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Local convexity and nonnegative curvature – Gromov's proof of the sphere theorem

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Dedicated to Wilhelm Klingenberg

1. Introduction

An immersed hypersurface S in a riemannian manifold M will be called *ε-convex* for some $\varepsilon > 0$ if all principal curvatures have the same sign and absolute value at least ε. Can one characterize all compact ε-convex immersed hypersurfaces of a complete manifold M? If M is euclidean n-space, $n \geqq 3$, this problem is solved by a theorem of Hadamard [14] generalized by Hopf [15]: S is embedded and bounds a convex n-disk (see 6.6, 6.3). If M is a flat space form, S is no longer necessarily embedded, but it still bounds an (immersed) convex n-disk. The aim of this paper is to show the same fact for a manifold M of curvature $K \geqq 0$:

Theorem A. *Let M be a complete riemannian manifold with nonnegative sectional curvature and dimension $n \geqq 3$. Let S be a compact connected C^∞-manifold of dimension $n-1$ and $y: S \to M$ an ε-convex immersion, for some $\varepsilon > 0$. Then there is an immersion $\hat{y}: D \to M$ where D is the standard n-disk, and a diffeomorphism $\phi: S^{n-1} = \partial D \to S$ such that $\hat{y}|_{\partial D} = y \circ \phi$ and the mean curvature vector of $y(S)$ is pointing towards $\hat{y}(D)$.*

Theorem A is not true for negative curvature; e.g. S could be the boundary of a tubular neighborhood around a closed geodesic. It is also false for $n=2$: Any locally strongly convex closed plane curve of winding number 2 or more provides a counterexample. The idea of the proof is to contract the hypersurface by pushing it along the gradient lines of a smoothed modification of its local distance function. The distance function is essentially strictly convex, due to $K \geqq 0$ ((Chap. 3), and the smoothing does not disturb the convexity (Chap. 4), and therefore, this motion is distance-decreasing with respect to the inner metric of S. In the case of the 2-dimensional counterexample mentioned above, S would eventually develop a cusp and the motion would stop there. A main step of the proof is to show that this cannot happen in higher dimension. Since this is a local question, it can be treated in euclidean space (Chap. 6), by means of suitable coordinates (Chap. 5). So the contraction ends with a point and the hypersurface bounds an immersed disk (Chap. 7).

Convexity methods have been used extensively in the geometry of nonnegative curvature (see [11, 4, 9]). The main difference is that we have to work with functions which are defined only locally.

A useful application of this theorem is a very short and direct proof of the sphere theorem of Berger and Klingenberg [2, 17, 10]:

Theorem B. *Let M be a complete connected riemannian manifold with bounded positive curvature $0<K_{\min}\leqq K\leqq K_{\max}$ and $K_{\max}/K_{\min}<4$. Then M is diffeomorphically covered by a twisted sphere.*

As usual, by a twisted sphere we mean the union of two discs D_+ and D_-, pasted together by a diffeomorphism between ∂D_+ and ∂D_- (see [3]). The proof of Theorem B uses neither Toponogov's theorem nor Klingenberg's estimate of the injectivity radius. In fact, this latter theorem is a consequence of the proof:

Theorem C. *Let M be complete, simply connected with $0<K_{\min}\leqq K\leqq K_{\max}$ and $K_{\max}/K_{\min}<4$. Then for any point p in M, the injectivity radius at p equals precisely the conjugate radius at p which is not less then $\pi/\sqrt{K_{\max}}$.*

2. Proof of the Theorems B and C

Let M be as in the assumption of Theorem B. Multiplying the metric of M by a suitable factor, we may assume $\frac{1}{4}\leqq K<1$. Choose an arbitrary point $p\in M$. Due to $K<1$, the conjugate distance r_0 of p is strictly larger than π (see remark in 3.4). So for any $r\in(\pi, r_0)$, the exponential map $e:=\exp_p$ has highest rank on the closed ball $\bar{B}_r(0)$ in T_pM. Let $S:=\partial B_r(0)$ and $y:=e|_S: S\to M$; this is an immersion. Let $N: S\to TM$ be the unit normal vector field along y which points towards the interior, $e(B_r(0))$. Then due to $K\geqq\frac{1}{4}$, there is an $\varepsilon>0$ such that $\langle D_X N, X\rangle\geqq\varepsilon\|X\|^2$ for any tangent vector X of S. (Just apply Lemma 3.4(b) to the manifold $B_{r_0}(0)$ with metric induced by $\exp_p$ and to the hypersurface $S=\partial B_r(0)$.) Thus the immersed hypersurface S in M is ε-convex with mean curvature vector pointing towards the exterior. By Theorem A, there is a diffeomorphism $\phi: S^{n-1}=\partial D\to S$ and an immersion $\hat{y}: D\to M$ with $\hat{y}|_{\partial D}=y\circ\phi$ such that the normal field N along y is pointing outside $\hat{y}(D)$. Let $D_+=B_r(0)$ and $D_-=D$ and consider the twisted sphere $S_\phi=D_+\bigcup_\phi D_-$. Then $\psi=e\bigcup_\phi \hat{y}$ is a local diffeomorphism of S_ϕ onto M and hence a covering map which proves Theorem B. If M is simply connected, ψ is even a global diffeomorphism and in particular, $e=\exp_p$ is injective on $\bar{B}_r(0)$ for all $r<r_0$. This proves Theorem C.

3. Hypersurfaces and distance function

3.1. Let M be a Riemannian manifold and S and $\bar{S}$ hypersurfaces in M which touch each other at some point $p\in M$. Let N and $\bar{N}$ be unit normal fields on S and $\bar{S}$ whith $N_p=\bar{N}_p$. Let t_0 and $\bar{t}_0$ be the focal distances of S and $\bar{S}$ in the direction of N_p.

Lemma 3.1. *If $\langle D_X N, X\rangle<\langle D_X\bar{N}, X\rangle$ for every $0\neq X\in T_pS$, then $t_0<\bar{t}_0$.*

Proof. Let c be the geodesic with $c(0)=p$, $c'(0)=N_p$. For any parameter t, we identify the subspace $c'(t)^\perp$ of $T_{c(t)}M$ with $T_pS=c'(0)^\perp$ via parallel transport along c. For any $x\in T_pS$ let $J_x(t)$ be the Jacobifield along c with $J_x(0)=x$ and $J'_x(0)=D_xN$ and similar $\bar{J}_x(t)$ with $\bar{J}'_x(0)=D_x\bar{N}$. Thus we defined two families of linear mappings $J(t)$, $\bar{J}(t)$ on T_pS by setting $J(t)x=J_x(t)$, $\bar{J}(t)x=\bar{J}_x(t)$, and these satisfy the Jacobi equation $J''+RJ=0$ where $R(t)$ is the symmetric linear map $R(t)x=R(x,c'(t))c'(t)$. By symmetry of R and $D\bar{N}$, we get that $\bar{J}'\bar{J}^*$ is also symmetric. It follows that for $0\leqq t<t_0$, we have $J(t)=\bar{J}(t)X(t)C$ with $C=DN-D\bar{N}$ and

$$X(t)=C^{-1}+\int_0^t(\bar{J}^*\bar{J})^{-1}(\tau)\,d\tau.$$

Note that C is negative definite on T_pS, in particular invertible, and that $\bar{J}$ and hence $\bar{J}^*\bar{J}$ is invertible on $[0,\bar{t}_0)$. For $t=0$, all eigenvalues of $X(t)$ are negative. If t comes close to $\bar{t}_0$, then $\int_0^t(\bar{J}^*\bar{J})(\tau)\,d\tau$ gets a very large eigenvalue: Since $\|\bar{J}(t)x\|^2\leqq k(t-t_0)^2\|x\|^2$ for some $x\neq 0$ and some $k>0$, we get

$$\operatorname{trace}(\bar{J}^*\bar{J})^{-1}(\tau)=\operatorname{trace}(\bar{J}^{-1*}\bar{J}^{-1})(\tau)\geqq\frac{\langle\bar{J}^{-1*}\bar{J}^{-1}Jx,Jx\rangle}{\langle Jx,Jx\rangle}(\tau)$$
$$\geqq k^{-1}(\tau-\bar{t}_0)^{-2},$$

and so the trace of the integral goes to ∞ as $t\to\bar{t}_0$. Thus for t_1 close enough to $\bar{t}_0$, $X(t_1)$ has a positive eigenvalue. So there is some $t_2\in(0,t_1)$ where $X(t_2)$ and hence $J(t_2)$ is not invertible. Since t_0 is the first parameter value where this happens, we have $t_0\leqq t_2<\bar{t}_0$.

