# The Fundamental Equations of Minimal Surfaces in $\mathbb{C} P^{2}$ 

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## 0. Introduction

The investigation of minimal surfaces in complex projective $n$-space recently gained interest in connection with physical applications. Din and Zakrzewski [5,6], Burns [2], and Glaser and Stora [11] classified all harmonic mappings of the 2 -sphere into complex projective $n$-space. This was proved rigorously by Eells and Wood $[7,8]$ (who also obtained results for surfaces of higher genus) and, on a different way, by Wolfson [25]. The main tool for showing the completeness of the classification was the non-existence of global holomorphic differentials on $S^{2}$.

Our intention was to give a characterization of minimal surfaces in terms of local invariants alone. In case of a closed oriented surface, integration of these local invariants should lead to topological restrictions for minimal surfaces. This could be carried out successfully in the case of real codimension 2 where the geometry of the normal bundle is given by a single curvature function, called normal curvature $K_{N}$. We were motivated by earlier work ([14], see also the forthcoming paper [9]) in which such a characterization was given for minimal surfaces in $S^{4}$ in terms of the Gaussian and the normal curvature. Replacing $S^{4}$ with $\mathbb{C} P^{2}$, we need an

[^0]additional invariant related to the complex structure: the Kähler angle which was introduced by Chern and Wolfson [4] (see Sect. 1 for definition). As a particular case, we characterize the induced metrics of holomorphic and totally real minimal immersions by a condition on the Gaussian curvature only. These local results are stated in Sects. 1 and 2 while the proofs of the main theorems are given in Sect. 8 and 9 . As a tool, we need an existence and congruence theorem for mappings into symmetric spaces which is derived in Sect. 7. We wish to mention that all local results can easily be generalized to Kähler 4-manifolds of arbitrary constant holomorphic sectional curvature $\kappa$. Since the projective plane is the most interesting space for global applications, we restrict our attention to the case $\kappa=4$.

Global applications are given in Sects. 4-6. We avoid refering to the RiemannRoch theorem but instead we use elementary properties of the Laplacian. In particular, we get constraints for compact minimal immersions in terms of the genus, the degree and the self-intersection number (Sects. 3 and 4). Curvature conditions are given in Sect. 6.

## 1. Generalities

Let ( $M, d s^{2}$ ) be an oriented 2-dimensional Riemannian manifold ("surface") and $(P,\langle\rangle$,$) an oriented Riemannian 4-manifold. The tangent bundles of M$ and $P$ are denoted by $T M$ and $T P$. Let $f: M \rightarrow P$ be an isometric immersion with differential $d f$. By means of $d f$ we may consider $T M$ as a subbundle of the induced bundle $f^{*} T P$ over $M$. This gives rise to the orthogonal decomposition $f^{*} T P=T M \oplus N M$, where $N M$ denotes the normal bundle of the immersion $f$. Let $\nabla$ denote the connection on $f^{*} T P$, induced by the Levi-Civita connection on TP, and let $\nabla=\nabla^{T}+\nabla^{N}$ be the corresponding decomposition. Then $\nabla^{T}$ is the LeviCivita connection of $M$ and $\nabla^{N}$ the so called normal connection. Let $R, R^{T}$, and $R^{N}$ be the curvature tensors of $\nabla, \nabla^{T}$, and $\nabla^{N}$. Choose oriented orthonormal local frames ( $e_{1}, e_{2}$ ) of $T M$ and ( $e_{3}, e_{4}$ ) of $N M$. The real valued functions

$$
K=\left\langle R^{T}\left(e_{1}, e_{2}\right) e_{2}, e_{1}\right\rangle \quad \text { and } \quad K_{N}=\left\langle R^{N}\left(e_{1}, e_{2}\right) e_{4}, e_{3}\right\rangle
$$

called Gaussian curvature and normal curvature, are independent of the choice of the frames. The Gauss-Bonnet-Chern theorem relates these to the Euler numbers $\chi$ of $T M$ and $\chi_{N}$ of $N M$ :

$$
\chi=2 \pi \int_{M} K d v, \quad \chi_{N}=2 \pi \int_{M} K_{N} d v
$$

where $d v$ denotes the volume element of $\left(M, d s^{2}\right)$.
The second fundamental form $A: T M \otimes T M \rightarrow N M$ is defined by $A(X, Y)$ $=\left(\nabla_{X} Y\right)^{N}$. Locally, $A$ can be expressed by the symmetric 2 -forms $A_{3}=\left\langle A, e_{3}\right\rangle$ and $A_{4}=\left\langle A, e_{4}\right\rangle$ with corresponding matrices $A_{a, i j}=\left\langle V_{e_{i}} e_{j}, e_{a}\right\rangle$ with $i, j \in\{1,2\}$, $a \in\{3,4\}$. These are related to the curvatures by Gauss and Ricci equations:

$$
K=\hat{K}+\operatorname{det} A_{3}+\operatorname{det} A_{4}, \quad K_{N}=\hat{K}_{N}+\left\langle\left[A_{3}, A_{4}\right] e_{2}, e_{1}\right\rangle,
$$

where

$$
\hat{K}=\left\langle R\left(e_{1}, e_{2}\right) e_{2}, e_{1}\right\rangle, \quad \hat{K}_{N}=\left\langle R\left(e_{1}, e_{2}\right) e_{4}, e_{3}\right\rangle .
$$

The immersion $f$ is called minimal if $\operatorname{trace}(A)=A_{11}+A_{22}$ vanishes. In this case, the matrices of $A_{3}$ and $A_{4}$ have the form

$$
A_{3}=\left(\begin{array}{rr}
a & b \\
b & -a
\end{array}\right) \quad \text { and } \quad A_{4}=\left(\begin{array}{rr}
c & d \\
d & -c
\end{array}\right)
$$

and the Gauss and Ricci equations become

$$
\begin{gather*}
K=\hat{K}-\left(a^{2}+b^{2}+c^{2}+d^{2}\right),  \tag{1.1}\\
K_{N}=\hat{K}_{N}+2(a d-b c) . \tag{1.2}
\end{gather*}
$$

Define the bundle mapping $\bar{A}: T M \rightarrow N M$ by $\bar{A}(X)=A(X, X)$. Let $T_{p}^{1} M$ denote the unit circle in $T_{p} M$. Then $\bar{A}\left(T_{p}^{1} M\right)$ is a (possibly degenerated) ellipse in $N_{p} M$, which is doubly covered by $T_{p}^{1} M$ and in the minimal case centered at the origin. This is called the ellipse of curvature. Namely, if $X=(\cos \theta) e_{1}+(\sin \theta) e_{2}$, then in the minimal case $A_{11}=-A_{22}$ we have $\bar{A}(X)=(\cos 2 \theta) A_{11}+\sin (2 \theta) A_{12}$. The oriented area surrounded by the ellipse is

$$
\pi \operatorname{det}\left(A_{11}, A_{12}\right)=\pi(a d-b c)=\frac{\pi}{2}\left(K_{N}-\hat{K}_{N}\right)
$$

Thus outside the zero section, $\bar{A}$ has degree 2 if $K_{N}>\hat{K}_{N}$ and degree -2 if $K_{N}<\hat{K}_{N}$. The ellipse of curvature is a circle if and only if $A_{11} \perp A_{12}$ and $\left\|A_{11}\right\|=\left\|A_{12}\right\|$, hence iff $a= \pm d, b= \pm c$.

Now assume further that the manifold $P$ is Kählerian, i.e. there is an orthonormal (1,1)-tensor field $J$ on $P$ with $\nabla J=0$ and $J^{2}=-I$. $J$ defines the structure of a complex vector space on each tangent space of $P$ : If $c=a+i b$ is an arbitrary complex number and $X \in T_{x} P$, then

$$
c X:=a X+b J X \in T_{x} P
$$

The Kähler form $\phi \in \Omega^{2}(P)$, defined by $\phi(X, Y)=\langle J X, Y\rangle$, represents a cohomology class in $H^{2}(P, \mathbb{R})$. Let $a$ be the smallest positive number such that $a \phi$ represents an integral class. If $P=\mathbb{C} P^{2}$ with holomorphic sectional curvature 4 (see below), then $a=\pi^{-1}$.

If an immersion $f: M \rightarrow P$ is given, $J$ is pulled back to an endomorphism of the bundle $f^{*} T P$ which will be called $J$, too. We define a smooth function $C: M \rightarrow \mathbb{R}$, called Kähler function, by

$$
C=\left\langle J e_{1}, e_{2}\right\rangle=\left(f^{*} \phi\right)\left(e_{1}, e_{2}\right)
$$

The last expression shows that this definition does not depend on the choice of the frame and hence $C$ is globally well defined. Therefore, if $M$ is compact and oriented,

$$
d:=a \int_{M} C d v
$$

is an integral number, namely the degree of the map $f$, defined by its $2^{\text {nd }}$ cohomology $f^{*}: H^{2}(P, \mathbb{Z}) \rightarrow H^{2}(M, \mathbb{Z})$.

It is convenient also to introduce the angle $\alpha \in[0, \pi]$ between $e_{1}$ and $J e_{2}$; this will be called Kähler angle [4]. Note that $C=\cos \alpha$. In general, the function $\alpha$ is not smooth at the points where $C= \pm 1$, however we will show that smoothness fails only at isolated points.

The immersion $f$ is called (totally) real if $C=0$ everywhere, i.e. if $J(T M)=N M$. It is called complex if $\sin ^{2} \alpha=1-C^{2}=0$ everywhere, i.e. if $J(T M)=T M$. Then either $C=1$ and $J e_{1}=e_{2}$ (holomorphic case) or $C=-1$ and $J e_{1}=-e_{2}$ (antiholomorphic case). Since $A$ is symmetric and $J$ commutes with $\nabla$, we have for complex immersions $A\left(e_{2}, e_{2}\right)=A\left(J e_{1}, J e_{1}\right)=J^{2} A\left(e_{1}, e_{1}\right)=-A\left(e_{1}, e_{1}\right)$, so a complex immersion is always minimal. Moreover, $A_{12}= \pm A\left(e_{1}, J e_{1}\right)= \pm J A_{11}$, hence the ellipse of curvature is a circle.

Now we restrict our attention to

$$
P=\mathbb{C} P^{2}=\mathbb{C}^{3}-\{0\} / \mathbb{C}^{*}=S^{5} / S^{1}=\left\{[x] ; x \in \mathbb{C}^{3},\|x\|=1\right\}
$$

We choose the Riemannian metric on $P$ so that the projection $\pi: S^{5} \rightarrow P$ becomes a Riemannian submersion. This is the Fubini-Study metric with holomorphic sectional curvature 4. For this metric, we have (e.g. see [19, Vol. II, p. 166])

$$
\begin{equation*}
\hat{K}=3 C^{2}+1, \quad \hat{K}_{N}=3 C^{2}-1 . \tag{1.3}
\end{equation*}
$$

The unitary group $U(3)$ acts on $P$ transitively by holomorphic isometries, more precisely, the group $P U(3)=U(3) / S^{1}$ with $S^{1}=\{\zeta I ; \zeta \in \mathbb{C},|\zeta|=1\}$ is the connected component of the isometry group of $P$.

It is well known that there is another distinguished family of minimal immersions on $P=\mathbb{C} P^{2}$ which can be constructed as follows $[7,8]$. Let $p \in P$ and $x \in \pi^{-1}(p) \subset S^{5} \subset \mathbb{C}^{3}$. Then $d \pi_{x}$ maps the horizontal subspace $H_{x}=\left\{v \in T_{x} S^{5} ; v \perp i x\right\}$ $=(\mathbb{C} x)^{\perp} \subset \mathbb{C}^{3}$ isometrically onto $T_{p} P$. At any other point $x^{\prime} \in \pi^{-1}(p)$ we have $x^{\prime}=e^{i \theta} x$ for some real $\theta$, thus $d \pi_{x^{\prime}}=e^{i \theta} d \pi_{x}$ on $H_{x}=H_{x^{\prime}}$. Therefore, any complex line $L$ in $T_{p} P$ (considered as $\mathbb{C}$-vector space) has a horizontal lift $\hat{L}$ which is a complex line in $\mathbb{C}^{3}$, defined independently of the preimage point $x \in \pi^{-1}(p)$. This line defines an element [ $\hat{L}$ ] of $P=\mathbb{C} P^{2}$. Now, if $\hat{f}: M \rightarrow P$ is a complex immersion, then $d \tilde{f}\left(T_{m} M\right)$ is a complex line $L_{m}$ in $T_{\tilde{f}(m)} P$ for any $m \in M$. Thus we get a mapping $f: M \rightarrow P, f(m)=\left[\hat{L}_{m}\right]$, called the associated map for $\widetilde{f}$ (notation of $[7,8]$ ). This map is still defined if $\tilde{f}$ is an arbitrary non-constant holomorphic or anti-holomorphic mapping from $M$ into $P[7,8]$. We will show that $f$ is an immersion if so is $\tilde{f}$, however, the converse is false. Following the notation of Eells and Wood, we will call an immersion isotropic if $f$ is either complex or associated to a complex immersion (called "associated" for short).

## 2. The Main Results

The results of this and the following chapter are local in the sense that we do not need assumptions about completeness or compactness of the surface.

