^{\pm}$ are uniquely determined at each point and fit together to a line.

Proof: Let $b^{\pm}_{p,r}$ be a support function of $b^{\pm}$ at p as in § 2. Then $b^{+}_{p,r} + b^{-}_{p,r}$ is a support function of $b^{+} + b^{-}$ at p and by the corollary of Lemma 1 we get $\Delta(b^{+}_{p,r} + b^{-}_{p,r})(p) \geq -2(n-1)/r$. Now the maximum principle implies $b^{+} + b^{-} \equiv 0$.

From $b^{+}_{p,r} \leq b^{+} = -b^{-} \leq -b^{-}_{p,r}$ with equality at p we get that $b^{\pm}$ is once differentiable at p, and $\text{grad } b^{\pm}(p) = \text{grad } b^{\pm}_{p,r}(p)$, in particular $\|\text{grad } b^{\pm}\| = 1$. Consequently, the asymptotes of γ^{+} and γ^{-} at any point $p \in M$ are uniquely determined and fit together to an unbroken geodesic. Since the restriction of an asymptote to any unbounded interval is an asymptote itself, such a geodesic must be a line. This completes the proof of Lemma 3.

Now, it follows from Lemma 2 that $\lim_{r\to\infty}$ Hess $b^{\pm}_{p,r}(p) = 0$. Thus, for any geodesic c, the functions $b^{\pm} \circ c$ have support funcions at any $t \in \mathbb{R}$ with arbitrarily small 2^{nd} derivative at t. If $l : [a,b] \to \mathbb{R}$ is any affine funcion, the same is true for $b^{\pm} \circ c - l$. Hence the functions $b^{\pm} \circ c$ are convex by the (trivial one-dimensional) maximum principle. Therefore, $b^{+} = - b^{-}$ is convex and concave and thus has totally geodesic level sets. Now grad b^{+} is a parallel vector field (in particular Killing), and it follows easily that the mapping

$$(b^{+})^{-1}(0) \times \mathbb{R} \to M \ , \quad (p,t) \mapsto \exp(t \cdot \mathrm{grad}\, b(p))$$

is an isometry.

5. Concluding Remarks

1. If one admits the use of the existence and regularity theory of elliptic equations, one does not need Lemma 2 and parts of Lemma 3. For one may apply the maximum principle directly to $b^{\pm} - h^{\pm}$ where $h^{\pm}$ is a harmonic function on some ball with the same boundary values as $b^{\pm}$. This shows that $b^{\pm} \le h^{\pm}$, i.e. $b^{\pm}$ is subharmonic. (This argument applies to *any* Busemann function.) From $b^{+} + b^{-} = 0$ in Lemma 3 (from the maximum principle again) it follows that b^{+} is sub- and superharmonic, hence harmonic and thus C^{∞}. By Lemma 1, Hess $b^{+} \equiv 0$. This argument also simplifies the proofs of the subharmonicity of the Busemann functions given in [3] and [9].

2. In case of nonpositive sectional curvature the proof simplifies considerably. The statements analogous to Lemma 1 and its corollary together with the 1-dimensional maximum principle immediately imply the convexity of any Busemann function as well as that of $b^+ + b^-$.

3. Using Remark 2 of §3 we finally indicate another elementary proof of the Splitting Theorem. By that remark it is only necessary to show $b^\pm \in C^2(M)$. As above we conclude $b^+ = -b^-$, hence $b^\pm \in C^1(M)$ (note that a limit of asymptotes is an asymptote) and the unique asymptotes extend to lines. Now, if $L_r^\pm(p) := \text{Hess}\, b_{p,r}^\pm(p)$ then $L^\pm := \lim_{r\to\infty} L_r^\pm$ are continuous, $L^+ = -L^-$ and $L_r^\pm \to L^\pm$ monotonously (cf. the first part of Lemma 2). Working locally (i.e. in $\mathbb{R}^n$), it is enough to show that $\text{Hess}\, b^\pm * \varphi_n \to L^\pm$ locally uniformly where $\{\varphi_n\}$ is a smooth approximation of the δ-function. This follows from

$$\text{Hess}\, b^\pm * \varphi_n(p) \geq \text{Hess}\, b_{p,r}^\pm \,\tilde{*}\, \varphi_n(p) = L_r^\pm * \varphi_n(p),$$

where $b_{p,r}^\pm \,\tilde{*}\, \varphi_n(x) = \int b_{p+u,r}^\pm(x+u)\, \varphi_n(u)\,du.$

6. The maximum principle

For completeness and to demonstrate its simplicity we give a proof of the Hopf-Calabi maximum principle.

<u>Lemma</u>: (E. Hopf [7], E. Calabi [2]).
Let M be a connected Riemannian manifold and $f \in C^0(M)$. If for each $p \in M$ and any $\varepsilon > 0$ there is a support function $f_{p,\varepsilon}$ of f at p which is C^2 in a neighborhood of p and satisfies $\Delta f_{p,\varepsilon}(p) \geq -\varepsilon$ then f attains no maximum unless it is constant.

<u>Proof</u>: If f attains a maximum at $p \in M$ and is not constant in any neighborhood of p, we choose a neighborhood U diffeomorphic to an open ball such that $\partial U \neq \partial' U := \{x \in \partial U;\ f(x) = f(p)\}$. Further pick a C^∞ function h with the properties (i) $h(p) = 0$, (ii) $\Delta h > 0$ in U and (iii) $h < 0$ on $\partial' U$.
h can be constructed easily in the form $e^{\alpha\varphi} - 1$ with sufficiently large $\alpha > 0$, since $\Delta(e^{\alpha\varphi}-1) = (\alpha^2 \, \|\text{grad}\ \varphi\|^2 + \alpha\,\Delta\varphi)\, e^{\alpha\varphi}$. If $\eta > 0$ is sufficiently small, we have $(f+\eta h)(x) < f(p) = (f+\eta h)(p)$ for all $x \in \partial U$. This shows that $f + \eta h$ attains a maximum in U, say at q. Since $f_{q,\varepsilon} + \eta h$ is a support function of $f + \eta h$ at q, also $f_{q,\varepsilon} + \eta h$ has maximum at q. But $\Delta(f_{q,\varepsilon} + \eta h)(q) > 0$ for sufficiently small ε contradicting the fact that the Hessean of a function at a maximum must be negative semidefinite.

Thus the set of points where f attains its maximum is open and closed.

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