Remark. The ideas of this proof go back to Green ([7], see also [5]).

3.2. For our purposes, the following form of the Jordan-Brouwer separation theorem is useful.

Theorem. *Let M be a simply connected smooth manifold and S a smooth closed connected hypersurface of M. Then $M\backslash S$ has exactly two connected components.*

Proof. Let $p\in S$ and U a small coordinate ball around p in M such that $U\backslash S$ has two connected components U_+ and U_-. Choose points $p_+\in U_+$, $p_-\in U_-$. Assume that $M\backslash S$ is connected. Then there is a smooth curve c_1 in $M\backslash S$ which connects p_+ to p_-. Choose a curve c_2 in U connecting p_- to p_+ and intersecting S transversally. Then $c=c_1\cup c_2$ is a closed curve which can be assumed to be smooth and which intersects S exactly once and transversally. By simple connectivity, c is homotopic to a closed curve $\bar{c}$ which does not intersect S. Since the intersection number mod 2 is a homotopy invariant (see [13], p. 78), this is a contradiction.

Moreover, if $M_+(M_-)$ denotes the connected component of $M\backslash S$ containing $U_+(U_-)$, then ∂M_+ and ∂M_- are open and closed in S. So there are no further components since S is connected.

3.3. Now let M be a Riemannian manifold and S a connected, two-sided hypersurface in M with unit normal vector field N. We will say that a point $q\in M$

projects onto S if there is a shortest geodesic from q to S. If M is complete and S is closed (as a subset of M), every point projects onto S. Let M''_+ (M''_-) denote the set of points which project onto the upper (lower) side of S, i.e. if $q \in M''_\pm$ and $c:[0,d] \to M$ is shortest with $c(0) \in S$, $c(d)=q$, $d=d(q,S)>0$, then $c'(0)=\pm N_{c(0)}$. Put $M''=M''_+ \cup S \cup M''_-$. E.g. if M is complete and $S=\partial B$ for some open subset B of M, and if N denotes the outer unit normal field, then $M''=M$, $M''_-=B$, $M''_+=M\setminus\bar{B}$. Further, let $M' \subset M''$ be the interior of the set of points where the shortest geodesic to S is unique. This is an open neighborhood of S. Put $M'_\pm = M' \cap M''_\pm$.

Lemma 3.3. *Let M be a complete Riemannian manifold and $S \subset M$ a twosided hypersurface. Let $p \in S$ and $0<\delta<i(p)$ where i denotes the injectivity radius function on M, and assume that $S \cap B_\delta(p)$ is connected and $S \cap \bar{B}_\delta(p)$ is closed in M. Then $B_{\delta/2}(p) \subset M''$.*

Proof. If $q \in B_{\delta/2}(p) =: B$, there is a shortest geodesic c from q to the compact set $S \cap \bar{B}_\delta(p)$. But since the length of c has to be smaller than $\delta/2$, the endpoint of c lies in the open subset $S \cap B_\delta(p)$ of S. Thus q projects onto S. Since $S \cap B$ is closed in B, the point q cannot project onto both sides of S, by 3.2, unless $q \in S$. So $B \subset M''$.

3.4. On M'', we may define the *signed distance function d* of S as follows: $|d(x)|$ is the distance $d(x,S)$ from x to S, and $d(x)$ is positive (negative) for $x \in M''_+$ ($x \in M''_-$). Then M' is the set of points where d is smooth. Its gradient $\hat{N} := \nabla d$ is the extension of N on M' with $D_{\hat{N}}\hat{N}=0$. Let $D^2 d$ denote the Hessean 2-form of d. The proof of the following facts is based on an idea of Green [7]:

Lemma 3.4(a). *Let $k, \lambda \in \mathbb{R}$. Let M be a Riemannian manifold with curvature $K \geqq k$ and let S be a hypersurface in M with unit normal field N and $\langle D_X N, X\rangle \geqq \lambda \|X\|^2$ for every nonzero tangent vector X of S. Then $D^2 d(X,X) \geqq v(d)\|X\|^2$ for all $0 \neq X \in TM'_-$ with $X \perp \nabla d$, where v is a solution of*

$$v' + v^2 + k = 0, \; v(0) = \lambda .$$

If $K>k$ or $DN>\lambda$, the inequality is strict.

Lemma 3.4(b). *Let M be as above, $\bar{p} \in M$ and $r<i(p)$. Let $S=\partial B_r(\bar{p})$ and N the inner unit normal field on S. Then $\langle D_X N, X\rangle \geqq v(0)\|X\|^2$ for any nonzero tangent vector X of S, where v is a solution of*

$$v' + v^2 + k = 0, \; \lim_{t\to r} 1/v(t) = 0 .$$

If $K>k$, the inequality is strict.

Proof. Let $\hat{N}=\nabla d$, where d is the signed distance function of S in both cases. If c is an integral curve of $\hat{N}$, i.e. a geodesic orthogonal to S, then as a consequence of the Jacobi equation, the familiy of linear maps $U(t)x := D_x\hat{N}$ for $x \in c'(t)^\perp$ satisfies the Riccati equation

$$U' + U^2 + R = 0$$

with $R(t)$ as in 3.1. Let us assume first that $R>k$, that means that $R(t)-kI$ is positive definite for all t. Let v be a solution of $v'+v^2+k=0$ and put $V=vI$. Then

$$(U-V)' < -(U+V)(U-V) .$$

If $(U-V)(t_1)>0$ for some t_1, then the same is true for all $t\in(t_0,t_1)$ where t_0 is the largest parameter smaller then t_1 where U or V has a pole. Namely, if $\bar{t}\in(t_0,t_1)$ was the largest parameter where this fails, there would be some $x\neq 0$ with $(U-V)(\bar{t})x=0$, hence we would have $\langle (U-V)x,x\rangle'(\bar{t})<0$ which contradicts $\langle (U-v)x,x\rangle(t)>0$ for $\bar{t}<t\leqq t_1$. (Here we identified x with its corresponding parallel field along c, as in 3.1.) In particular, if t_0 is finite, it must be a pole of U since otherwise we would get a contradiction from $\lim_{t\downarrow t_0} v(t)=+\infty$ and $U>V$.

This proves immediately 3.4(a) for $K>k$, $DN>\lambda$. Since v depends continuously on k and λ, the result follows also for the weaker assumption.

If $S=\partial B_r(p)$ and N the inner normal field, note that $U(t)^{-1}\to 0$ as $t\to r$, and $(U^{-1})'=I+U^{-1}RU^{-1}$. Therefore, the singularities of U^{-1} and V^{-1} at $t=r$ are removable, and $(U^{-1})'(r)=(V^{-1})'(r)=I$, $(U^{-1})''(r)=(V^{-1})''(r)=0$, $(U^{-1})'''(r)=2R(r)>2k=(V^{-1})'''(r)$ if $R>k$. Thus for $t_1<r$ sufficiently near to r, we have $U(t_1)^{-1}<V(t_1)^{-1}$, hence $(U-V)(t_1)<0$. So by the previous argument we get $U(0)>v(O)I$. The result for $R\geqq k$ follows by continuity, as above.

Remark. Exactly the analogous arguments are valid under the assumption $K\leqq k$ which implies the opposite inequalities. In particular it follows that then the conjugate distance on M is larger than on a sphere of curvature k.