Let $\left(M, d s^{2}\right)$ be a connected oriented surface. It is well known that any point in $M$ has a neighborhood $U$ which can be mapped conformally onto an open subset of $\mathbb{C}$, i.e. there exists an oriented coordinate chart (called conformal coordinate) $z: U \rightarrow \mathbb{C}$ such that $d s^{2}=\lambda^{2} d z d \bar{z}$ with some positive function $\lambda$ which we will call the conformal factor of $z$. By means of these charts, $M$ gets the structure of a complex curve (Riemann surface), so that we may talk about holomorphic functions on $M$. Recall that the Laplace-Beltrami operator $\Delta$ on $M$ is related to the euclidean Laplacean in the $z$-coordinate, $\Delta^{0}=4 \partial_{z} \partial_{z}$, by

$$
\Delta^{0}=\lambda^{2} \Delta
$$

Also recall

$$
\Delta \log \lambda=-K,
$$

where $K$ denotes the Gaussian curvature of $M$.
A smooth complex valued function $t$ on $M$ is called of holomorphic type if locally $t=t_{0} t_{1}$ where $t_{0}$ is holomorphic and $t_{1}$ smooth without zeros. A nonnegative function $a$ on $M$ is called of absolute value type if there is a function $t$ of holomorphic type on $M$ with $a=|t|$. The zero set of such a function is either isolated or the whole of $M$, and outside its zeros, the function is smooth.

Now let $f: M \rightarrow P=\mathbb{C} P^{2}$ be an isometric minimal immersion. Subtracting Eq. (1.1) from (1.2) and inserting (1.3), we see that the function

$$
k:=K_{N}-K+2=(a+d)^{2}+(b-c)^{2}
$$

is non-negative so that $\sqrt{k}$ is defined. Also, consider the non-negative functions

$$
\begin{aligned}
& c:=\cos \frac{\alpha}{2}=\left(\frac{1}{2}(1+C)\right)^{1 / 2}, \\
& s:=\sin \frac{\alpha}{2}=\left(\frac{1}{2}(1-C)\right)^{1 / 2} .
\end{aligned}
$$

Note that $f$ is (anti-)holomorphic if and only if $s=0(c=0)$ everywhere.
Theorem A. (i) Let $f: M \rightarrow P$ be an isometric minimal immersion. Then the functions $c, s, \sqrt{k}$ defined above are of absolute value type and satisfy

$$
\begin{align*}
\Delta \log c & =\frac{1}{2}\left(K+K_{N}\right)-3 C,  \tag{2.1}\\
\Delta \log s & =\frac{1}{2}\left(K+K_{N}\right)+3 C,  \tag{2.2}\\
\Delta \log \sqrt{k} & =2 K-K_{N} \tag{2.3}
\end{align*}
$$

outside the corresponding zero sets. Moreover, $f$ is associated if and only if $k=0$ everywhere.
(ii) Let $\left(M, d s^{2}\right)$ be a simply connected surface with curvature $K$ and Laplacean $\Delta$. Let $c, s, \sqrt{k}$ be functions of absolute value type on $M$ with $c^{2}+s^{2}=1$ and assume that Eqs. (2.1), (2.2), (2.3) are satisfied. Then there exists an isometric minimal immersion $f: M \rightarrow P$ with Kähler function $C=2 c^{2}-1$ and normal curvature $K_{N}=k$ $+K-2$.

Remark. Equations (2.1), (2.2), (2.3) will be called the fundamental equations of the immersion $f$. Adding and subtracting (2.1) and (2.2), we get

$$
\begin{align*}
& \Delta \log \sin \alpha=K+K_{N},  \tag{2.4}\\
& \Delta \log \tan \frac{\alpha}{2}=6 \cos \alpha, \tag{2.5}
\end{align*}
$$

adding also (2.3) we have

$$
\begin{equation*}
\Delta \log (\sqrt{k} \sin \alpha)=3 K \tag{2.6}
\end{equation*}
$$

Clearly, (2.4), (2.5), (2.6) are equivalent to the fundamental equations. Moreover, we can rewrite (2.1), (2.2) in the form

$$
\begin{aligned}
& -\frac{1}{1+C}\|\nabla C\|^{2}+\Delta C=\left(K+K_{N}-6 C\right)(1+C) \\
& -\frac{1}{1-C}\|\nabla C\|^{2}-\Delta C=\left(K+K_{N}+6 C\right)(1-C)
\end{aligned}
$$

where $\nabla C$ denotes the gradient of the function $C$. Adding these two equations gives

$$
\begin{equation*}
\|\nabla C\|^{2}=\left(1-C^{2}\right)\left(6 C^{2}-\left(K+K_{N}\right)\right), \tag{2.7}
\end{equation*}
$$

while subtraction and application of (2.7) gives

$$
\begin{equation*}
\Delta C=2 C\left(K+K_{N}-3\left(1+C^{2}\right)\right) \tag{2.8}
\end{equation*}
$$

By continuity, (2.7) and (2.8) hold on all of $M$.
Theorem B. (i) Let $\left(M, d s^{2}\right)$ be a simply connected surface and $f: M \rightarrow P$ an isometric minimal immersion. Then there exists a one-parameter family $f_{\theta}: M \rightarrow P, \theta \in S^{1}$, of isometric minimal immersions with the same normal curvature $K_{N}$ and Kähler function $C$ as $f=f_{1}$. Moreover, any isometric minimal immersion with the same normal curvature and Kähler function belongs to this family, up to isometries of $P$. The immersion $f$ is isotropic if and only if $f_{\theta}=f$ for all $\theta \in S^{1}$.
(ii) The family $\left(f_{\theta}\right)_{\theta \in S^{1}}$ is constant up to reparametrization, i.e. there is a oneparameter group $\left(\psi_{t}\right)_{t \in \mathbb{R}}$ of isometries of $M$ with $f_{\theta}=f \circ \psi_{t}$ for $\theta=e^{i t}$, if and only if $\left(\psi_{t}\right)_{t \in \mathbb{R}}$ is a group of rotations with a common fixed point and $C$ and $K_{N}$ are invariant functions.

Remark. Examples of the last type will be constructed later on (see Remark 3 following Theorem 3.11).
Theorem C. Let $\tilde{f}: M \rightarrow P$ be a complex immersion. Then the associated map $f: M \rightarrow P$ is also an immersion and the induced metrics $d s^{2}$ of $f$ and $d \tilde{s}^{2}$ of $\tilde{f}$ are conformal, more precisely $d \tilde{s}^{2}=\mu^{2} d s^{2}$ with $\mu=s=\sin \frac{\alpha}{2}\left(\mu=c=\cos \frac{\alpha}{2}\right)$ if $\tilde{f}$ is holomorphic (antiholomorphic), where $\alpha$ denotes the Kähler angle of $f$.

Theorem D. Let $f: M \rightarrow P$ be a minimal immersion. Then the ellipse of curvature is $a$ circle everywhere if and only if $f$ is either isotropic or real.

The proof of these theorems will be given in Sect. 9.

## 3. Local Consequences

Theorem 3.1. Let $f: M \rightarrow P$ be a minimal immersion with curvature $K$, normal curvature $K_{N}$ and Kähler function $C$. Then

$$
K+K_{N} \leqq 6 C^{2}
$$

with equality if and only if $f$ is complex or real.

Proof. Since $1-C^{2}=4 s^{2} c^{2}$ is an absolute value function, by (2.7) we have either $C^{2}=1$ everywhere or $K+K_{N} \leqq 6 C^{2}$. In the first case, we get either $C=1, c=1$ or $C=-1, s=1$ and either (2.1) or (2.2) imply the equality. Conversely, if $K+K_{N}$ $=6 C^{2}$, then $C=$ const by (2.7), and from (2.5) we get either $C=0$ or $s=0$ or $c=0$, thus $f$ is either real or complex.

Corollary 3.2. Let $f: M \rightarrow P$ be a minimal immersion with curvature $K$ and normal curvature $K_{N}$. Then

$$
K-2 \leqq K_{N} \leqq 6-K
$$

and $K_{N}=K-2$ everywhere if and only if $f$ is associated, and $K_{N}=6-K$ everywhere if and only if $f$ is complex.

Proof. The lower bound together with the equality discussion follows from Theorem A(i). From Theorem 3.1 we get

$$
6-\left(K+K_{N}\right) \geqq 6 C^{2}-\left(K+K_{N}\right) \geqq 0
$$

with equality if and only if $C^{2}=1$.
The following lemma is due to a remark of Kenmotsu:
Lemma 3.3. Let $\left(M, d s^{2}\right)$ be a surface and $C: M \rightarrow \mathbb{R}$ a smooth function with

$$
\|\nabla C\|^{2}=u(C), \quad \Delta C=v(C)
$$

for smooth functions $u, v: \mathbb{R} \rightarrow \mathbb{R}$. Then on $M^{\prime}:=\{\nabla C \neq 0\}$ the Gaussian curvature $K$ satisfies

$$
\begin{align*}
K & =-w(C) / u(C) \\
w: & =\left(v-u^{\prime}\right)\left(v-\frac{1}{2} u^{\prime}\right)+u\left(v^{\prime}-\frac{1}{2} u^{\prime \prime}\right) \tag{*}
\end{align*}
$$

Proof. Put $D=r \circ C$ where $r$ is a principal function of $u^{-1 / 2}$, defined on $C\left(M^{\prime}\right)$. Then $\|\nabla D\|=1$ and hence the integral curves of the vector field $X=\nabla D$ are geodesics which intersect the level curves $S_{t}=\{C=t\}$ orthogonally. Thus, the geodesic curvature $\kappa(m)$ of $S_{t}$ at $m \in S_{t}$ equals $\Delta D(m)$. On the other hand,

$$
\Delta D=r^{\prime \prime}(C)\|\nabla C\|^{2}+r^{\prime}(C) \Delta C=\left(u^{-1 / 2}\left(v-\frac{1}{2} u^{\prime}\right)\right) \circ C .
$$

Note further that the Jacobi fields $J$ separating the integral curves of $X$ satisfy $\nabla_{X} J=\kappa J$, hence $\nabla_{X} \nabla_{X} J=\left(X \kappa+\kappa^{2}\right) J$, and so the curvature is

$$
K=-\left(X \kappa+\kappa^{2}\right)=-\left(\kappa^{\prime} u^{1 / 2}+\kappa^{2}\right)
$$

with

$$
\kappa=u^{-1 / 2}\left(v-\frac{1}{2} u^{\prime}\right) .
$$

Inserting proves the lemma.
Theorem 3.4. Let $f: M \rightarrow P$ be a minimal immersion. Then $f$ is real if and only if $K+K_{N}=0$ everywhere.

Proof. If $f$ is real minimal, then $C=0$ and the equation follows from (2.7). Conversely, if $K+K_{N}=0$, then by (2.7) and (2.8) the assumptions of Lemma 3.3 are satisfied with

$$
u(C)=6 C^{2}\left(1-C^{2}\right), \quad v(C)=-6 C\left(1+C^{2}\right)
$$

Hence either $C=$ const which implies $C=0$ everywhere, i.e. $f$ is real, or $K$ satisfies (*) on some open non-empty subset $M^{\prime}$. Moreover we have $w=24 u$, so we get $K=-24$ on $M^{\prime}$. But this implies $\sqrt{k}=$ const $\neq 0$ and $2 K-K_{N}=3 K \neq 0$ which contradicts (2.3).

Theorem 3.5. Let $f: M \rightarrow P$ be an associated immersion with constant curvature $K$. Then $f(M)$ is totally geodesic, i.e. $f(M) \subset \mathbb{C} P^{1}$ or $f(M) \subset \mathbb{R} P^{2}$ up to isometries of $P$.

Proof. Since $K_{N}=K-2$, we have $K+K_{N}=2 K-2=$ const. Setting $\kappa:=\frac{1}{6}\left(K+K_{N}\right)$, we get again the assumptions of Lemma 3.3, but this time,

$$
u(C)=6\left(-C^{4}+(\kappa+1) C^{2}-\kappa\right), \quad v(C)=6\left(-C^{3}+(2 \kappa-1) C\right)
$$

Thus (*) says that some polynomial in $C$ vanishes identically. Comparison of coefficients gives a contradiction unless $C$ is constant. Now by (2.5) either $C=0$ or $c=0$ or $s=0$, so $f$ is real or complex. In the first case $K=-K_{N}=1$ and $C=0$ as for $\mathbb{R} P^{2}$, thus by Theorem $\mathbf{B}, f(M)$ is a part of $\mathbb{R} P^{2}$, up to isometries of $P$. In the second case we get $K+K_{N}=6$, hence $K=4, K_{N}=2$ and we get in the same way $f(M) \subset \mathbb{C} P^{1}$ up to isometries of $P$ (see below).
Theorem 3.6. Let $\left(M, d s^{2}\right)$ be a simply connected surface, $K$ its Gaussian curvature. Then there is a complex isometric immersion $f: M \rightarrow P$ if and only if $K \leqq 4$ and the function $\sqrt{4-K}$ is of absolute value type satisfying

$$
\begin{equation*}
\Delta \log \sqrt{4-K}=3(K-2) \tag{3.1}
\end{equation*}
$$

outside its zero set. This immersion is unique up to isometries of $P$.
Proof. Under the assumption $C^{2}=1, K+K_{N}=6$ (see Corollary 3.2) Eq. (2.3) becomes (3.1) while one of the conditions (2.1), (2.2) is trivial, the other void. Thus the result follows from Theorem $A$ and Theorem $B(i)$.