3.5. *Remark.* The Rauch comparison theorems are an easy consequence of the previous section. E.g. if J is a Jacobi field along a geodesic c with $J'(0)=0$, then J belongs to the normal flow of any hypersurface S through $c(0)$ with $N_{c(0)}=c'(0)$ and $DN|_{c(0)}=0$. Therefore, if d is the signed distance function of S and $U(t):=DVd|_{c(t)}$ its Hessean, then $J'=UJ$. If $K\geqq 0$, then $U(t)\geqq 0$ for $t\leqq 0$ up to the focal distance, by 3.4(a). Therefore, $\|J\|'=\langle UJ,J\rangle/\|J\|\geqq 0$, hence $\|J(t)\|\leqq\|J(0)\|$ for $t\leqq 0$. Reversing the orientation of c we get the same for $t\geqq 0$.

3.6. Let M be an arbitrary Riemannian manifold and $S\subset M$ a hypersurface which is ε-convex with respect to a unit normal vector field N, that means $\langle D_X N,X\rangle\geqq\varepsilon\|X\|^2$ for any $X\in TS$. Let $d:M''\to\mathbb{R}$ be its signed distance function. Then for any $q\in M''$ and $\eta<\varepsilon$ we get a support function $\bar{d}=d_{q,\eta}$ of d at q as follows (see [18, 20]): Let c be a shortest geodesic from q to S and $p\in S$ its end point. Let $\bar{S}$ be another hypersurface through p with normal field $\bar{N}$ and suppose that $\bar{N}_p=N_p$ and $D_X\bar{N}=\eta X$ for any $X\in T_pS$. We may choose

$$\bar{S}=\exp_p(\partial B_{\bar{R}}(-\bar{R}N_p)\cap V),$$

where $\bar{R}=1/\eta$ and V an open neighborhood of O_p in T_pM which lies in the injectivity domain of $\exp_p$. By Lemma 3.1, applied to the normal fields $-N$ and $-\bar{N}$, the first focal point of $\bar{S}$ along c comes behind q. So the signed distance function $\bar{d}$ of $\bar{S}$ is defined and smooth in a small neighborhood U of q, if V is small enough to exclude cut locus points near q. Moreover, if $\gamma:[0,\delta)\to\bar{S}$ is a geodesic in $\bar{S}$ with $\gamma(0)=p$, and if we put $\phi=d\circ\gamma$, then $\phi(0)=0$, $\phi'(0)=0$, $\phi''(0)\geqq\varepsilon-\eta>0$. Thus we have $d\geqq 0$ on a neighborhood of p in $\bar{S}$. Making $\bar{S}$ even smaller if necessary, we may assume $\bar{S}\subset M''$ and $d|_{\bar{S}}\geqq 0$. Let $x\in M''_-$ be in the domain of $\bar{d}$ and let $\bar{p}$ be a point in $\bar{S}$ with shortest distance to x. Then $|d(x)-d(\bar{p})|\leqq d(x,\bar{p})=|\bar{d}(x)|$, hence $d(x)\geqq\bar{d}(x)$ because $d(\bar{p})\geqq 0$. So we have shown:

Lemma 3.6. *If $q \in \operatorname{Int}(M''_-)$ and $\eta < \varepsilon$, then $\bar{d} = d_{q,\eta}$ is a smooth support function of d in q, more precisely, $\bar{d}$ is defined and smooth on a neighborhood U of q with $\bar{d} \leqq d$ and $\bar{d}(q) = d(q)$.*

4. ε-convex functions and smoothing

In the following chapter, we use ideas of various authors [1, 6, 8, 9, 12, 18, 20] to describe the smoothing of a certain type of convex functions. We discuss details since our notion of ε-convexity is sligtly different.

4.1. Let M be a Riemannian manifold and ε any real number. A continuous real valued function f on M is called *ε-convex* if for any $q \in M$ and any $\eta < \varepsilon$ there is a smooth support function $f_{q,\eta}$ of f in q (defined near q, $f_{q,\eta} \leqq f$, $f_{q,\eta}(q) = f(q)$), such that

$$D^2 f_{q,\eta}(X, X) \geqq \eta \|X\|^2 \quad \text{for all} \quad X \in T_q M .$$

It is easy to see that ε-convexity implies convexity for $\varepsilon \geqq 0$. (In fact, for $\varepsilon > 0$, ε-convexity implies strict convexity in the sense of [1, 20].) Namely, for a curve $c: [a, b] \to M$ let $\phi_{\eta, c, f} = \phi_\eta$ be the real quadratic polynomial with

$$\phi_\eta(a) = f(c(a)), \qquad \phi_\eta(b) = f(c(b)), \qquad \phi_\eta'' = \eta .$$

If c is a geodesic (parametrized by arc length), then $f \circ c \leqq \phi_\varepsilon$: Otherwise $f \circ c - \phi_\eta$ for some $\eta < \varepsilon$ would attain an interior maximum at some point $u \in (a, b)$, and this would contradict to $(f_{c(u),\eta'} \circ c - \phi_\eta)''(u) \geqq \eta' - \eta > 0$ for any $\eta' \in (\eta, \varepsilon)$. Moreover, a similar argument still holds if c is slightly curved:

Lemma 4.1. *Let f be an ε-convex function on M with $\|\nabla f_{q,\eta}(q)\| \leqq L$ for any $q \in M$ and $\eta < \varepsilon$. Let $c: [a, b] \to M$ be a curve with $\|D_{c'} c'\| \leqq \gamma$ and $\|c'\|^2 \geqq 1 - \beta$ for small positive β, γ. Let $\eta = \eta(\beta, \gamma) = \varepsilon - \varepsilon\beta - L\gamma$. Then $f \circ c \leqq \phi_\eta$.*

4.2. Clearly, a smooth function f is ε-convex if and only if $D^2 f \geqq \varepsilon$; this follows from 4.1. If S is a regular level hypersurface of a smooth function f with $D^2 f \geqq \varepsilon$ and $\|\nabla f\| \leqq L$ along S, then S is (ε/L)-convex with respect to the unit normal vector field $N = \nabla f / \|\nabla f\|$.

4.3. Let U be an open subset of M such that curvature and injectivity radius are bounded on U. Let f be a continuous real valued function on U. For any $r > 0$ which is smaller than the convexity radius on U, we may approximate f by a smooth function f_r defined on $U_r := \{x \in M;\ B_r(x) \subset U\}$ as follows (see [8, 12]):

$$\begin{aligned} f_r(x) &= \int_{T_x M} f(\exp_x(u)) \psi_r(\|u\|) d^n u \\ &= \int_{B_r(x)} f(y) \psi_r(d(x, y)) d\mu_x(y), \end{aligned}$$

where $d^n u$ denotes the volume element on $T_x M$ and $d\mu_x$ the measure on $B_r(x)$ with $\exp_x^*(d\mu_x) = d^n u$, and $\psi_r: \mathbb{R}_+ \to \mathbb{R}_+$ is a smooth function with $\psi_r|_{[0, r/2]} = \text{const} > 0$,

$\psi_r|_{[r,\infty)}=0$ and $\int_0^r \psi_r(t)t^{n-1}dt=1/\omega_{n-1}$ (where ω_{n-1} denotes denotes the volume of the unit sphere in $\mathbb{R}^n$).

If f is a Lipschitz function with Lipschitz constant L, we get immediately $|f-f_r|\leqq Lr$. (Note that $\int \psi_r(\|u\|)d^n u=1$.) Moreover, if $K\geqq 0$ on U, then f_r has the same Lipschitz constant L: If $x, y\in U_r$ are sufficiently near and $P: T_xM\to T_yM$ denotes the parallel displacement along the shortest geodesic between x and y, then

$$\begin{aligned}|f_r(x)-f_r(y)| &\leqq \int_{T_xM} |f(\exp_x(u))-f(\exp_y(Pu))|\psi_r(\|u\|)d^n u\\ &\leqq L\int_{T_xM} d(\exp_x(u),\exp_y(Pu))\psi_r(\|u\|)d^n u\\ &\leqq Ld(x,y)\end{aligned}$$

by Rauch's theorem (see 3.5).