Corollary 3.7. Any complex immersion $f: M \rightarrow P$ of constant curvature $K$, up to isometries of $P$ is a parametrization of an open subset of the Veronese surface

$$
V:=\left\{[x] \in P ; x_{0}^{2}+x_{1}^{2}+x_{2}^{2}=0\right\}
$$

with $K=2, K_{N}=4$, or of the totally geodesic projective line

$$
\mathbb{C} P^{1}:=\left\{[x] \in P ; x_{2}=0\right\}
$$

with $K=4, K_{N}=2$.
Proof. By the preceding theorem, we have $K=4$ if and only if $f(M) \subset \mathbb{C} P^{1}$ up to isometries of $P$. If $K=\mathrm{const} \neq 4$ then $K=2$ by (3.1). So there exists a unique complex immersion with $K=2, K_{N}=4$. On the other hand, the inclusion map of the complex curve $V$ is a holomorphic immersion, and $V$ is an orbit of the subgroup
$O(3)$ of $U(3)$ acting by isometries, so $V$ has constant curvature. But as an algebraic curve of degree 2 , it is not congruent to the line $\mathbb{C} P^{1}$. So it must be the surface in question.
Remark. To get a simple parametrization, it is sometimes more convenient to consider the algebraic curve

$$
V^{\prime}:=\left\{[x] \in P ; x_{1}^{2}=2 x_{0} x_{2}\right\}
$$

instead of $V$. Since the defining quadratic forms are conjugate under $U(3)$, the surfaces $V$ and $V^{\prime}$ are congruent. We have $V^{\prime}=f(\tilde{\mathbb{C}})$ for the immersion $f(z):=\left[1, z, \frac{1}{2} z^{2}\right]$ (cf. [7]).

Theorem 3.8. Let $\left(M, d s^{2}\right)$ be a simply connected surface with Gaussian curvature $K$. There exists an isometric realminimal immersion $f: M \rightarrow P$ if and only if $K \leqq 1$ and the function $\sqrt{1-K}$ is of absolute value type satisfying

$$
\begin{equation*}
\Delta \log \sqrt{1-K}=3 K \tag{3.2}
\end{equation*}
$$

outside its zero set. In fact, there is exactly a one-parameter family of such immersions (up to isometries of $P$ ) which has constant image if and only if $\left(M, d s^{2}\right)$ is a surface of revolution.
Proof. This is another special case of the Theorems A and B under the assumption $C=0, K+K_{N}=0$ (see Theorem 3.4).

Considering the real minimal immersions of constant curvature, we already know two examples. The first is the real projective plane

$$
\mathbb{R} P^{2}:=\left\{[x] \in P ; \bar{x}_{a}=x_{a} \text { for } a=0,1,2\right\} .
$$

This is the fixed point set of the complex conjugation of the coordinates which is an isometry of $P$; hence $\mathbb{R} P^{2}$ it totally geodesic. The second is the surface introduced in [22] which we would like to call Clifford torus $C T$ :

$$
C T:=\left\{[x] \in P ; x_{0} \bar{x}_{0}=x_{1} \bar{x}_{1}=x_{2} \bar{x}_{2}\right\} .
$$

This is an orbit of the subgroup $U(1) \times U(1) \times U(1)$ of $U(3)$, hence $K, K_{N}, C$ and the length of the mean curvature vector field $\eta$ are constant. Now the permutation group of the coordinates, $S_{3} \subset U(3)$, acts linearly on $T_{p} P$ for $p=[1,1,1]$ with $\eta(p)$ fixed which implies $\eta(p)=0$; hence $C T$ is minimal. Since $C=$ const and $C T$ is not complex, we have $C=0$ by (2.5), and $K=0$ because $C T$ is a torus. It is well known that the inclusion of $C T$ is the only flat real minimal immersion in $P[22,18]$. The following is a generalization:
Corollary 3.9. Any real minimal immersion $f: M \rightarrow P$ of constant curvature $K$, up to isometries of $P$ is a parametrization of an open subset of $\mathbb{R} P^{2}$ or of CT.

Proof. By (3.2) either $K=1$ or $K=0$, and the corresponding minimal surfaces are uniquely determined up to isometries of $P$, due to Theorem 3.6. So the two surfaces introduced above are the only examples.

A submanifold $M$ of $P$ which is an orbit of a subgroup of the isometry group of $P$ is called homogeneous. For homogeneous surfaces in $P$, the functions $K, K_{N}$, and
$C$ are constant, and in particular $C=0$ or $C^{2}=1$ by (2.5). Therefore, we have also proved:

Corollary 3.10. The only homogeneous minimal surfaces in $P$ are $\mathbb{C} P^{1}, V, \mathbb{R} P^{2}$, and $C T$, up to isometries of $P$.

The following theorem shows how to construct locally all non-complex minimal immersions in $P$. We use the following notion: If $a, b$ are functions of absolute value type on a surface $M$, then $a$ dominates $b$ if $a / b$ is bounded near any zero of $b$.

Theorem 3.11. Let $U$ be an open, simply connected subset of $\mathbb{C}, h$ a holomorphic function on $U$ and $a:=|h|$. Let $p, q$ be nonzero functions of absolute value type on $U$ which have no common zeros and which are dominated by a. Suppose that p, q satisfy the following equations:

$$
\begin{align*}
& \Delta^{0} \log p=a^{2} /\left(2 p^{2} q^{2}\right)-4 p^{2}+2 q^{2}  \tag{3.3}\\
& \Delta^{0} \log q=a^{2} /\left(2 p^{2} q^{2}\right)+2 p^{2}-4 q^{2} \tag{3.4}
\end{align*}
$$

Then there exists a conformal minimal immersion $f: U \rightarrow P$ with Kähler angle $\alpha=2 \arctan (p / q)$ and conformal factor $\lambda=\left(p^{2}+q^{2}\right)^{1 / 2}$. In fact, there is exactly a oneparameter family of such immersions. The immersion $f$ is associated iff $a=0$ everywhere, and $f$ is real iff $p=q$.

Conversely, any conformal local parametrization of a non-complex immersed minimal surface in $P$ is of this type.

Proof. We start proving the last statement. If a minimal immersion $f: M \rightarrow P$ is given, we may pull back all functions on $M$ to an open ball $U$ in $\mathbb{C}$ by means of a coordinate $z$ which is conformal with respect to the induced metric on $M$. Put $p=\lambda c, q=\lambda s, a=\lambda \sqrt{k} p q$. Now by (2.6) we have

$$
\Delta \log a=\Delta \log \sqrt{k} c s+3 \Delta \log \lambda=0 .
$$

Therefore, $a=|h|$ for some holomorphic function $h$ : See the subsequent Lemma 3.12. Moreover, $a$ dominates $p$ and $q$. Now (3.3) and (3.4) follow from (2.1) and (2.2) since $\lambda^{2}=p^{2}+q^{2}, k=a^{2} /\left(p^{2} q^{2} \lambda^{2}\right)$, and $A^{0}=\lambda^{2} \Delta$.

Conversely, assume that functions $p, q$ on $U$ are given which satisfy the assumptions of the theorem. Put $\lambda^{2}=p^{2}+q^{2}>0, c=p / \lambda, s=q / \lambda$, and let $K$ be the curvature of the metric $d s^{2}=\lambda^{2} d z d \bar{z}$ on $U$. Set $k=a^{2} /\left(p^{2} q^{2} \lambda^{2}\right)$ and $K_{N}=k+K-2$. Then (3.3) and (3.4) imply (2.1) and (2.2), and (2.6) and hence (2.3) follow from $\Delta \log \lambda=-K$ unless $a=0$ everywhere. In the latter case we get $k=0$ which is the associated case. So the result follows from the Theorems A and B. Clearly, $p=q$ iff $c=s$ which is the case of a real minimal immersion.

Remarks. 1. Locally, we may always assume that $U$ is a disk around 0 and $a=r^{n}$ for some-non-negative integer $n$, where $r(z):=|z|$, unless $a=0$ everywhere. Namely, if $f: M \rightarrow P$ is given and $m \in M$ fixed, the absolute value type function $a=\lambda^{3} \sqrt{k} c s$ is of some order $n \geqq 0$ in $m$, i.e. for any conformal coordinate $z$ around $p$ with $z(m)=0$ there is a holomorphic function $h$ with $h(0) \neq 0$ such that $a=\left|z^{n} h(z)\right|$ (see Lemma 3.12). If $z=\psi(w)$ is a biholomorphic transformation, for the new
coordinate $w$ we have $d z=\psi^{\prime}(w) d w$, so $d s^{2}=\tilde{\lambda}^{2} d w d \bar{w}$ with $\tilde{\lambda}=\lambda\left|\Psi^{\prime}\right|$. Hence, with respect to the coordinate $w$ we get a new function $\tilde{a}=\left|\psi^{\prime}\right|^{3} \cdot a \circ \psi$ instead of $a$. We are looking for a transformation $\psi$ with $\tilde{a}=|w|^{n}$, i.e. we have to solve the differential equation

$$
\psi^{\prime}(w)^{3} \cdot \psi(w)^{n} \cdot h(\psi(w))=w^{n}
$$

Instead, we look for the inverse $\operatorname{map} \eta=\psi^{-1}, \eta(z)=w$. This satisfies the differential equation

$$
\eta^{\prime}(z)^{3} \eta(z)^{n}=z^{n} h(z)
$$

which has a nonsingular solution around $z=0$, according to Lemma 3.13 below.
2. Since $p$ and $q$ have no common zeros, we may assume that $p$ is nowhere zero around 0 while $q=r^{k} q_{1}$ with $k$ a nonegative integer and $q_{1}$ nowhere vanishing. Putting

$$
u=\log \left(q_{1} p\right), \quad v=\log \left(q_{1} / p\right)
$$

and assuming $a=r^{k+m}$ with $m \geqq 0$, we get from (3.3), (3.4) the equivalent equations

$$
\begin{align*}
& \Delta^{0} u=r^{2 m} e^{-2 u}-2 e^{u}\left(e^{-v}+r^{2 k} e^{v}\right),  \tag{3.5}\\
& \Delta^{0} v=6 e^{u}\left(e^{-v}-r^{2 k} e^{v}\right), \tag{3.6}
\end{align*}
$$

while in the case $a=0$ (3.5) is replaced with

$$
\begin{equation*}
\Delta^{0} u=-2 e^{u}\left(e^{-v}+r^{2 k} e^{v}\right) \tag{3.5}
\end{equation*}
$$

Note that we may apply the Cauchy-Kowalewski Theorem to these equations [17] so that we get local solutions for any $k$ and $m$.
3. In particular, we may consider the case where all functions are circular symmetric. For a circular symmetric function $t=t(r)$ we have

$$
\Delta^{0} t=t^{\prime \prime}+\frac{1}{r} t^{\prime}=\frac{1}{r}\left(r t^{\prime}\right)^{\prime}=: D t
$$

Note that the equation $D t(r)=g(r)$ has a unique solution $t$ with $t(0)=t^{\prime}(0)=0$, for any given function $g$, namely

$$
t(r)=\int_{0}^{r} \frac{1}{s}\left(\int_{0}^{s} \sigma g(\sigma) d \sigma\right) d s
$$

and if $|g| \leqq \beta$ on $[0, r]$, then $|t| \leqq \frac{\beta}{4} r^{2}$. This shows that the Eq. (3.5),(3.6) have always circular symmetric solutions $t=(u(r), v(r))$ with $t(0)=t^{\prime}(0)=0$. The corresponding minimal surface in $P$ has a rotation-invariant metric, and the rotations leave also invariant the functions $c, s, k$. So its one-parameter family of minimal surfaces has constant image in the sense of Theorem B(ii).

Lemma 3.12. Let a be an absolute value type function on an open, simply connected subset $U$ of $\mathbb{C}$, and assume $\Delta^{0} \log a=0$. Then there exists a holomorphic function hon $U$ with $|h|=a$.

Proof. In a neighborhood of any $z_{0} \in U$ we have $a=\left|z-z_{0}\right|^{k} a_{1}$ for a positive function $a_{1}$ and some nonnegative integer $k$. Let $b=\log a_{1}$, then $\Delta^{0} b=0$. Therefore, $b$ is the real part of a holomorphic function $g$. So the function $h:=\left(z-z_{0}\right)^{k} e^{g}$ is holomorphic and satisfies $|h|=a$. Any other holomorphic function with this property differs from $h$ only by a constant factor of unit length. Since $U$ is simply connected, we may choose this factor constant on the whole of $U$ and thus extend $h$ to all of $U$.

Lemma 3.13. Let $h$ be a holomorphic function defined in a neighborhood of 0 in $\mathbb{C}$, with $h(0) \neq 0$. Then the complex differential equation

$$
\begin{equation*}
y^{3} y^{n}=x^{n} h(x) \tag{*}
\end{equation*}
$$

has a holomorphic solution $y$ around 0 with $y(0)=0, y^{\prime}(0) \neq 0$ (in fact exactly $n+3$ such solutions).

Proof. Let $h_{1}$ be a holomorphic function, defined on some open neighborhood $U_{1}$ of 0 , with $h_{1}^{3}=h$. There is an analytic solution $y_{1}$ of the equation

$$
\frac{3}{n+3} x y_{1}^{\prime}+y_{1}=h_{1}
$$

defined on some neighborhood $U_{2}$ of 0 , contained in $U_{1}$. This solution is obtained easily as a convergent power series. In particular, $y_{1}(0)=h_{1}(0) \neq 0$. On an even smaller neighborhood $U_{3}$, there is a holomorphic function $y_{2}$ with $y_{2}^{n+3}=y_{1}^{3}$. Now $y:=x y_{2}$ is a solution of the Eq. $(*)$ with $y(0)=0, y^{\prime}(0)=y_{2}(0) \neq 0$.

## 4. Topological Restrictions

Throughout this section, let $\left(M, d s^{2}\right)$ be a compact oriented surface. If $a$ is a positive function on $M$, then $\int_{M} \Delta \log a=0$ by the divergence theorem. If $a$ is a nonzero function of absolute value type, i.e. locally $a=\left|t_{0}\right| \cdot a_{1}$ with $t_{0}$ holomorphic and $a_{1}$ smooth, positive, then $\Delta \log a$ is still bounded on $M-\{a=0\}$, and the integral can be computed as follows: For any $p \in M$ with $a(p)=0$, the order $k \geqq 1$ of $a$ at $p$, by definition is the order of $t_{0}$ at $p$. Let $N(a)$ be the sum of all orders for all zeros of $a$.

Lemma 4.1. $\int_{M} \Delta \log a=-2 \pi N(a)$.
Proof. Let $Z:=a^{-1}(0)$; this is a finite set. For any $p_{0} \in Z$, choose a conformal coordinate $z$ around $p_{0}$ with conformal factor $\lambda$, and let

$$
U_{\varepsilon}\left(p_{0}\right)=\left\{p \in M ;\left|z(p)-z\left(p_{0}\right)\right|<\varepsilon\right\} .
$$

For small $\varepsilon$, the $U_{\varepsilon}\left(p_{0}\right), p_{0} \in Z$, are disjoint. Put $M_{\varepsilon}:=M-\bigcup_{p_{0} \in Z} U_{\varepsilon}\left(p_{0}\right)$. Then by the divergence theorem,

$$
\int_{M_{\varepsilon}} \Delta \log a d v=\sum_{p_{0} \in Z} \int_{\partial U_{\varepsilon}\left(p_{0}\right)}\langle V \log a, v\rangle d s,
$$

where $d v$ denotes the volume element of $M, d s$ the line element of $\partial U_{\varepsilon}\left(p_{0}\right)$ and $v$ the unit normal vector of $\partial U_{\varepsilon}\left(p_{0}\right)$ pointing inward. Fix a zero $p_{0} \in Z$ and let $r:=\left|z-z\left(p_{0}\right)\right|$. If $a$ has order $k$ at $p_{0}$, we have $a=r^{k} a_{1}$ around $p_{0}$ for some smooth positive function $a_{1}$. Now $\left\langle V \log a_{1}, v\right\rangle$ is bounded, and so its integral over $\partial U_{\varepsilon}\left(p_{0}\right)$ gets arbitrarily small as $\varepsilon \rightarrow 0$. So we are left with the remaining integrand which is

$$
\left\langle\nabla \log r^{k}, v\right\rangle=k v(\log r)=-\frac{k}{\lambda} \frac{d}{d r}(\log r)=-\frac{k}{\lambda r}
$$

So we get

$$
\int_{\partial U_{\varepsilon}\left(p_{0}\right)}\langle V \log a, v\rangle d s=\int_{\{r=\varepsilon\}} \frac{-k}{\lambda r} d s=-\frac{k}{\varepsilon} \int_{\{r=\varepsilon\}}|d z|=-2 \pi k,
$$

and the result follows as we let $\varepsilon$ go to 0 .
Note that the orientability was not needed in this proof. Applying this lemma to the functions $c, s, \sqrt{k}$ defined in Sect. 2, we get immediately from Theorem A(i) and the integral formulas of Sect. 1:

Theorem 4.2. Let $f: M \rightarrow P$ be a minimal immersion with degree $d$ and normal characteristic $\chi_{N}$. Then

$$
\begin{equation*}
\frac{1}{2}\left(\chi+\chi_{N}-3 d\right)=-N(c) \tag{4.1}
\end{equation*}
$$

unless $f$ is anti-holomorphic,

$$
\begin{equation*}
\frac{1}{2}\left(\chi+\chi_{N}+3 d\right)=-N(s) \tag{4.2}
\end{equation*}
$$

unless $f$ is holomorphic, and

$$
\begin{equation*}
2 \chi+\chi_{N}=-N(\sqrt{k}) \tag{4.3}
\end{equation*}
$$

unless $f$ is associated.
Adding these equations, we see
Corollary 4.3. Either $f$ is isotropic or

$$
\begin{equation*}
\chi(M)=-N(c s \sqrt{k}) . \tag{4.4}
\end{equation*}
$$

In particular, if $M$ is a sphere, $f$ must be isotropic which is well known [2, 4-8, 25].
Moreover, inserting (4.3) in (4.1), (4.2) we get

$$
\begin{equation*}
3(\chi-d)=-N\left(c^{2} \sqrt{k}\right) \tag{4.5}
\end{equation*}
$$

unless $f$ is antiholomorphic or associated, and

$$
\begin{equation*}
3(\chi+d)=-N\left(s^{2} \sqrt{k}\right) \tag{4.6}
\end{equation*}
$$

unless $f$ is holomorphic or associated. In particular, we get
Corollary 4.4. Either $f$ is isotropic or

$$
\chi(M) \leqq-|d|
$$

This has also been proved by Eells and Wood [7], using the Riemann-Roch theorem.

Furthermore, integrating the inequalities of Corollary 3.2 we get
Theorem 4.5. Let $f: M \rightarrow P$ be a minimal immersion. Then

$$
\begin{equation*}
\operatorname{area}(M) \geqq \pi\left(\chi-\chi_{N}\right) \tag{4.7}
\end{equation*}
$$

with equality if and only if $f$ is associated, and

$$
\begin{equation*}
\operatorname{area}(M) \geqq \frac{\pi}{3}\left(\chi+\chi_{N}\right) \tag{4.8}
\end{equation*}
$$

with equality if and only if $f$ is complex.
In particular, for isotropic minimal immersions there are no other minimal immersions in the same isotopy class which have smaller area.

By means of the following lemma, we often may remove $\chi_{N}$ in the equations above, using the self-intersection number $I_{f}$ :
Lemma 4.6. Let $f: M \rightarrow P$ be an immersion of degree $d$ and self-intersection number $I_{f}$ which has only regular self intersections which are of multiplicity 2 . Then

$$
\begin{equation*}
\chi_{N}=d^{2}-2 I_{f} . \tag{4.9}
\end{equation*}
$$

Proof. Let $S \subset f(M)$ be the set of points which have two pre-images under $f . S$ is a finite subset of $P$. Let $s \in S$ and $x, y \in M$ the two pre-images. We define $s$ to have the weight $w(s)=+1$ if $d f\left(T_{x} M\right)$ and $d f\left(T_{y} M\right)$ together define the positive orientation on $T_{s} P$, otherwise $w(s)=-1$. The self-intersection number is defined as

$$
I_{f}=\sum_{s \in S} w(s) .
$$

Put $X:=f^{-1}(S) \subset M$, and for $x \in X$ let $w(x):=w(f(x))$. To $X$, we assign the zerocycle

$$
[X]=\sum_{x \in X} w(x) x=2 I_{f} g \in H_{0}(M),
$$

where $g$ denotes the generator of $H_{0}(M)$ dual to $1 \in H^{0}(M)$. Call $D_{P}: H^{*} P \rightarrow H_{*} P$ and $D_{M}: H^{*} M \rightarrow H_{*} M$ the Poincaré duality maps and $[M] \in H_{2}(M)$ the fundamental class of $M$. Let $e \in H^{2} M$ denote the Euler class of the normal bundle $N M$ of $f$. By Herbert [16, pp. ix, x] and Lashof and Smale [21],

$$
[X]=D_{M}\left(f^{*} D_{P}^{-1} f_{*}[M]-e\right) .
$$

If we denote the pairing $H^{*} \otimes H_{*} \rightarrow \mathbb{Z}$ by $\langle$,$\rangle , we have$

$$
\left\langle 1, D_{M} f^{*} D_{P}^{-1} f_{*}[M]\right\rangle=\left\langle f^{*} D_{P}^{-1} f_{*}[M],[M]\right\rangle=\left\langle D_{P}^{-1} f_{*}[M], f_{*}[M]\right\rangle .
$$

Let $w$ be a generator of $H_{2}(P)$. Then $f_{*}[M]= \pm d \cdot w$ and $\left\langle D_{P}^{-1} w, w\right\rangle=1$. Moreover, $D_{M} e=\chi_{N} \cdot g$. Therefore $2 I_{f}=d^{2}-\chi_{N}$ which finishes the proof.

In particular, the last result yields necessary conditions for a minimal embedding where we have $\chi_{N}=d^{2}$. By choice of the orientation of $M$, we may always assume $d \geqq 0$.

Theorem 4.7. Let $M$ be a compact orientable surface of genus $g$ and $f: M \rightarrow P a$ minimal embedding of degree $d \geqq 0$.
(i) If $f$ is not isotropic, then $2 g \geqq d^{2}+3 d+2$.
(ii) If $f$ is complex, then $2 g=d^{2}-3 d+2$.
(iii) If $f$ is associated, then $g=0$ and $f(M)=\mathbb{C} P^{1}$.

Proof. (i) and (ii) are immediate from (4.1) and (4.2) since $\chi=2-2 g, \chi_{N}=d^{2}$ and in the holomorphic case $c=1$, hence $N(c)=0$. If $f$ is associated and not complex, then $\chi-\chi_{N}>0$ by (4.7) and $\chi+\chi_{N} \leqq 0$ by (4.1) $+(4.2)$. Hence $d^{2}=\chi_{N}<0$, a contradiction. If $f$ is associated and complex, it has constant curvature $K=4$ by Corollary 3.2, and so $f(M)=\mathbb{C} P^{1}$ up to isometries of $P$, by Corollary 3.7 .

Statement (ii) is well known (see [12]). Statement (iii) can be expressed as follows: Any full associated immersion $f: M \rightarrow P$ has nontrivial self-intersections.

## 5. Complex Maps and Associated Immersions

Let $\left(M, d s^{2}\right)$ be a compact orientable surface and $\tilde{f}: M \rightarrow P$ a holomorphic or antiholomorphic mapping. By the choice of orientation on $M$ we may always assume that $\tilde{f}$ is in fact holomorphic. Let $f: M \rightarrow P$ be the associated map, and assume that $f$ is an immersion. Then the holomorphic map $\tilde{f}$ is not necessarily an immersion, too; there may be isolated points where $d f f^{f}$ vanishes. An example is provided by the Neil Parabola where $M=\widehat{\mathbb{C}}$ and $\widetilde{f}(z)=\left[1, z^{2}, z^{3}\right]$. The associated map is

$$
f(z)=\left[-(2+3 r) \bar{z}^{2}, 2-r^{6},\left(3+r^{4}\right) z\right] \quad \text { with } \quad r:=|z| .
$$

Note that $f$ has no singularity at $z=0$ while $\tilde{f}$ has.
Let $d s^{2}, d v, K, K_{N}$ etc. be induced metric, volume element, curvatures etc. with respect to $f$, while the corresponding quantities for $\tilde{f}$ are denoted by $d \tilde{s}^{2}, d \tilde{v}, \tilde{K}, \tilde{K}_{N}$. By Theorem C we have

$$
\begin{equation*}
d \tilde{s}^{2}=s^{2} d s^{2}, \tag{5.1}
\end{equation*}
$$

where $s:=\sin \frac{\alpha}{2}, \alpha$ the Kähler angle of $f$. As above, let $N(s)$ denote the number of zeros of $s$, counted with multiplicities.
Lemma 5.1. $\int_{M} \tilde{K} d \tilde{v}=2 \pi(\chi(M)+N(s))$.
Proof. Comparing the curvatures, we get from (5.1)

$$
\begin{equation*}
s^{2} \tilde{K}=K-\Delta \log s \tag{5,2}
\end{equation*}
$$

hence by means of Lemma 4.1

$$
\begin{aligned}
\int_{M}^{\tilde{K} d \tilde{v}} & =\int s^{2} \tilde{K} d v=\int K d v-\int \Delta \log s d v \\
& =2 \pi(\chi(M)+N(s))
\end{aligned}
$$

Note that a holomorphic curve $\tilde{f}: M \rightarrow P$ has a well defined complex tangent line even at the points where $d \tilde{f}=0$, by Taylor expansion. Therefore also the
normal bundle $\tilde{N} M$ is defined everywhere, and it inherits a connection and a curvature from $\tilde{f}^{*} T P$ as defined in Sect. 1. The curvature form is still expressed by $\tilde{K}_{N} d \tilde{v}$, by continuity. Hence we still have the Gauss-Bonnet formula $\chi_{N}=2 \pi \int \tilde{K}_{N} d \tilde{v}$. Denote the degrees of $f$ and $\tilde{f}$ by $d$ and $\tilde{d}$. Note that area $(\tilde{f})=\pi \tilde{d}$ since $\tilde{C}=1$.