Next we want to estimate the derivatives (see [12]):

Lemma 4.3. *Let $0\leqq K\leqq k$ on U and f be a smooth function with $\|Df\|\leqq L$, $\|D^2f\|\leqq C$. Then*

$$\|Df_r-Df\|\leqq Cr+\tfrac{1}{2}Lkr^2\,.$$

Proof. Fix $x\in U_r$, $v\in T_xM$ with $\|v\|=1$. Let c be a geodesic with $c(0)=x$, $c'(0)=v$. For any $u\in T_xM$ with $\|u\|\leqq r$ let $a_u(s,t)=\exp_{c(t)}sP_tu$ where P_t denotes the parallel displacement along c. Let $U=\frac{d}{ds}a_u$, $V=\frac{d}{dt}a_u$. Then $V_t(s)=V(s,t)$ is the Jacobi field along the geodesic $c_t(s)=a_u(s,t)$ with $V_t(0)=c'(t)$, $V_t'(0)=\frac{D}{dt}U(0,t)=0$. Since $\|U\|=\|u\|\leqq r$ and $\|V\|\leqq\|v\|=1$, by 3.5, we have $\|V''\|=\|R(V,U)U\|\leqq kr^2$, and so $\|V'\|'\leqq\|V''\|\leqq kr^2$, hence $\|V_t'(s)\|\leqq kr^2s$. Now

$$Df_r(v)-Df(v)=\left.\frac{d}{dt}\right|_{t=0}\int_{T_xM}(f(a_u(1,t))-f(a_u(0,t))\psi_r(\|u\|)d^n u$$

and

$$\begin{aligned}\frac{d}{dt}(f(a_u(1,t))-f(a_u(0,t))&=\int_0^1\frac{d}{dt}\frac{d}{ds}f(a_u(s,t)ds\\ &=\int_0^1[D^2f(U(s,t),V(s,t))+Df(V_t'(s))]\,ds\,,\end{aligned}$$

so the result follows.

4.4. Now we want to show that ε-convexity is almost preserved by smoothing. Let M be any Riemannian manifold, M_0 a relatively compact open subset, and let $r_0>0$ be smaller than the convexity radius on M_0. The following lemma is essentially due to Greene and Wu [8, 9]:

Lemma 4.4. *For any $\varepsilon, L > 0$ there is a monotonely decreasing function $\eta: (0, r_0) \to \mathbb{R}$ with $\eta(r) \to \varepsilon$ as $r \to 0$ with the following property: If f is any ε-convex function defined on some open convex subset U of M_0, with $\|\nabla f_{q,\bar{\varepsilon}}(q)\| \leqq L$ for all $q \in U$, $\bar{\varepsilon} < \varepsilon$, then the smoothing f_r of f is $\eta(r)$-convex, for any $r \in (0, r_0)$.*

Proof. For any $x \in \bar{M}_0$ and $u, v \in T_x M$ with $\|u\| \leqq r$, $\|v\| = 1$ let $c = c_{v0}$ be the geodesic with $c(0) = x$, $c'(0) = v$; further let $c_{vu}(t) := \exp_{c(t)}(P_t u)$ where P_t denotes the parallel displacement along c. Let

$$\beta(v, u) = 1 - \|c'_{vu}(0)\|^2, \qquad \gamma(v, u) = \|D_{c'_{vu}} c'_{vu}(0)\|,$$

and let $\beta(r)$, $\gamma(r)$ be the maxima of these functions (note that the set of all (u, v) is compact). We have $\beta(r), \gamma(r) \to 0$ as $r \to 0$. Hence, if $c: [a, b] \to U_r$ is a geodesic segment, then for any $u \in T_{c(a)}M$ with $\|u\| \leqq r$, we have by Lemma 4.1 for $t \in [a, b]$

$$f(\exp_{c(t)}(P_t u)) = f(c_{vu}(t)) \leqq \phi_{\eta(r), c_{vu}, f}(t),$$

where $\eta(r) = \varepsilon - \varepsilon\beta(r) - L\gamma(r)$, and so $f_r \circ c \leqq \phi_{\eta(r), c, f_r}$. Since f_r is smooth, this implies $D^2 f_r \geqq \eta(r)$.

4.5. Let M be any Riemannian manifold and S an ε-convex hypersurface. Then its signed distance function d fails to be ε-convex along S since we have $D^2 d(X, X) \geqq \varepsilon \|X\|^2$ only for $X \in TS$. Therefore, we consider the function $f = \chi_\varepsilon \circ d$ instead (compare [1]) with

$$\chi_\varepsilon(t) := t + \frac{\varepsilon}{2} t^2.$$

Now f is ε-convex with $\|\nabla f\| = 1$ along S.

If $K \geqq 0$ on M, then on M'_- we have $D^2 d(X, X) \geqq \dfrac{\varepsilon}{1 + \varepsilon d} \|X\|^2$ for any $X \perp \nabla d$, by Lemma 3.4(a). Hence

$$D^2 f = \varepsilon Dd \cdot Dd + (1 + \varepsilon d) D^2 d \geqq \varepsilon$$

on M'_-. So for $q \in M''_-$ with $d(q) > -R := -1/\varepsilon$, the function $f_{q,\eta} := \chi_\varepsilon \circ d_{q,\eta}$ (compare 3.6) is a smooth support function of f at q with $D^2 f_{q,\eta}(X, X) \geqq \eta \|X\|^2$ for any $X \in T_q M$. Thus we have shown

Lemma 4.5. *If $K \geqq 0$ on M and S is an ε-convex hypersurface with signed distance function $d > -R$, then $f = \chi_\varepsilon \circ d$ is ε-convex on M''_-.*

Further note that f has Lipschitz constant $L_t = 1 + \varepsilon t$ on the set $\{d \leqq t\}$ and we have $d \leqq f \leqq d/2$ on $\{0 \geqq d \geqq -R\}$.

Remark. Since the focal distance of S is not bigger than R, by Lemma 3.4(a), we have always $d \geqq -R$, and it is not difficult to show that $d(q) = -R$ for some q occurs only if $S \subset \partial B_R(q)$ and M''_- is flat (see [6]). However, one may avoid this argument by choosing ε sligtly smaller, if necessary; then $R = 1/\varepsilon$ gets larger and we have $d > -R$ for the new R.

4.6. In particular we have shown: If $K \geqq 0$ on M and B is a relatively compact open subset with smooth boundary $S = \partial B$ which is ε-convex with respect to the outer unit normal field, then $f=\chi_\varepsilon \circ d$ is ε-convex on $\bar{B}$. This remains true if $M=\bar{B}$, that means that M is a compact manifold with boundary S. Namely, f is ε-convex on the subset M'_- where d is smooth. Moreover, the parallel hypersurfaces $S_r=\{d=-r\}$ are smooth for small positive r, and for the signed distance function d_r of S_r we have $d_r=d+r$. Since S_r is $\bar{\varepsilon}$-convex for $\bar{\varepsilon}=(R-r)^{-1}$ with $R=1/\varepsilon$ (see Lemma 3.4(a)), the function $f=\chi_\varepsilon \circ (d_r-r)$ is ε-convex on $\{d \leqq -r\}$.

5. Coordinates preserving convexity

5.1. Let M be a Riemannian manifold and (U, ϕ) a coordinate chart, i.e. U is an open subset of M and ϕ a diffeomorphism of U onto an open subset V of $\mathbb{R}^n$. Let $ds^2=\| \ \|^2$ be the given metric on U and $ds_0^2=\| \ \|_0^2$ the euclidean metric induced by ϕ, and let D, D^0 denote the corresponding Levi-Civita connections. Assume that

$$\|D-D^0\| \leqq \frac{\varepsilon}{4} \quad \text{and} \quad \frac{1}{4} ds^2 < ds_0^2 < 4 ds^2 .$$

Lemma 5.1. *If $S \subset U$ is an ε-convex hypersurface, then $\phi(S) \subset V \subset \mathbb{R}^n$ is $\frac{\varepsilon}{16}$-convex.*

Proof. Let d be the signed distance function of S and $f=\chi_\varepsilon \circ d$. Then for any $p \in S$ we have $\|Df|_p\|_0 < 2\|Df|_p\| = 2$, and for all $X \in T_pM$,

$$|(D^0 Df - DDf)(X, X)| = |Df(D_X X - D_X^0 X)| \leqq \|Df|_p\| \, \|D_X X - D_X^0 X\| \leqq \frac{\varepsilon}{2} \|X\|^2 .$$

On the other hand, $DDf(X, X) \geqq \varepsilon \|X\|^2$ (see 4.5) and so

$$D^0 Df \geqq \frac{\varepsilon}{2} \|X\|^2 \geqq \frac{\varepsilon}{8} \|X\|_0^2 .$$

Therefore, S is $\frac{\varepsilon}{16}$-convex with respect to ds_0^2, by 4.2.