## Theorem 5.2.

$$
\begin{gather*}
\chi_{N}+\tilde{\chi}_{N}=\chi,  \tag{5.3}\\
d+\tilde{d}=\chi+N(s) . \tag{5.4}
\end{gather*}
$$

Proof. From (5.2) and (2.2) we have

$$
s^{2} \tilde{K}=K-\Delta \log s=\frac{1}{2}\left(K-K_{N}\right)-3\left(1-2 s^{2}\right)=6 s^{2}-2
$$

since $K-K_{N}=2$ for associated immersions $f$ (Corollary 3.2). So

$$
s^{2}=2 /(6-\tilde{K})=2 / \tilde{K}_{N},
$$

by Corollary 3.2 again. It follows that $d s^{2}=\frac{1}{2} \tilde{K}_{N} d \tilde{s}^{2}$, hence $d v=\frac{1}{2} \tilde{K}_{N} d \tilde{v}$ and by integration

$$
\begin{equation*}
\operatorname{area}(f)=\pi \tilde{\chi}_{N} . \tag{5.5}
\end{equation*}
$$

On the other hand area $(f)=\pi\left(\chi-\chi_{N}\right)$ [equality case of (4.7)], whence we get (5.3). Further, integrating $6-\tilde{K}=\tilde{K}_{N}$ (Corollary 3.2), we get from Lemma 5.1 and (5.5)

$$
3 \operatorname{area}(\tilde{f})-\pi(\chi+N(s))=\pi \tilde{\chi}_{N}=\operatorname{area}(f)
$$

On the other hand, we integrate $d \tilde{v}=s^{2} d v=\frac{1}{2}(1-C) d v$ to get

$$
\operatorname{area}(\mathrm{f})=\pi d+2 \operatorname{area}(\tilde{f})
$$

Inserting into the equation above yields (5.4).

## 6. Curvature Restrictions

Theorem 6.1. Let $\left(M, d s^{2}\right)$ be a complete surface, $f: M \rightarrow P$ an isometric minimal immersion. Assume either
(a) $K \geqq 0$
or
(b) $K \leqq 0$ and $b:=k \sin ^{2} \alpha \geqq \delta>0$.

Then either $f$ is isotropic or $M$ is flat with constant $b$.
Proof. Assume that $f$ is not isotropic. Then $b$ is positive outside a discrete set, and from (2.6) we have $\Delta \log b=6 K$, hence $\Delta b=\|\nabla b\|^{2} / b+6 K b$. In case (a) this is nonnegative, so $b$ is a subharmonic function. Moreover recall that $K_{N} \leqq 6-K$ (Corollary 3.2), hence $k=2+K_{N}-K \leqq 2(4-K) \leqq 8$. Thus $b$ is bounded and therefore constant (see Huber [15]). Consequently, $K=\frac{1}{6} \Delta \log b=0$.

In case (b), we may proceed as in Yau [26] and Klotz and Ossermann [20]. We may assume that $M$ is simply connected. The metric $d s_{0}^{2}:=b^{1 / 3} d s^{2}$ on $M$ is flat and complete since $b$ is bounded away from zero. If $\Delta^{0}$ denotes the Laplacean of $d s_{0}^{2}$, we
have $\Delta^{0}=b^{-1 / 3} \Delta$, and so we get from (2.6) $\Delta^{0} \log b=6 b^{-1 / 3} K \leqq 0$. Therefore, $-\log b$ is a bounded subharmonic function on the euclidean plane which must be a constant. This proves (b).

Remark. Case (b) is a generalization of Theorem 7 in Yau [26]. Namely, if $f$ is real, then $b=k=2(1-K)$. In this case, the assumption $K \leqq 0$ implies $b \geqq 2$.
Theorem 6.2. Let $M$ be a compact oriented surface and $f: M \rightarrow P$ a minimal immersion.
(i) If $K_{N}+K \geqq 0$, then $f$ is complex or real.
(ii) If $K_{N}+K<0$, then $M$ is a sphere of area $6 \pi$ and $f$ is associated with degree $d=0$.
(iii) If $K_{N} \leqq 2 K$, then $f$ is associated with $K \geqq-2$ or $f(M)=V$ or $f(M)=C T$.

Proof. (i) From (2.4) and Lemma 4.1 we see that either $f$ is complex or

$$
\int_{M}\left(K+K_{N}\right)=-2 \pi N(\sin \alpha) \leqq 0
$$

which would imply that $K+K_{N}=0$ everywhere, hence $f$ real by Theorem 3.4.
(ii) From (2.7) we can see that the only critical points of the function $C$ occur where $C^{2}=1$; hence they are maxima or minima. By a standard Morse theory argument we see that there is exactly one maximum and one minimum and that $M$ is diffeomorphic to a sphere. In particular, $f$ is isotropic, by Corollary 4.4. So $f$ is associated since in the complex case we would have $K_{N}+K=6$ (Corollary 3.2). Hence $K-K_{N}=2$ which implies $K_{N}<-1$. Since $\hat{K}_{N}=3 C^{2}-1 \geqq-1$ [see (1.3)], the oriented area of the ellipse of curvature (in this case a circle) is everywhere strictly negative (cf. Sect. 1). Therefore the bundle map $\bar{A}: T M \rightarrow N M, \bar{A}(X)=A(X, X)$, has degree -2 outside the zero section, as explained in Sect. 1. We now use an argument of Asperti et al. [1, Theorem 1]: The index formula for the Euler number, applied to a generic tangent vector field $X$ and to the normal field $A \backsim X$ yields $\chi_{N}=\operatorname{deg}(\bar{A}) \chi=-2 \chi=-4$. Now from (4.1), (4.2) we get $\chi+\chi_{N}+3|d| \leqq-2$, thus $d=0$. Moreover, the equality case of (4.7) shows area $(M)=6 \pi$.
(iii) If $f$ is associated, i.e. $K_{N}=K-2$, then $K_{N} \leqq 2 K$ if and only if $K \geqq-2$. Otherwise we see from (2.3) and Lemma 4.1 that $\int_{M}\left(2 K-K_{N}\right)=-N(\sqrt{k}) \leqq 0$, hence $K_{N}=2 K$ everywhere and $k=$ const. So both $K$ and $K_{N}$ are constant. Since $K+K_{N}<0$ is excluded by (ii), we are in case (i). Then the result follows from the Corollaries 3.7 and 3.9 , since $\mathbb{C} P^{1}$ and $\mathbb{R} P^{2}$ are also associated.

Remark. In case (ii), we have $N(c)=N(s)=1$, by (4.1), (4.2). Thus Theorem 5.2 implies that $f$ is associated to a holomorphic curve of degree 3 with one singular point on $\mathbb{C}$. E.g. the Neil parabola $\widetilde{f}(z)=\left[1, z^{2}, z^{3}\right]$ has this property. It is an open question whether the assumption of (ii) can be realized by such a curve.

## 7. Structure Equations and an Imbedding Theorem

Let $(P,\langle\rangle, J$,$) be a 4-dimensional Kähler manifold. On any tangent space T_{p} P$ we define an hermitean inner product (, ) by

$$
(X, Y):=\langle X, Y\rangle+i\langle X, J Y\rangle .
$$

A unitary frame at $p$ is a pair of vectors $E_{1}, E_{2} \in T_{p} P$ such that $\left(E_{i}, E_{j}\right)=\delta_{i j}$. Let $U P$ denote the bundle of all unitary frames on $P$. A local unitary frame is a local section $E$ of $U P$, denoted by $E(p)=\left(p ; E_{1}(p) ; E_{2}(p)\right)$, defined on some open subset $U$ of $P$. By means of $E$ we may define complex valued exterior forms $\omega_{i}, \omega_{i j} \in \Omega_{\mathbb{C}}^{1}(U)$ and $\Omega_{i j} \in \Omega_{\mathbb{C}}^{2}(U)$ :

$$
\begin{aligned}
\omega_{i}(X) & =\left(X, E_{i}\right), \\
\omega_{i j}(X) & =\left(\nabla_{X} E_{i}, E_{j}\right), \\
\Omega_{i j}(X, Y) & =\left(R(X, Y) E_{i}, E_{j}\right) .
\end{aligned}
$$

These satisfy

$$
\bar{\omega}_{i j}=-\omega_{j i}, \quad \bar{\Omega}_{i j}=-\Omega_{j i}
$$

and the Cartan structure equations

$$
\begin{equation*}
d \omega_{i}=\sum_{k} \omega_{k} \wedge \omega_{k i}, \quad d \omega_{i j}=\Omega_{i j}+\sum_{k} \omega_{i k} \wedge \omega_{k j} \tag{7.1}
\end{equation*}
$$

In our case $P=\mathbb{C} P^{2}$, the curvature form is well known from the curvature tensor (see e.g. [19, Vol. II, p. 166])

$$
\begin{equation*}
\Omega_{i j}=-\bar{\omega}_{i} \wedge \omega_{j}+\delta_{i j} \sum_{j} \omega_{k} \wedge \bar{\omega}_{k} \tag{7.2}
\end{equation*}
$$

Note that the sign convention for the curvature tensor is different in [19]. Alternatively, one could use the Maurer-Cartan equations of the group $P U(3)$ to derive (7.2).

It is well known that the forms $\omega_{i}, \omega_{i j}, \Omega_{i j}$ are pull backs of corresponding forms $\hat{\omega}_{i}, \hat{\omega}_{i j}, \hat{\Omega}_{i j}$ on $U P$ via the section $E$, and these forms satisfy equations formally identical to (7.1), (7.2) (e.g. see [19, Vol. I]). Moreover, real and imaginary part of $\hat{\omega}_{i}$ and $\hat{\omega}_{i j}$ span the cotangent bundle of $U P$. For $P=\mathbb{C} P^{2}$, the Lie group $P U(3)$ $=U(3) / S^{1}$ (see Sect. 1) acts simply transitively on $U P$ and may be identified with $U P$. In particular, $U P$ has (real) dimension 8.

Now let $M$ be a surface and $f: M \rightarrow P$ a smooth mapping. Let $E=\left(E_{1} ; E_{2}\right)$ be a unitary frame along $f$, i.e. a map $E: M \rightarrow U P$ with $\tau \circ E=f$ where $\tau: U P \rightarrow P$ denotes the bundle projection. This defines complex valued 1 -forms on $M$, namely

$$
\omega_{i}=\left(d f, E_{i}\right), \quad \omega_{i j}=\left(\nabla E_{i}, E_{j}\right)
$$

where $\nabla$ and (,) are now pulled back to the bundle $f^{*} T P$. Since $\omega_{i}=E^{*} \hat{\omega}_{i}$, $\omega_{i j}=E^{*} \hat{\omega}_{i j}$, these forms satisfy the same structure equations

$$
\begin{align*}
d \omega_{i} & =\sum_{k} \omega_{k} \wedge \omega_{k i}  \tag{7.3}\\
d \omega_{i j} & =\sum_{k} \omega_{i k} \wedge \omega_{k j}-\bar{\omega}_{i} \wedge \omega_{j}+\delta_{i j} \sum_{k} \omega_{k} \wedge \bar{\omega}_{k}
\end{align*}
$$

The following theorem shows the converse:
Theorem 7. Let $M$ be a simply connected surface and $\omega_{i}, \omega_{i j}$ for $i, j \in\{1,2\}$ complex valued 1 -forms on $M$ satisfying $\omega_{j i}=-\bar{\omega}_{i j}$ and (7.3). Then there exist a smooth mapping $f: M \rightarrow P$ and a unitary frame $E=\left(E_{1} ; E_{2}\right)$ along $f$ such that

$$
\omega_{i}=\left(d f, E_{i}\right), \quad \omega_{i j}=\left(\nabla E_{i}, E_{j}\right)
$$

Moreover, $(f, E)$ is uniquely determined up to isometries of $P$.
Remark. Consider the quadratic form $d s^{2}:=\omega_{1} \bar{\omega}_{1}+\omega_{2} \bar{\omega}_{2}$ on $M$. The theorem implies that $d s^{2}=f^{*} d \hat{s}^{2}$ where $d \hat{s}^{2}$ denotes the Riemannian metric on $P$. Thus $f$ is an immersion if and only if $d s^{2}$ is everywhere positive, and then $f$ is isometric with respect to $d s^{2}$.

Proof of the theorem. We proceed as in Spivak [23, p. 67]. It suffices to construct a $\operatorname{map} E: M \rightarrow U P$ with $\omega_{i}=E^{*} \hat{\omega}_{i}, \omega_{i j}=E^{*} \hat{\omega}_{i j}$. This will be done by constructing the graph of $E$, called $\Gamma \subset M \times U P$. We consider $\omega_{i}, \omega_{i j}, \hat{\omega}_{i}, \hat{\omega}_{i j}$ as being forms on $M \times U P$, by pulling back via the projections $\pi_{1}, \pi_{2}$ of $M \times U P$ onto $M$ and $U P$. Consider the difference forms

$$
\chi_{i}=\omega_{i}-\hat{\omega}_{i}, \quad \chi_{i j}=\omega_{i j}-\hat{\omega}_{i j}
$$

which define 8 real valued 1-forms: the real and imaginary parts of $\chi_{1}, \chi_{2}, \chi_{12}$ and $i^{-1} \chi_{11}, i^{-1} \chi_{22}$. These are linearly independent since real and imaginary parts of $\hat{\omega}_{i}, \hat{\omega}_{i j}$ span the 8 -dimensional cotangent spaces of $U P$. Thus the subbundle $\left\{\chi_{i}=0, \chi_{i j}=0\right\}$ of $T(M \times U P)$ is a 2-dimensional distribution on $M \times U P$ since it has codimension 8 . Now by the structure equations we have

$$
\begin{aligned}
d \chi_{i}= & \sum_{k}\left(\omega_{k} \wedge \omega_{k i}-\hat{\omega}_{k} \wedge \hat{\omega}_{k i}\right) \\
d \chi_{i j}= & \sum_{k}\left(\omega_{i k} \wedge \omega_{k j}-\hat{\omega}_{i k} \wedge \hat{\omega}_{k j}\right)-\left(\bar{\omega}_{i} \wedge \omega_{j}-\overline{\hat{\omega}}_{i} \wedge \hat{\omega}_{j}\right) \\
& -\delta_{i j} \sum_{k}\left(\omega_{k} \wedge \bar{\omega}_{k}-\bar{\omega}_{k} \wedge \overline{\hat{\omega}}_{k}\right)
\end{aligned}
$$

These 2 -forms belong to the ideal generated by $\chi_{i}, \chi_{i j}$ since in any ring we have the identity

$$
2(a b-c d)=(a+c)(b-d)+(a-c)(b+d)
$$

Thus, by the Frobenius theorem, the distribution is integrable.
Let $\Gamma$ be a maximal integral leaf through some point $u=\left(m ; E_{1} ; E_{2}\right) \in M \times U P$ and let $x \in T_{u} \Gamma$. If $d \pi_{1}(x)=0$, we have $0=\chi_{i}(x)=\hat{\omega}_{i}\left(d \pi_{2}(x)\right)$ and $0=\chi_{i j}(x)$ $=\hat{\omega}_{i j}\left(d \pi_{2}(x)\right)$, so $d \pi_{2}(x)=0$ and thus $x=0$. Therefore, $\pi_{1} \mid \Gamma$ is an immersion and $\Gamma$ is locally a graph over $M$. Hence there exists a neighborhood $U$ of $m$ in $M$ and a $\operatorname{map} E: U \rightarrow U P$ such that $\operatorname{graph}(E)$ is an open subset of $\Gamma$.

We now extend the action of the group $G:=P U(3)$ on $U P$ to $M \times U P$ by letting $G$ act trivially on the factor $M$. If $\left(E_{1}^{\prime} ; E_{2}^{\prime}\right) \in U P$ is another frame, there exists a transformation $g \in G$ with $E_{i}^{\prime}=d g\left(E_{i}\right)$. Thus $g(\Gamma)$ is an integral leaf through the point ( $m ; E_{1}^{\prime} ; E_{2}^{\prime}$ ), and by the uniqueness part of the Frobenius theorem, any integral leaf over $m$ arises in this way. On the other hand, the graph of any map $E: U \rightarrow P$ which satisfies the assumption of the theorem, is an integral leaf. So $E$ is uniquely determined up to the action of the holomorphic isometry group $G$. This uniqueness together with the paracompactness and simple connectivity of $M$ allows to extend $E$ to all of $M$. This proves the theorem.
Remark. We did not use the fact that $M$ is a surface. Moreover, a similar argument is valid for any symmetric space instead of $P$. A similar result was obtained by Wettstein [24].

## 8. A Special Unitary Frame for Immersions

We will prove Theorem A by showing that the fundamental equations (2.1), (2.2), (2.3) are equivalent to the structure equations for some special unitary local frame $E_{1}, E_{2}$ along the immersion $f$.

Let $M$ be an oriented surface and $f: M \rightarrow P$ an immersion. Let $\left(e_{1} ; e_{2}\right)$ be an oriented orthonormal local tangent frame on $M$ (with respect to the induced metric) and put

$$
\begin{equation*}
s_{1}=\frac{1}{2}\left(e_{1}-J e_{2}\right), \quad s_{2}=\frac{1}{2}\left(e_{1}+J e_{2}\right), \tag{8.1}
\end{equation*}
$$

where $J$ denotes the complex structure of $P$ as above. $s_{1}$ and $s_{2}$ are sections of $f^{*} T P$ which are no longer tangential in general. Note that ( $s_{1}, s_{2}$ ) $=0$, where (,) denotes the hermitean product introduced above. The sections $s_{1}$ and $s_{2}$ cannot have a common zero. So we may define two complex line bundles $L_{1}, L_{2}$ (subbundles of $f^{*} T P$ ) as follows:

$$
\begin{aligned}
& L_{1}=\mathbb{C} s_{1}, \quad L_{2}=\left(\mathbb{C} s_{1}\right)^{\perp} \quad \text { whereever } \quad s_{1} \neq 0 \\
& L_{1}=\left(\mathbb{C} s_{2}\right)^{\perp}, \quad L_{2}=\mathbb{C} s_{2} \quad \text { whereever } \\
& s_{2} \neq 0
\end{aligned}
$$

(Recall that we defined a $\mathbb{C}$-scalar multiplication on $T P$.) If we rotate our frame $e_{1}, e_{2}$ by an angle $\tau$ varying on $M$, we just have to multiply $s_{1}$ by $e^{i \tau}, s_{2}$ by $e^{-i \tau}$, so $\mathbb{C} s_{1}$ and $\mathbb{C} s_{2}$ are independent of the choice of the frame $e_{1}, e_{2}$. Therefore, the bundles $L_{1}$ and $L_{2}$ are well defined globally on $M$.

Remark. If $T^{\prime} P \subset T P \otimes \mathbb{C}$ denotes the set of $(1,0)$-vectors, i.e. the holomorphic tangent bundle, and ' the projection on this subbundle, i.e. $X^{\prime}=\frac{1}{2}(X-\sqrt{-1} J X)$, then $L_{1}^{\prime}$ is spanned by $(\partial f / \partial z)^{\prime}$ and $L_{2}^{\prime}$ by $(\partial f / \partial \bar{z})^{\prime}$ for any conformal coordinate $z$, whereever these derivatives are nonzero. However, we will not pass to the complexified tangent bundle.

Now choose local sections $E_{1}$ of $L_{1}$ and $E_{2}$ of $L_{2}$ which have unit length. Then $\left(E_{1} ; E_{2}\right)$ is a unitary frame along $f$ which will be called special. Since $\left|s_{1}\right|=\frac{1}{2} \sqrt{2+2 C}=c$ and $\left|s_{2}\right|=\frac{1}{2} \sqrt{2-2 C}=s$ (see Sect. 2), we have

$$
\begin{equation*}
s_{1}=u E_{1}, \quad s_{2}=\bar{v} E_{2} \tag{8.2}
\end{equation*}
$$

for some $\mathbb{C}$-valued smooth functions $u, v$ on $M$ with

$$
\begin{equation*}
|u|=c, \quad|v|=s \tag{8.3}
\end{equation*}
$$

In turn, we have

$$
\begin{equation*}
e_{1}=u E_{1}+\bar{v} E_{2}, \quad e_{2}=i\left(u E_{1}-\bar{v} E_{2}\right) \tag{8.4}
\end{equation*}
$$

Further, we get an oriented orthonormal frame $\left(e_{3} ; e_{4}\right)$ of the normal bundle $N M$ by putting

$$
\begin{equation*}
e_{3}=-\bar{v} E_{1}+u E_{2}, \quad e_{4}=i\left(\bar{v} E_{1}+u E_{2}\right) \tag{8.5}
\end{equation*}
$$

Let $\omega_{i}=\left(d f, E_{i}\right), \omega_{i j}=\left(\nabla E_{i}, E_{j}\right)$ as above and put

$$
\theta_{i}=\left\langle d f, e_{i}\right\rangle, \quad \phi=\theta_{1}+i \theta_{2}
$$

A complex valued 1 -form on $M$ is a $(1,0)$-form if it is pointwise a multiple of $\phi$. Now by (8.1), (8.2) we have

$$
\begin{equation*}
\omega_{1}=u \phi, \quad \bar{\omega}_{2}=v \phi, \tag{8.6}
\end{equation*}
$$

thus these forms are $(1,0)$.
Lemma 8.1. Let $f: M \rightarrow P$ be an immersion and $E_{1}, E_{2}$ a unitary frame along $f$. The frame $\left(E_{1} ; E_{2}\right)$ is special if and only if $\omega_{1}$ and $\bar{\omega}_{2}$ are $(1,0)$-forms.
Proof. We only have to prove the "if"-part yet. Let $e_{1}, e_{2}$ be an orthonormal tangent frame on $M$ with coframe $\theta_{1}, \theta_{2}$ and $\phi=\theta_{1}+i \theta_{2}$. Assume $\omega_{1}=u \phi, \bar{\omega}_{2}=v \phi$ for some complex functions $u, v$. Put $s_{1}=u E_{1}, s_{2}=\bar{v} E_{2}$ and $f_{1}=s_{1}+s_{2}$, $f_{2}=J\left(s_{1}-s_{2}\right)$. We claim that $f_{1}=e_{1}, f_{2}=e_{2}$.

Define $\tilde{\omega}_{i}: f^{*} T P \rightarrow \mathbb{C}$ by $\tilde{\omega}_{i}(X)=\left(X, E_{i}\right)$. Then $\tilde{\omega}_{1}, \tilde{\omega}_{2}$ form a local basis of the bundle $\operatorname{Hom}_{\mathbb{C}}\left(f^{*} T P, \mathbb{C}\right)$, and $\tilde{\omega}_{i} \mid T M=\omega_{i}$. Since $v \omega_{1}=u \bar{\omega}_{2}$, the form $v:=v \tilde{\omega}_{1}-u \tilde{\omega}_{2}$ has kernel $T M$. Note that $v\left(f_{1}\right)=v\left(f_{2}\right)=0$, so $f_{1}$ and $f_{2}$ are tangent vectors. Further note that $u \phi\left(f_{i}\right)=\tilde{\omega}_{1}\left(f_{i}\right)=u \phi\left(e_{i}\right)$ and $v \phi\left(f_{i}\right)=\bar{\omega}_{2}\left(f_{i}\right)=v \phi\left(e_{i}\right)$. Since $\phi \bar{\phi}=d s^{2}=\omega_{1} \bar{\omega}_{1}+\omega_{2} \bar{\omega}_{2}=(u \bar{u}+\bar{v} v) \phi \bar{\phi}$, the functions $u$ and $v$ have no common zero, and so we conclude $\phi\left(f_{i}\right)=\phi\left(e_{i}\right)$ which means $f_{i}=e_{i}$. Looking back we see that $s_{1}=\left(e_{1}-J e_{2}\right) / 2, s_{2}=\left(e_{1}+J e_{2}\right) / 2$, and therefore $E_{1}$ and $E_{2}$ span $L_{1}$ and $L_{2}$.
Remark. In fact we only have to assume that either $\omega_{1}$ or $\bar{\omega}_{2}$ is a ( 1,0 )-form. E.g. assume that $\omega_{1}=u \phi$ for some function $u \neq 0$. Since $\omega_{2} \bar{\omega}_{2}=\phi \bar{\phi}-\omega_{1} \bar{\omega}_{1}=(1-u \bar{u}) \phi \bar{\phi}$, the form $\omega_{2}$ must be a multiple of $\phi$ or $\bar{\phi}$. But if $\omega_{2}=v \phi$, then $T M$ would be the kernel of the complex linear form $v \tilde{\omega}_{1}-u \tilde{\omega}_{2}$; thus $f$ would be holomorphic and $v=0$. So $\omega_{2}$ is always a multiple of $\bar{\phi}$.

Next we compute the second fundamental form $A$ of $f$ with respect to the frame $\left(e_{1}, e_{2}, e_{3}, e_{4}\right)$ defined above. Let

$$
\psi:=u d v-v d u-u v\left(\omega_{11}+\omega_{22}\right)
$$

Then we get from (8.4) and (8.5)

$$
\begin{align*}
& \left\langle\nabla e_{1}, e_{3}\right\rangle-i\left\langle\nabla e_{2}, e_{3}\right\rangle=\omega_{12}+\psi,  \tag{8.7}\\
& \left\langle\nabla e_{2}, e_{4}\right\rangle-i\left\langle\nabla e_{1}, e_{4}\right\rangle=\omega_{12}-\psi .
\end{align*}
$$

Let $h_{3}, h_{4}$ denote the mean curvatures of $f$, i.e.

$$
h_{3} e_{3}+h_{4} e_{4}=\frac{1}{2}\left(A_{11}+A_{22}\right)
$$

Then the matrices $A_{3}=\left\langle A, e_{3}\right\rangle, A_{4}=\left\langle A, e_{4}\right\rangle$ have the form

$$
A_{3}=\left(\begin{array}{rr}
a & b \\
b & -a
\end{array}\right)+h_{3} I, \quad A_{4}=\left(\begin{array}{rr}
c & d \\
d & -c
\end{array}\right)+h_{4} I .
$$

Putting

$$
\begin{equation*}
w:=\frac{1}{2}((a+d)+i(c-b)), \quad \tilde{w}:=\frac{1}{2}((a-d)+i(c+b)), \tag{8.8}
\end{equation*}
$$

and $h:=h_{3}+i h_{4}$, we get from (8.7)

$$
\omega_{12}=w \phi+\frac{1}{2} h \bar{\phi}, \quad \psi=\tilde{w} \phi+\frac{1}{2} h \bar{\phi} .
$$

Note that $f$ is minimal if and only if $h=0$. So we get in particular:
Lemma 8.2. Let $f: M \rightarrow P$ be an immersion and $E_{1}, E_{2}$ local unit sections of $L_{1}, L_{2}$ (special unitary frame). Then $f$ is minimal if and only if $\omega_{12}$ is a $(1,0)$-form, more precisely,

$$
\begin{equation*}
\omega_{12}=w \phi \tag{8.9}
\end{equation*}
$$

with $w$ defined by (8.8).
Now assume $f$ to be minimal. Then we obtain the absolute values of $w, \tilde{w}$ by Gauss and Ricci equations (1.1), (1.2) together with (1.3):

$$
\begin{align*}
& |2 w|^{2}=K_{N}-K+2=k \\
& |2 \tilde{w}|^{2}=6 C^{2}-\left(K+K_{N}\right) . \tag{8.10}
\end{align*}
$$