5.2. A coordinate system satisfying the assumptions of 5.1 will be called a *good coordinate system*. If $M_0 \subset M$ is a relatively compact, open subset of M, then by continuity, there is a radius $\varrho > 0$ such that the exponential coordinates in $B_\varrho(p)$ have this property, for any $p \in M_0$. A more explicit lower bound for the radius of a good coordinate patch in terms of the injectivity radius and the curvature bounds has been given by Jost and Karcher [16] using almost-linear coordinates.

5.3. Let $y: S \to M$ be an ε-convex immersion. For every $s \in S$ let $p=y(s)$ and $(B_\varrho(p), \phi_p)$ be the good coordinate system of 5.2. Let S' be the connected component of $y^{-1}(B_\varrho(p))$ through s. Then $x := \phi_p \circ y|_{S'} : S' \to \mathbb{R}^n$ is an $\frac{\varepsilon}{16}$-convex immersed hypersurface in $\mathbb{R}^n$. Thus on a small scale, the properties of ε-convex immersions can be studied in euclidean space.

6. ε-convexity in euclidean space

6.1. Throughout this chapter, we let $M=\mathbb{R}^n$ be the euclidean n-space. Let S be a connected hypersurface which is ε-convex with respect to the unit normal field N on S, and let d denote its signed distance function. In the following, we always put $R:=1/\varepsilon$. A special property of the flat space is

$$D^2 d(X,X) \geqq (d+R)^{-1} \|X\|^2 \quad \text{for any} \quad X \perp \nabla d$$

at any point where d is smooth, also on M'_+. Hence, by 4.5, the function $f=\chi_\varepsilon \circ d$ is ε-convex on $M' \cup M''_-$. Moreover, we have canonical support functions: For any $p \in S$ let $B_p := B_R(p - RN_p)$ and $S_p := \partial B_p$. Let d_p be the signed distance function of S_p and $f_p = \chi_\varepsilon \circ d_p$. The function f_p is defined and smooth everywhere with $D^2 f_p(X,X) = \varepsilon \|X\|^2$ for every tangent vector X. Hence $g := f - f_p$ is 0-convex with $g(p)=0$, $\nabla g(p)=0$. So g attains a local minimum at p and consequently, $f \geqq f_p$ on any convex neighborhood U of p in $M' \cup M''_-$. It follows that $d \geqq d_p$ and therefore, $S \cap U \subset \bar{B}_p$.

6.2. Lemma. *Let f be a continuous function on $\mathbb{R}^n$ which is convex on a neighborhood U of the closed set $\bar{B} = \{f \leqq 0\}$, and assume that $\bar{B}$ is connected. Then $\bar{B}$ is convex.*

Proof. Let p be an arbitrary point in $\bar{B}$. Let C be the set of all $q \in \bar{B}$ such that the straight line segment $\overline{pq}$ lies in $\bar{B}$. Clearly, C is closed. We show that C is also open in $\bar{B}$. Since $\overline{pq} \subset \bar{B}$ for $q \in C$ and since U is a neighborhood of $\bar{B}$, there is a neighborhood V of q such that $\overline{xp} \subset U$ for any $x \in V$. By convexity, f takes its maximum on $\overline{xp}$ at the end points, therefore $\overline{xp} \subset B$ whenever $x \in V \cap \bar{B}$. So $V \cap \bar{B} \subset C$ and therefore, C is open. Since $p \in C$, we have $C = \bar{B}$ by connectivity which finishes the proof.

6.3. Now let $S \subset M = \mathbb{R}^n$ be a compact, ε-convex hypersurface. By the Jordan-Brouwer separation theorem (see 3.2), S bounds an open domain $B \subset \mathbb{R}^n$ which lies on the side of the normal field $-N$ on S. Then $B = M''_-$ (see 3.3), and by 6.2, $\bar{B}$ is convex. Consequently, for any $q \in \mathbb{R}^n \setminus \bar{B}$, there is a unique shortest line segment from q to S, and therefore, $\mathbb{R}^n \setminus \bar{B} = M'_+$. So by 6.1 we have $d \geqq d_p$ on all of $\mathbb{R}^n$, for every $p \in S$, thus $d \geqq \max_{p \in S} d_p$. On the other hand, for any $q \in \mathbb{R}^n$ there is a closest point $p \in S$ for which $d(q) = d_p(q)$, so we get in fact $d = \max_{p \in S} d_p$. Consequently, $\bar{B} = \bigcap_{p \in S} \bar{B}_p$.

More generally, a connected open subset B of $\mathbb{R}^n$ (with smooth boundary or not) will be called *ε-convex* for some $\varepsilon \geqq 0$ if for any $p \in \partial B$ there is a neighborhood U of p and a ball B_p of radius $R = 1/\varepsilon$ with $p \in \partial B_p$ (support ball or support half space) such that $B \cap U \subset B_p$. Applying the same arguments as above to the signed distance function d of ∂B which is negative on B and positive outside, we see again the convexity of B, more precisely: $B = \bigcap_{p \in \partial B} B_p$ as above.

6.4. Lemma. *Let $\varepsilon = 1/R > 0$ and B a connected, ε-convex open domain containing a line segment of length a. Then B contains a ball of radius $a^2/8R$.*

Proof. If B is a ball of radius R containing a line segment of length a with center q, then B contains the ball $B_r(q)$ with $r=R-(R^2-a^2/4)^{1/2} \geqq a^2/8R$. Hence for an arbitrary ε-convex open set B we have $B_r(q) \subset B_p$ for any $p \in \partial B$ and so $B_r(q) \subset \bigcap_{p \in \partial B} B_p = B$ (see 6.3).

6.5. Let $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n;\ x_n > 0\}$ and $\bar{\mathbb{R}}^n_+$ its closure. Let $S \subset \mathbb{R}^n$ be an ε-convex hypersurface such that $S \cap \mathbb{R}^n_+$ is connected and $S \cap \bar{\mathbb{R}}^n_+$ compact. Thus $S \cap \mathbb{R}^n_+$ is closed in $\mathbb{R}^n_+$, and hence it bounds an open set B in $\mathbb{R}^n_+$ which lies on the side of the normal field $-N$ (see 3.2). So the full boundary of B in $\mathbb{R}^n$ is contained in $S \cup \mathbb{R}^{n-1}$ and therefore, B is 0-convex and hence convex (6.3). However, in general B is no more contained in its support ball B_p for arbitrary $p \in S \cap \mathbb{R}^n_+$. Nevertheless, there is one point p for which $B \subset B_p$ remains true:

Lemma 6.5. *Let $p \in S$ be the point where the coordinate x_n attains its maximum on S. Then $B \subset B_p$.*

Proof. Let d, d_p, f, f_p be the functions defined in 6.1. Then $g := f - f_p$ is convex on $M' \cup M''_-$. Since B is convex, every point of $\mathbb{R}^n \backslash B$ has a unique projection onto ∂B from which we conclude $M''_+ = M'_+$. So g is convex on M'' with local minimum 0 on the line $L''_p := (p + \mathbb{R}e_n) \cap M''$. All we have to show is that every point of $S_+ := S \cap \mathbb{R}^n_+$ can be connected to some point of L''_p by a straight line segment in M''. Then by convexity we have $g \geqq 0$ on S_+ and hence $S_+ \subset \{d_p \leqq 0\} = \bar{B}_p$ which implies $B \subset B_p$.

Let $T = \bar{B} \cap \mathbb{R}^{n-1}$. Then $\partial B = S_+ \cup T$. To examine the size of M'', let $\bar{d}$ be the signed distance function of ∂B which is defined on all of $\mathbb{R}^n$. Put

$$A := \{\bar{d} - x_n < 0\} \cap \mathbb{R}^n_+, \qquad C := \{\bar{d} + x_n < 0\} \cap \mathbb{R}^n_+.$$

These sets are convex since $\bar{d}$ is a convex function. We have $S \subset A \backslash \bar{C}$. Moreover, on $A \backslash \bar{C}$ we have $|\bar{d}| < x_n$. So the points of this set project on S_+ and therefore $A \backslash \bar{C} \subset M''$ with $d = \bar{d}$ on $A \backslash \bar{C}$.