Moreover, due to the fact that $\omega_{1}, \bar{\omega}_{2}, \omega_{12}$ are $(1,0)$-forms, the structure equations (7.3) for ( $f ; E_{1} ; E_{2}$ ) simplify as follows:

$$
\begin{align*}
d \omega_{1} & =\omega_{1} \wedge \omega_{11},  \tag{8.11a}\\
d \omega_{2} & =\omega_{2} \wedge \omega_{22},  \tag{8.11b}\\
d \omega_{12} & =\omega_{12} \wedge\left(\omega_{22}-\omega_{11}\right),  \tag{8.11c}\\
d \omega_{11} & =\bar{\omega}_{12} \wedge \omega_{12}-2 \bar{\omega}_{1} \wedge \omega_{1}+\omega_{2} \wedge \bar{\omega}_{2},  \tag{8.11d}\\
d \omega_{22} & =-\bar{\omega}_{12} \wedge \omega_{12}-\bar{\omega}_{1} \wedge \omega_{1}+2 \omega_{2} \wedge \bar{\omega}_{2} . \tag{8.11e}
\end{align*}
$$

Remark. From these equations it is easy to compute the Euler numbers of the line bundles $L_{1}$ and $L_{2}$ in case that $M$ is compact: Using $E_{k}, J E_{k}$ as a frame for $L_{k}$ ( $k=1,2$ ), the connection form is $i^{-1} \omega_{k k}$ and therefore its curvature form $i^{-1} d \omega_{k k}$. From (8.11d), (8.11e) and (8.3), (8.6), (8.9), (8.10) we get

$$
\begin{align*}
& d \omega_{11}=\frac{1}{4}\left(K_{N}-K-6 C\right) \bar{\phi} \wedge \phi  \tag{8.12a}\\
& d \omega_{22}=\frac{1}{4}\left(-K_{N}+K-6 C\right) \bar{\phi} \wedge \phi \tag{8.12b}
\end{align*}
$$

Integrating and using the Gauss-Bonnet-Chern theorem we get

$$
\begin{aligned}
& \chi\left(L_{1}\right)=\frac{1}{2}\left(3 d+\chi+\chi_{N}\right), \\
& \chi\left(L_{2}\right)=\frac{1}{2}\left(3 d-\chi+\chi_{N}\right) .
\end{aligned}
$$

[We already knew from (4.1), (4.2) that these numbers are integers.] In the holomorphic case $L_{1}=T M$, and we get again $\chi+\chi_{N}=3 d$ as in the equality case of (4.8).

## 9. Proof of the Main Results

Choose a conformal coordinate $z=x+i y$ on $M$ with conformal factor $\lambda$. We will use the orthonormal frame

$$
e_{1}=\partial_{x} / \lambda, \quad e_{2}=\partial_{y} / \lambda
$$

Let $\theta_{1}, \theta_{2}$ be the corresponding coframe and put $\phi=\theta_{1}+i \theta_{2}$ as above; we have $\phi=\lambda d z$. (In fact we have $\phi=\mu d z$ for any orthonormal frame, where $\mu$ is $\mathbb{C}$-valued with $|\mu|=\lambda$.) Putting

$$
p=\lambda u, \quad q=\lambda v, \quad r=\lambda w,
$$

we have

$$
\omega_{1}=p d z, \quad \bar{\omega}_{2}=q d z, \quad \omega_{12}=r d z .
$$

Now the first three structure equations (8.11a) (8.11c) become

$$
\begin{align*}
\left(d p+p \omega_{11}\right) & \wedge d z  \tag{9.1a}\\
\left(d q-q \omega_{22}\right) & \wedge d z=0,  \tag{9.1b}\\
\left(d r+r\left(\omega_{22}-\omega_{11}\right)\right) & \wedge d z=0 . \tag{9.1c}
\end{align*}
$$

In particular, $d(p q r) \wedge d z=0$. Therefore, $p q r$ is a holomorphic function, and

$$
\begin{equation*}
\Lambda:=\omega_{1} \bar{\omega}_{2} \omega_{12}=p q r d z^{3} \tag{9.2}
\end{equation*}
$$

is a holomorphic cubic form. Note that the definition of $A$ does not depend on the choice of the special unitary frame $E_{1}, E_{2}$. In fact, a global definition is

$$
\Lambda(x)=\left(V_{x} x_{1}, x_{2}\right)
$$

for any tangent field $x$ of $M$, where $x=x_{1}+x_{2}$ denotes the decomposition with respect to the subbundles $L_{1}$ and $L_{2}$ of $f^{*} T P$. This global holomorphic form has been used before, e.g. see [4] and [7]. However, (9.1) has stronger consequences:
Lemma 9.1. Let $\left(M, d s^{2}\right)$ be a surface and $z$ a conformal coordinate on some open subset $U$ of $M$. Let t be a smooth complex valued function and $\omega$ a purely imaginary 1 -form on $U$, i.e. $i^{-1} \omega \in \Omega_{\mathbb{R}}^{1}(U)$. Assume

$$
\begin{equation*}
(d t+t \omega) \wedge d z=0 \tag{*}
\end{equation*}
$$

Then
(i) $t$ is of holomorphic type,
(ii) $\omega=2 i \operatorname{Im}\left(\frac{\partial}{\partial z} \log \bar{t} d z\right)$,
(iii) $d \omega=\frac{1}{2} \Delta \log |t| \Phi \wedge \phi$,
where $\phi=\lambda d z$ as above and $\Delta$ the Laplacean of $M$.
Remark. Note that, by (i), the expression in (ii) and (iii) make still sense at a zero of $t$ unless $t=0$ everywhere. Part (i) goes back to Chern ([3], see also [25]).

Proof. By (*), the $d \bar{z}$-part of the form $d t+t \omega$ vanishes. Since $\omega$ is purely imaginary, we have

$$
\omega=b d z-\bar{b} d \bar{z}==2 i \operatorname{Im}(b d z)
$$

for some complex valued function $b$. Hence ( $*$ ) implies

$$
\begin{equation*}
\bar{\partial} t=t \bar{b}, \tag{**}
\end{equation*}
$$

where $\partial:=\partial / \partial z$ and $\bar{\partial}:=\partial / \partial \bar{z}$. Let $u$ be any solution of the inhomogeneous Cauchy-Riemann equation

$$
\bar{\partial} u=\bar{b}
$$

on $U$ (e.g. see $[10,13])$. Then $t_{1}=e^{u}$ is a nowhere vanishing solution of $(* *)$, and so $\bar{\partial}\left(t / t_{1}\right)=0$. Thus $t_{0}:=t / t_{1}$ is holomorphic which proves (i). By ( $* *$ ) we have $\bar{b}=\bar{\partial} t / t$ $=\bar{\delta}(\log t)$, hence $b=\partial(\log \bar{t})$ which gives (ii). Moreover,

$$
\begin{aligned}
d \omega & =d b \wedge d z-d \bar{b} \wedge d \bar{z}=(\bar{\partial} b+\partial \bar{b}) d \bar{z} \wedge d z \\
& =(\bar{\partial} \partial(\log \bar{t})+\partial \bar{\partial}(\log t)) d \bar{z} \wedge d z \\
& =2 \partial \bar{\partial}(\log |t|) d \bar{z} \wedge d z \\
& =\frac{1}{2} \Delta \log |t| \bar{\phi} \wedge \phi
\end{aligned}
$$

which proves (iii).
Replacing (*) with either of the equations (9.1a)-(9.1c), we see that $p, q, r$ and hence $u, v, w$ are of holomorphic type, so $c, s, \sqrt{k}$ are of absolute value type [see (8.3), (8.10)]. Furthermore, we can compute $d \omega_{11}, d \omega_{22}, d\left(\omega_{11}-\omega_{22}\right)$ on two different ways: By means of Lemma 9.1 (iii) and by the structure equations, see (8.12). Equating the two results yields the fundamental equations (2.1), (2.2), (2.3) in Theorem A.

To prove the converse [part (ii) of Theorem A], assume first that none of the given functions $c, s, \sqrt{k}$ vanishes identically. Let $z$ be a conformal coordinate on an arbitrary open disk $U$ in $M$, and let $\lambda$ denote the conformal factor. Adding (2.1), (2.2), (2.3), we see that the absolute value type function $a:=\frac{1}{2} \lambda^{3} c s \sqrt{k}$ satisfies $\Delta(\log a)=0$. Hence by Lemma 3.12, we have $a=|h|$ for some holomorphic function $h$ on $U$ which is uniquely determined up to a constant phase factor $e^{i \tau}$.

Now choose holomorphic type functions $p, q$ on $U$ with $|p|=\lambda c,|q|=\lambda s$ which exist since $c, s$ are of absolute value type. Put $r=h /(p q)$. Since $|r|=\frac{1}{2} \lambda \sqrt{k}$ is bounded near the zeros of $p$ and $q$, the singularities of $r$ are removable and $r$ is a holomorphic type function on all of $U$. Now we set

$$
\omega_{1}=p d z, \quad \bar{\omega}_{2}=q d z, \quad \omega_{12}=r d z
$$

and Lemma 9.1(ii) motivates us to put

$$
\begin{align*}
& \omega_{11}=2 i \operatorname{Im}(\partial(\log \bar{p}) d z)  \tag{9.3}\\
& \omega_{22}=-2 i \operatorname{Im}(\partial(\log \bar{q}) d z) \tag{9.4}
\end{align*}
$$

As a consequence of $\bar{\partial}(\log p q r)=0$, we also get

$$
\begin{equation*}
\omega_{22}-\omega_{11}=2 i \operatorname{Im}(\partial(\log \vec{r}) d z) \tag{9.5}
\end{equation*}
$$

It is now easy to check that these 1 -forms on $M$ satisfy the structure equations (8.11) and hence (7.3). By Theorem 7, they define a mapping $f: U \rightarrow P$ together with a unitary frame $E_{1}, E_{2}$ along $f$ such that $\omega_{i}=\left(d f, E_{i}\right), \omega_{i j}=\left(\nabla E_{i}, E_{j}\right)$. Since $\omega_{1} \bar{\omega}_{1}$ $+\omega_{2} \bar{\omega}_{2}=\left(c^{2}+s^{2}\right) \phi \bar{\phi}=d s^{2}$, this map $f$ is an isometric immersion. By Lemma 8.1, the unitary frame $E_{1}, E_{2}$ is special, hence $f$ is minimal, by Lemma 8.2. Note that $\Lambda=h d z^{3}$ is well determined up to the constant factor $e^{i \tau}$.

In case $k=0$ everywhere, we choose $p$ and $q$ as above and define $\omega_{1}, \omega_{2}, \omega_{11}, \omega_{22}$ as before while setting $\omega_{12}=0$. In case $s=0$ everywhere, we put $\omega_{1}, \omega_{11}$ as above, choose a holomorphic type function $r$ with $|r|=\frac{1}{2} \sqrt{k}$, put $\omega_{2}=0$, $\omega_{12}=r d z$ and define $\omega_{22}$ by (9.5). If both $s$ and $k$ vanish identically, we put $\omega_{1}, \omega_{11}$ as above, $\omega_{2}=0, \omega_{12}=0$, and choose an arbitrary imaginary form $\omega_{22}$ with

$$
d \omega_{22}=-\bar{\omega}_{1} \wedge \omega_{1}=-\lambda^{2} d \bar{z} \wedge d z
$$

If $c$ instead of $s$ vanishes identically, the roles of $\omega_{1}, \omega_{11}$ and $\omega_{2}, \omega_{22}$ are interchanged. In all these cases the structure equations (8.11) and (7.3) are satisfied, and $\omega_{1} \bar{\omega}_{1}+\omega_{2} \bar{\omega}_{2}=d s^{2}$, so by Theorem 7 and Lemma 8.1, 8.2, we get an isometric minimal immersion $f: U \rightarrow P$ together with a special unitary frame $E_{1}, E_{2}$ along $f$, and $f$ is either complex ( $c=0$ or $s=0$ ) or satisfies $k=0$, i.e. $\omega_{12}=0$. This finishes the local part of the existence theorem.