Let $Z = T + \bar{\mathbb{R}}_+ e_n \subset \bar{\mathbb{R}}^n_+$ be the cylinder over T; this is a closed convex set. We claim that $C \subset Z \cap B$. In fact, $C \subset B$ since $\bar{d}$, $x_n \geqq 0$ on $\bar{\mathbb{R}}^n_+ \backslash B$. Moreover, for any $q \in B \backslash Z$, the vertical ray $L_q^- = q - \mathbb{R}_+ e_n$ starting at q intersects ∂B at some point $q' \in \partial B \backslash T = S_+$, so $x_n(q') > 0$. Therefore, $-\bar{d}(q) \leqq d(q, q') = x_n(q) - x_n(q') < x_n(q)$ and hence $q \notin C$ which proves the claim.

Now for $q \in S_+$ the vertical rays $L_q^+ = q + \mathbb{R}_+ e_n$ do not meet the set $Z \cap B$ since either $q \notin Z$ or the line $L_q = q + \mathbb{R}e_n$ leaves B at q. In both cases there is an open cone C_q with vertex q around L_q^+ which does not meet $Z \cap B$; in the first case this is because $Z \cap B$ is contained in the truncated cylinder of hight $x_n(p)$ over T. So there is a line segment L from q to some point of L_p^+ within C_q. On the other hand, $L_p^+ \subset A$, so $L \subset A \cap C_q \subset A \backslash \bar{C} \subset M''$ which finishes the proof.

6.6. Lemma. [14, 15]: *Let S be a compact connected manifold of dimension $n-1$ and $x: S \to \mathbb{R}^n$ an ε-convex immersion. If $n=2$, assume further that the closed plane curve x has winding number ± 1. Then S is diffeomorphic to the $(n-1)$-sphere and x is an embedding.*

Proof. Let $\nu: S \to S_1^{n-1}$ be the Gauss mapping of the immersion x. Due to the ε-convexity, this is a local diffeomorphism and in particular a covering map. So it must be a global diffeomorphism since S_1^{n-1} is simply connected for $n \geqq 3$ and the degree of ν is ± 1 for $n=2$. Consequently, for every $v \in S_1^{n-1} \subset \mathbb{R}^n$ the hight function $h_v(s) = \langle v, x(s) \rangle$, $s \in S$, has exactly two critical points: one maximum and one minimum. Therefore, x is an embedding: If $s \in S$ and $v = \nu(s)$ its outer normal vector, then h_v attains its maximum only at s and so we have $x(s') \neq x(s)$ for every $s' \neq s$ in S.

6.7. We now can prove the main result of this section. For any immersion $x: S \to \mathbb{R}^n$ and any $s \in S$, $r > 0$ let $U_r(s)$ be the connected component of $x^{-1}(B_r(x(s)))$ containing s.

Lemma 6.7. *Let $x: S \to \mathbb{R}^n$ be an ε-convex hypersurface immersion, for $n \geqq 3$. Let $s_0 \in S$ and assume that $S' := U_\varrho(s_0)$ is relatively compact in S, for some $\varrho > 0$. Let $\delta = \frac{1}{2}\varepsilon\varrho^2$ and $S'' := U_\delta(s_0)$. Then $x|_{S''}$ is an embedding.*

Proof. Let $p := x(s_0)$. We may assume that the n^{th} basis vector e_n of $\mathbb{R}^n$ is the outer normal vector of x at s_0 so that the hight function $x_n = \langle x, e_n \rangle$ on S has a local maximum $h := x_n(s_0) = p_n$ at s_0. Since $x(S)$ lies locally on one side of each of its tangent hyperplanes, every critical point of x_n is either a maximum or a minimum, so the set C of critical points is isolated.

Let U be a neighborhood of s_0 in S such that $x|_U$ is an embedding with $x(U) \subset B_\varrho(p)$. For every $t < h$ let S_t denote the connected component of $\{s \in S; x_n(s) \geqq t\}$ through s_0. For t sufficiently close to h we have $S_t \subset U \subset S'$. Let $u := \inf\{t < h;\ S_t \subset S'\}$. The set S_u is a closed subset of $\overline{S'}$ and therefore compact, and S_u is invariant under the flow ϕ_t, $t \geqq 0$, of the vector field ∇x_n. Every flow line ends at a maximum, so every point in $S_u \setminus C$ lies in the domain of attraction of some maximum. Since these domains are open and $S_u \setminus C$ is connected (here we need $\dim S \geqq 2$), there is no other local maximum then s_0 on S_u. Likewise, there is at most one local minimum on S_u, and if there exists such a minimum, its domain of attraction under the flow of $-\nabla x_n$ is $S_u \setminus \{s_0\}$. In this case we have $S' = S_u$, so S' is compact and connected and therefore embedded by 6.6. So we may assume that the interval $[u, h)$ contains no critical values for x_n. In particular, $u < -\infty$, and by choice of coordinates we may assume $u = 0$, so $S_u = S_0$.

For $0 \leqq t < h$ let $S^t := \{s \in S_0; x_n(s) = t\}$. This is a compact regular hypersurface of S and the map $x^t: S^t \to \mathbb{R}^{n-1}$, $x^t(s) = x(s) - te_n$ is an ε-convex immersion, by Meusnier's theorem. So for $n \geqq 4$, the immersions x^t are embeddings (6.6), and so the same is true for $x|_{S_0}$. For $n=3$, note that the flow ψ_t of the vector field $\nabla x_n / \|\nabla x_n\|^2$ provides a diffeomorphism of S^0 onto S^t, so we have a smooth family of closed plane curves $x^t \circ \psi_t: S^0 \to \mathbb{R}^2$. For t sufficiently close to h, this is an embedding and so the winding number is 1. Since the winding number is constant for all $t \in [0, h)$, we get the same conclusion as in the case of higher dimension, by 6.6.

Now by 6.5, the hypersurface $x(S_0) \subset \bar{\mathbb{R}}^n_+$ is contained in the closure of the support ball $B_p := B_R(p - Re_n)$ of radius $R = \frac{1}{\varepsilon}$, and $B_p \cap \mathbb{R}^n_+ \subset B_r(p)$ with $r = (2Rh)^{1/2}$. Since $0 = \inf\{t < h;\ x(S_t) \subset B_\varrho(p)\}$, we have $r \geqq \varrho$ and therefore $h \geqq \frac{1}{2}\varrho^2\varepsilon = \delta$. So $S'' \subset U_h(s_0) \subset S_0$ is embedded and the proof is finished.

7. Proof of Theorem A

Throughout this chapter, let M be a complete Riemannian manifold of dimension $n \geqq 3$ with nonnegative sectional curvature and $y: S \to M$ a compact, connected, ε-convex hypersurface immersion, for $\varepsilon = \frac{1}{R} > 0$. Let $M_0 := \{q \in M;\ d(q, y(S)) < 10R\}$. The contraction of S which we want to construct will take place within this set M_0. Since we also want to consider parallel hypersurfaces, let us assume more generally for the following sections 7.2–7.5 that M_0 is an arbitrary relatively compact open subset of M with $y(S) \subset M_1 := \{q \in M;\ B_R(q) \subset M_0\}$. Let $\varrho \in (0, R)$ be a radius for good coordinates around any point of M_0 (see 5.2).

7.2. Lemma. *For every $s \in S$, there is an open, connected neighborhood S'' of s in S such that $y|_{S''}$ is an embedding and $y(S'') \cap \bar{B}_\delta(y(s))$ is compact for $\delta := 2^{-8}\varepsilon\varrho^2$.*

Proof. Put $p = y(s)$. Let $\phi: B_\varrho(p) \to \mathbb{R}^n$ be the good coordinate system around p. Let S' be the connected component of $y^{-1}(B_\varrho(p))$ through s. Then $x = \phi \circ y|_{S'}$ is an $\frac{\varepsilon}{16}$-convex immersion (5.1). Since $\bar{B}^0_{\varrho/2}(p) \subset B_\varrho(p)$, where the suffix ^{0} refers to the euclidean metric induced by ϕ, the set $x^{-1}(\bar{B}_{\varrho/2}(\phi(p)))$ is compact. So we may apply 6.7 for $\varepsilon/16$ and $\varrho/2$ instead of ε and ϱ, and so the s-component S'' of $x^{-1}(B_{2\delta}(\phi(p)))$ for $\delta = 2^{-8}\varepsilon\varrho^2$ is embedded. Moreover, $y(S'') \cap \bar{B}_\delta(p)$ is compact since $\bar{B}_\delta(p) \subset B^0_{2\delta}(p)$.

7.3. As before let M'' be the subset of M where the signed distance function d of the hypersurface $y(S'')$ is defined. By Lemma 3.3 we have $B_{\delta/2}(p) \subset M''$ for $p = y(s)$.

Lemma 7.3. *If $y(S)$ is not entirely contained in $B_{\delta/2}(p)$, then there is a point $q \in B_{\delta/2}(p)$ with $d(q) \leqq -\alpha$ for $\alpha = 2^{-12}\delta^2\varepsilon$.*

Proof. We have $\bar{B}^0_{\delta/4}(p) \subset B_{\delta/2}(p)$, and $B^0 := B^0_{\delta/4}(p) \cap M''_-$ is an $\frac{\varepsilon}{16}$-convex domain with respect to the euclidean metric induced by ϕ since $\partial B^0 \subset y(S'') \cup \partial B^0_{\delta/4}(p)$ (see 6.3). Moreover, $\partial B^0 \cap \partial B^0_{\delta/4}(p) \neq \emptyset$, hence B^0 containes a euclidean straight line of length $\delta/4$ and by 6.4 a euclidean ball of radius $r = \frac{1}{8}\frac{\delta^2}{16}\frac{\varepsilon}{16} = 2\alpha$. Thus the center of this ball is a point $q \in B_{\delta/2}(p) \cap M''_-$ with Riemannian distance $d(q, y(S)) > r/2$ and therefore $d(q) < -\alpha$.

7.4. For $s \in S$ let $U(s)$ and $V(s)$ be the connected components through s of the sets $y^{-1}(B_\delta(y(s)))$ and $y^{-1}(B_{\delta/8}(y(s)))$. We saw that $U(s)$ is relatively compact and $y|_{U(s)}$ is an embedding. Let us assume that $U(s) \neq S$ for every $s \in S$, that means that $y(S)$ is contained in no ball of radius δ. Put $\lambda = \frac{1}{16}\alpha = 2^{-16}\delta^2\varepsilon$. Since $\delta < R$, we have $\lambda < 2^{-16}\delta$.

Lemma 7.4. *For every $s \in S$ there is a smooth function $g = g_s$ defined on a neighborhood M_s of $y(V(s))$ with the following properties:*

(i) $y(V(s)) \subset g^{-1}(0) \subset y(U(s))$,

(ii) $\|\nabla g\| \leqq 2,\ D^2 g \geqq \varepsilon/2$,

(iii) $[-\lambda, 0]$ is a regular interval for g, and $g^{-1}(-\lambda)$ is an ε_1-convex hypersurface with $\varepsilon_1 := 1/(R-\lambda/4) < \varepsilon$.

(iv) Let ψ_t denote the flow of the vector field $X = -\nabla g/\|\nabla g\|^2$. Then $\psi_t(x) \in M_s$ for every $x \in y(V(s))$, $t \in [0, \lambda]$.

Moreover, if $V(s) \cap V(s') \neq \emptyset$ for $s, s' \in S$, then $g_s = g_{s'}$ on $M_s \cap M_{s'}$.

Proof. Let d be the signed distance function of $y(U(s))$ defined on $B_{\delta/2}(p)$ for $p = y(s)$, and let $f = \chi_\varepsilon \circ d$. The function f is ε-convex with $d \leqq f \leqq \frac{1}{2}d$ on $\{d \leqq 0\}$. Moreover, f is smooth on $\{|d| \leqq r_1\}$ where r_1 is the focal distance of the immersed hypersurface $y(S)$, and we have $\|\nabla f\| = 1 + \varepsilon d$. Therefore, if $\lambda < r_1/2$, we may choose $g = f$ and $M_s \subset B_{\delta/3}(p)$ an open set containing $\{0 \geqq d \geqq -2\lambda\} \cap B_{\delta/3}(p)$. If s' is another point in S with $V(s) \cap V(s') \neq \emptyset$, then $d(p, p') < \delta/8$ for $p' = y(s')$. So the signed distance functions of $y(U(s))$ and $y(U(s'))$ agree on $B_{\delta/3}(p) \cap B_{\delta/3}(p')$ since the endpoint of a shortest geodesic from $q \in B_{\delta/3}(p)$ to $y(U(s))$ lies in $B_{2\delta/3}(p) \cap y(U(s)) \subset B_\delta(p') \cap y(U(s)) \subset y(U(s'))$ and vice versa. Therefore, g_s agrees to $g_{s'}$ on $M_s \cap M_{s'}$.

Now assume $\lambda \geqq r_1/2$. Put $r_0 = r_1/6$. For $r < r_0 \leqq \lambda/3$, we consider the smoothing f_r of f (see 4.3) on $B := B_{\delta/4}(y(s))$. Since the Lipschitz constant of f is $L_t = 1 + \varepsilon t$ on $\{d \leqq t\}$ and in particular $L_0 = 1$ on $\{d \leqq 0\}$, we have $|f - f_r| \leqq r$ and $\|\nabla f_r\| \leqq 1$ on $B \cap \{d \leqq -r\}$ (see 4.3). Moreover, the support functions $f_{q,\eta}$ of f satisfy $\|\nabla f_{q,\eta}(q)\| \leqq 1$ for all $q \in \{d \leqq 0\}$ and $\eta < \varepsilon$. Applying Lemma 4.4 we get a function $\eta(r)$ independent of $s \in S$ with $\eta(r) \uparrow \varepsilon$ as $r \downarrow 0$, such that f_r is $\eta(r)$-convex.

Let $q \in B_{\delta/4}(p)$ with $f_r(q) = -\lambda$. Then $d(q) \leqq f(q) \leqq -\lambda + r \leqq -\frac{2}{3}\lambda$ and hence $\|\nabla f_r(q)\| \leqq 1 + \varepsilon(d(q) + r) < 1 - \frac{1}{3}\varepsilon\lambda = \varepsilon(R - \lambda/3)$. Now we choose r so small that

$$\eta(r) \geqq \frac{R - \lambda/3}{R - \lambda/4}\,\varepsilon .$$

Then $f_r^{-1}(-\lambda)$ is an ε_1-convex hypersurface provided that $-\lambda$ is a regular value (4.2).

To satisfy (i), we have to connect f and f_r. Let $\phi: \mathbb{R} \to [0, 1]$ be a smooth function with $\phi(t) = 1$ for $t \leqq -2r_0$ and $\phi(t) = 0$ for $t \geqq -r_0$. Put $g = f$ on $\{|d| \leqq r_0\}$ and

$$g = f + \phi(d)(f_r - f)$$

on $\{d \leqq -r_0\}$. Since $|f - f_r| < r$ and $|D^2 d|$ is bounded from above on $\{|d| \leqq 2r_0\}$ independently of $s \in S$, we may assume $\|\nabla g\| \leqq 2$, $D^2 g \geqq \varepsilon/2$ on $\{-r_0 \geqq d \geqq -2r_0\}$ by choosing r still smaller if necessary. Since f is ε-convex with $\|\nabla f\| \leqq 1$ on $\{0 \geqq d \geqq -r_0\}$ and f_r is $\eta(r)$-convex with $\eta(r) > \frac{2}{3}\varepsilon$ and $\|\nabla f_r\| \leqq 1$ on $\{d \leqq -2r_0\}$, the function g satisfies (ii) on an open set $M_s \subset B_{\delta/4}(p)$ containing $\{d \leqq 0\} \cap B_{\delta/4}(p)$. If $q \in g^{-1}(-\lambda)$, then $d(q) \leqq f(q) \leqq -\frac{2}{3}\lambda \leqq -2r_0$. So $g^{-1}(-\lambda) = f_r^{-1}(-\lambda)$.