It remains to show that the last case $\omega_{12}=0$ occurs if and only if $f$ is associated. This is well known [4,7]. For completeness we present the argument: Let $Z_{0}: U$ $\rightarrow S^{5} \subset \mathbb{C}^{3}$ be a lift of $f$, i.e. $\pi \circ Z_{0}=f$, where $\pi: S^{5} \rightarrow P$ is the canonical projection. Let $Z_{1}, Z_{2}: U \rightarrow S^{5}$ be horizontal lifts of $E_{1}, E_{2}$ at $Z_{0}$, i.e. $Z_{1}, Z_{2}$ are pointwise orthogonal to $\mathbb{C} Z_{0}$ and $d \pi_{Z_{0}}\left(Z_{i}\right)=E_{i}$. We claim that $f_{1}=\pi \circ Z_{1}$ in an antiholomorphic and $f_{2}=\pi \circ Z_{2}$ a holomorphic map from $U$ to $P$. In fact, we have

$$
\begin{aligned}
& d f_{1}=\left(d Z_{1}, Z_{0}\right) d \pi_{Z_{1}}\left(Z_{0}\right)+\left(d Z_{1}, Z_{2}\right) d \pi_{Z_{1}}\left(Z_{2}\right) \\
& d f_{2}=\left(d Z_{2}, Z_{0}\right) d \pi_{Z_{2}}\left(Z_{0}\right)+\left(d Z_{2}, Z_{1}\right) d \pi_{Z_{2}}\left(Z_{1}\right)
\end{aligned}
$$

where (, ) denotes also the hermitean inner product on $\mathbb{C}^{3}$. Since $\pi$ is a Riemannian submersion, we have

$$
\begin{aligned}
-\left(d Z_{i}, Z_{0}\right) & =\left(Z_{i}, d Z_{0}\right)=\left(E_{i}, d f\right)=\bar{\omega}_{i} \\
\left(d Z_{i}, Z_{j}\right) & =\left(\nabla^{s} Z_{i}, Z_{j}\right)=\left(\nabla E_{i}, E_{j}\right)=\omega_{i j}
\end{aligned}
$$

where $\nabla^{S}$ denotes the Levi-civita connection of the sphere $S^{5}$. Hence $\left(d Z_{1}, Z_{0}\right)$ $=-\bar{p} d \bar{z},\left(d Z_{2}, Z_{0}\right)=-q d z,\left(d Z_{1}, Z_{2}\right)=0=\left(d Z_{2}, Z_{1}\right)$ and so

$$
\begin{equation*}
d f_{1}=-\bar{p} d \bar{z} d \pi_{Z_{1}}\left(Z_{0}\right), \quad d f_{2}=-q d z d \pi_{Z_{2}}\left(Z_{0}\right) \tag{9.6}
\end{equation*}
$$

Thus $f_{1}$ is antiholomorphic and $f_{2}$ holomorphic, and the complex lines $\mathbb{C} Z_{0}$ which define $f$ are the horizontal lifts of the tangent lines of $f_{1}$ and $f_{2}$.

Conversely, let $f: M \rightarrow P$ be associated to an antiholomorphic map $f_{1}: M \rightarrow P$ and let $Z_{0}, Z_{1}: U \rightarrow S^{5}$ be local lifts of $f$ and $f_{1}$. Then the horizontal lift of $d f_{1}\left(T_{m} M\right)$ is the complex line $\mathbb{C} Z_{0}(m), m \in U$. In particular, $\left(Z_{0}, Z_{1}\right)=0$, and the image of $d Z_{1}$ is contained in the $\mathbb{C}$-linear span of $Z_{0}$ and $Z_{1}$. Choosing a unit vector $Z_{2}$ ( $C^{\infty}$-dependent on $m \in U$ ) orthogonal to this subspace, we define a unitary frame $E_{1}, E_{2}$ along $f$ by setting $E_{i}=d \pi_{z_{0}}\left(Z_{i}\right)$. Since $f_{1}$ is anitholomorphic, the corresponding 1 -form

$$
\omega_{1}=\left(d f, E_{1}\right)=\left(d Z_{0}, Z_{1}\right)=-\left(Z_{0}, d Z_{1}\right)
$$

is $(1,0)$, and so, by Lemma 8.1 and the subsequent remark, the unitary frame $E_{1}, E_{2}$ is special. Moreover,

$$
\omega_{12}=\left(\nabla E_{1}, E_{2}\right)=\left(d Z_{1}, Z_{2}\right)=0
$$

which proves $k=0$. The case where $f$ is associated to a holomorphic map $f_{2}$ is obtained by reversing orientation.

We also get the proof of Theorem $C$ from (9.6). Namely, for any $x \in T_{m} M$, $m \in U$, we have

$$
\begin{aligned}
& \left\|d f_{1}(x)\right\|=\left|p(m)\left\|d \bar{z}_{m}(x) \mid=s(m)\right\| x \|,\right. \\
& \left\|d f_{2}(x)\right\|=\left|q(m)\left\|d z_{m}(x) \mid=c(m)\right\| x \|,\right.
\end{aligned}
$$

which proves Theorem C .
To prove Theorem B, for a given surface ( $M, d s^{2}$ ) and smooth functions $C, K_{N}$ on $M$ we consider the set $F\left(M, C, K_{N}\right)$ of all isometric minimal immersions $f: M \rightarrow P$ with Kähler function $C$ and normal curvature $K_{N}$. We claim that the holomorphic 3 -form $A$ introduced in (9.2) separates the congruence classes in $F\left(M, C, K_{N}\right):$

Lemma 9.2. Let $M$ be any connected surface and $f_{1}, f_{2} \in F\left(M, C, K_{N}\right)$ with corresponding 3 -forms $\Lambda_{1}, \Lambda_{2}$. Then $f_{2}=g \circ f_{1}$ for some holomorphic isometry $g$ of $P$ if and only if $\Lambda_{1}=\Lambda_{2}$.

Proof. For $k=1,2$ let $E_{1}^{k}, E_{2}^{k}$ be a special unitary frame along $f_{k}$, defined on some open subset $U$ of $M$, and let $\omega_{i}^{k}, \omega_{i j}^{k}$ be the corresponding 1 -forms. We have $\left|\omega_{i}^{1}\right|=\left|\omega_{i}^{2}\right|,\left|\omega_{12}^{1}\right|=\left|\omega_{12}^{2}\right|$ by (8.3), (8.6), (8.9), (8.10). Multiplying $E_{1}^{2}$ and $E_{2}^{2}$ by suitable phase functions $e^{i \tau_{1}}, e^{i \tau_{2}}$ if necessary, we may assume $\omega_{i}^{1}=\omega_{i}^{2}, \omega_{i i}^{1}=\omega_{i i}^{2}$; this follows from Lemma 9.1(ii) applied to the equations (9.1) in case $\omega_{i}^{k} \neq 0$, and otherwise from the transformation rule $\tilde{\omega}_{i i}^{2}=\omega_{i i}^{2}+i d \tau_{i}$. Now $\Lambda_{1}=\Lambda_{2}$ if and only if $\omega_{12}^{1}=\omega_{12}^{2}$, and in this case we have $\left(f_{2} ; E_{1}^{1} ; E_{2}^{1}\right)=g \circ\left(f_{1} ; E_{1}^{2} ; E_{2}^{2}\right)$ for some holomorphic isometry $g$ of $P$, by the uniqueness part of Theorem 7. So the subset of $M$ where $f_{2}$ and $g \circ f_{1}$ agree, is closed and open, since we can do the same construction around a possible boundary point, and it contains $U$. Therefore $f_{2}=g \circ f_{1}$ on the whole of $M$. The converse statement is clear.

In particular, it follows from Lemma 9.2 that all members of $F\left(M, C, K_{N}\right)$ are congruent in case $\Lambda=0$ which is the isotropic case $s=0$ or $c=0$ or $k=0$. If $\Lambda \neq 0$, then $\left|\Lambda_{1}\right|=\left|\Lambda_{2}\right|$ implies $\Lambda_{2}=e^{i \tau} \Lambda_{1}$ for some constant $\tau \in \mathbb{R}$, and any of these forms is possible, as we saw earlier. So for any $f \in F\left(M, C, K_{N}\right)$ for a simply connected surface $M$, we get a one-parameter family of immersions $f_{\theta} \in F\left(M, C, K_{N}\right), \theta \in S^{1}$, which are incongruent among each other, and any $\tilde{f} \in F\left(M, C, K_{N}\right)$ is congruent to one of the $f_{\theta}$. This proves Theorem $\mathrm{B}(\mathrm{i})$.

To prove (ii), assume that $M$ is simply connected and $f: M \rightarrow P$ a minimal immersion with $A \neq 0$, i.e. $f$ is not isotropic. It follows from Theorem $\mathrm{A}(\mathrm{i})$ that $M$ is not diffeomorphic to the sphere [see (4.4)]. Hence we may assume $M \subset \mathbb{C}$. Now let $\left(\psi_{t}\right)_{t \in \mathbb{R}}$ be a group of isometries of $M$ such that $f_{\theta}=f \circ \psi_{t}$ for $\theta=e^{i t}$. Hence $e^{i t} \Lambda=\Lambda_{\theta}=\psi_{t}^{*} \Lambda$. We have $\Lambda=h d z^{3}$ for some holomorphic function $h$. Then $\psi_{t}^{*}=\mathrm{h} \circ \psi_{t} \cdot\left(\psi_{t}^{\prime}\right)^{3} d z^{3}$, hence

$$
\begin{equation*}
h \circ \psi_{t} \cdot\left(\psi_{t}^{\prime}\right)^{3}=e^{i t} h \tag{*}
\end{equation*}
$$

(Note that $\psi_{t}$ is conformal and orientation preserving, hence holomorphic.) Either $h$ has a zero $z_{0} \in M$; then (*) implies $\psi_{t}\left(z_{0}\right)=z_{0}$ for small $t$ since the zero set of $h$ is isolated and preserved by $\psi_{t}$. Or $h$ has no zeros. Then there is a holomorphic $3^{\text {rd }}$
root $g$ of $h$. Let $w$ be a primitive function of $g$. Then $\Lambda=d w^{3}$. So by a change of the conformal coordinate, we may assume $h=1$. Now (*) becomes $\left(\psi_{t}\right)^{3}=e^{i t}$, hence $\psi_{t}^{\prime}=e^{i t / 3}$. Thus $\left(\psi_{t}\right)$ is a group of rotations around some point $z_{0} \in \mathbb{C}$ which must lie in $M$, by simple connectivity. In either case, $\left(\psi_{t}\right)$ is a group of rotations around some common fixed point $z_{0} \in M$. Since $f \circ \psi_{t}=f_{\theta} \in F\left(M, C, K_{N}\right)$ for all $t$, the functions $C$ and $K_{N}$ are invariant under this group.

Conversely, let $\left(\psi_{i}\right)_{\in \mathbb{R}}$ be a group of rotations of $M$ with fixed point $m_{0}$, and assume that $f \circ \psi_{t} \in F\left(M, C, K_{N}\right)$ for all $t$. Let $z$ be a conformal coordinate with $z\left(m_{0}\right)=0$ such that $r:=|z|$ is a $\psi_{t}$-invariant function. [This is possible since $M$ is biholomorphically equivalent to $\mathbb{C}$ or the unit disk and $\left(\psi_{t}\right)$ is a group of conformal transformations of $M$ with a common fixed point.] Then the functions $\lambda, k, c, s$ depend only on $r$. Thus if $\Lambda=h d z^{3}$, we get that also $|h|=\frac{1}{2} \lambda^{3} c s \sqrt{k}$ is a function of $r$ and therefore, $h(z)=\beta z^{n}$ for some nonnegative integer $n$ and a complex constant $\beta$. Moreover, $\psi_{t}(z)=e^{i \sigma t} \cdot z$ for some non-zero $\sigma \in \mathbb{R}$, so

$$
\psi_{t}^{*} \Lambda=h \circ \psi_{t} \cdot\left(\psi_{t}^{\prime}\right)^{3} d z^{3}=e^{i(n+3) \sigma t} \Lambda .
$$

Choosing $\sigma=\frac{1}{n+3}$, we get the result, and Theorem B is proved.
Now we can finish the proof of the existence part of Theorem A. We cover the surface $M$ by open coordinate discs ( $U, z$ ), and doing the local construction described earlier in any of the $U$ 's, we choose the holomorphic function $h$ so that $A=h d z^{3}$ is well defined globally. This is possible since $\Lambda$ is uniquely determined up to a constant factor and $M$ is simply connected. Now two local immersions $f_{U}: U \rightarrow M, f_{V}: V \rightarrow M$ are congruent on $U \cap V$, by Lemma 9.2, therefore the local immersions $f_{U}$ can be patched together to give a global immersion $f: M \rightarrow P$.

It remains to prove Theorem D. The ellipse of curvature of a minimal immersion $f: M \rightarrow P$ is a circle everywhere if and only if either $a+d=b-c=0$ or $a-d=b+c=0$ (see Sect. 1). By (8.8) this means either $w=0$ or $\tilde{w}=0$ everywhere (recall that $w$ is a function of holomorphic type). Now by (8.10), $w=0$ iff $f$ is associated and $\tilde{w}=0$ iff $f$ is complex or real, see Theorem 3.1. This finishes the proof of Theorem D.

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Note added in proof. If $P=\mathbb{C} P^{\mathbf{2}}$ is replaced with an arbitrary Kähler surface $P_{\sigma}$ of constant holomorphic curvature $\kappa=4 \sigma$, the values of the curvature forms $\Omega_{i j}$ in (7.2) have to be multiplied by $\sigma$. Therefore, the fundamental equations (2.1), (2.2), (2.3) in Theorem A remain valid if we replace $C$ by $\sigma C$ and put $k=K_{N}-K+2 \sigma$. If $P_{\sigma}$ is compact, Theorem 4.2 is still true with $d=\frac{\sigma}{\pi} \int_{M} C d v$ (Sect. 1). However, $d$ is not necessarily an integer, but $3 d$ is since the first Chern form is

$$
c_{1}=\frac{1}{2 \pi i}\left(\Omega_{11}+\Omega_{22}\right)=\frac{3 \sigma}{\pi} \phi,
$$

where $\phi$ denotes the Kähler form of $P_{\sigma}$, and so $3 \frac{\sigma}{\pi} \phi$ represents an integral cohomology class of $P_{\sigma}$.


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