By 7.3 there is a point $q \in B_{\delta/4}(p)$ with $d(q) \leqq -\alpha$, $\alpha = 16\lambda$. Thus $f(q) \leqq -\dfrac{\alpha}{2} = -8\lambda$ and $g(q) \leqq f(q) + r \leqq -7\lambda$. So for all $x \in B_{\delta/4}(p) \cap \{g \geqq -\lambda\}$ we have $g(x) - g(q) \geqq 6\lambda$ and $d(x, q) \leqq \delta/2$. Using the convexity of g along the geodesic between x and q in $B_{\delta/4}(p)$, we get $\|\nabla g(x)\| \geqq \dfrac{6\lambda}{\delta/2} > 8\lambda/\delta$. In particular, the interval $[-\lambda, 0]$ contains no critical values for g which finishes the proof of (iii).

If c is an integral curve of the vector field $X=-\nabla g/\|\nabla g\|^2$ with $c(0)\in y(V(s)) \subset B_{\delta/8}(p)\cap\{g=0\}$, then $g(c(t))=-t$ and $\|c'(t)\|=\|\nabla g(c(t))\|^{-1}<\delta/8\lambda$, for $t\leqq\lambda$. So the curve $c(t)$ stays within $B_{\delta/4}(p)$ for $0\leqq t\leqq\lambda$. In particular, c is defined on $[0,\lambda]$ with $c([0,\lambda])\subset M_s$. This proves (iv).

Note that the choice of r was uniform for all $s\in S$. If $V(s)\cap V(s')\neq\emptyset$, then as above the signed distance functions of $y(U(s))$ and $y(U(s'))$ agree on $B_{\delta/3}(p)\cap B_{\delta/3}(p')$ for $p'=y(s')$. Since $r<\delta/12$, the smoothed functions f_r agree on $B_{\delta/4}(p)\cap B_{\delta/4}(p')$, hence $g_s=g_{s'}$ on $M_s\cap M_{s'}$.

7.5. Now we define an immersion $y^1: S\times[0,\lambda]\to M$ as follows: For $s\in V(s_0)$ let $y^1(s,t)=\psi_t(y(s))$ where ψ_t denotes the flow of the vector field $X=-\nabla g/\|\nabla g\|^2$ for $g=g_{s_0}$. In 7.4 we have shown that this is well defined. Let ds_t^2 be the metric on S induced by the immersion $y_t^1:=y^1|_{S\times\{t\}}$. Put $\kappa:=e^{-\varepsilon\lambda/4}$.

Lemma 7.5. $ds_\lambda^2\leqq\kappa^2 ds_0^2$.

Proof. Let $s\in V(s_0)$, $s_0\in S$. For $a\in T_sS$ put $A(t)=Dy_t^1(a)$; this is a vector field along the curve $c(t)=\psi_t(y(s))$ with derivative $A'(t)=D_{A(t)}X$. So

$$\|A\|'=\langle D_AX,A\rangle/\|A\|=-\langle D_A\nabla g,A\rangle/(\|\nabla g\|^2\,\|A\|)\leqq-\frac{\varepsilon}{4}\|A\|$$

by 7.4 (ii). Integrating, we get $\|A(\lambda)\|\leqq\kappa\|A(0)\|$ which proves the lemma.

7.6. We now may replace the given immersion y with y_λ^1. By Lemma 7.4 (iii) this is an ε_1-convex immersion of S. Since $\varepsilon_1>\varepsilon$ and $y_\lambda^1(S)\subset M_0$ (see 7.1), we may repeat the argument getting an immersion $y^2: S\times[\lambda,2\lambda]\to M$ such that the immersion $y_{2\lambda}^2=y^2|_{S\times\{2\lambda\}}$ of S is ε_2-convex for $\varepsilon_2=(R-2\lambda/4)^{-1}$ and the induced metric $ds_{2\lambda}^2$ satisfies $ds_{2\lambda}^2\leqq\kappa_1^2 ds_\lambda^2$ for $\kappa_1=e^{-\varepsilon_1\lambda/4}$ and so on. Since we proved $\|\nabla g\|\leqq 2$, any point of $y_{k\lambda}^k(S)$ has distance $\leqq 2\lambda$ from $y_{(k-1)\lambda}^{k-1}(S)$, so we do not leave M_0 before k exceeds $5R/\lambda$. On the other hand, $\varepsilon_k=(R-k\lambda/4)^{-1}$ is finite only for $k<4R/\lambda$. So after, say, m steps with $m<4R/\lambda$, the set $y_{m\lambda}^m(S)$ is contained in a ball of radius $\delta<\varrho$ in M_0 and in particular in the domain of a good coordinate system ϕ. Therefore, $x=\phi\circ y_{m\lambda}^m$ is an $\frac{\varepsilon}{16}$-convex immersion of S into euclidean n-space. By Lemma 6.6, this is an embedding and $x(S)$ bounds a convex disk (6.3). So $y_{m\lambda}^m(S)$ bounds a closed embedded disk B_{m+1} in M. Providing $B_k:=S\times[(k-1)\lambda,k\lambda]$ with the metric induced by y^k and gluing together B_k and B_{k+1} at their common boundary, for $0\leqq k\leqq m$, we get a compact Riemannian manifold D with boundary (S,ds_0^2), and an isometric immersion $\hat{y}$: $D\to M$ with $\hat{y}|_S=y$. In particular, we have nonnegative curvature on D and the boundary S is an ε-convex hypersurface.

7.7. It remains to construct a diffeomorphism of D onto the standard n-disk. Consider the ε-convex function $f=\chi_\varepsilon\circ d$ where d is the negative distance to S on D (see 4.6). Let f_r be the smoothing of f for small enough r and put $g=f$ on $\{|d|\leqq r_0\}$ and $g=f+\phi(d)(f_r-f)$ on $\{|d|\geqq r_0\}$ as in 7.4, but this time, g is defined globally on D. Thus $g\leqq 0$ with $S=g^{-1}(0)$, and g is $\frac{\varepsilon}{2}$-convex if r is small enough. By strong convexity, the set of critical points, C, contains only minima, and the domain of attraction of each minimum is a connected component of $D \backslash C$; so there is exactly one minimum $q \in \mathrm{Int}(D)$. By the Morse lemma (see [19]), for small $\gamma > 0$, the set $D_\gamma = \{x \in D; g(x) - g(q) \leqq \gamma\}$ is diffeomorphic to the standard disk. Using the flow of $X = -\nabla g / \|\nabla g\|^{-2}$, we get a diffeomorphism of D onto D_γ. This finishes the proof of Theorem A.

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References

1. Bangert, V.: Über die Approximation von lokal konvexen Mengen. Manuscr. Math. **25**, 397–420 (1978)
2. Berger, M.: Les variétés riemanniennes (1/4)-pincées. Ann. Sc. Norm. Super. Pisa, III. **14**, 161–170 (1960)
3. Cheeger, J., Ebin, D.G.: Comparison theorems in Riemannian geometry. Amsterdam: North Holland 1975
4. Cheeger, J., Gromoll, D.: On the structure of complete manifolds of nonnegative curvature. Ann. Math. **96**, 413–443 (1972)
5. Eschenburg, J.H., O'Sullivan, J.J.: Jacobi tensors and Ricci curvature. Math. Ann. **252**, 1–26 (1980)
6. Eschenburg, J.H., Heintze, E.: An elementary proof of the Cheeger-Gromoll Splitting Theorem. Ann. Glob. Analysis and Geometry **2**, 141–151 (1984)
7. Green, L.W.: A theorem of E. Hopf. Mich. Math. J. **5**, 31–34 (1958)
8. Greene, R.E., Wu, H.: On the subharmonicity and plurisubharmonicity of geodesically convex functions. Indiana Univ. Math. J. **22**, 641–653 (1973)
9. Greene, R.E., Wu, H.: C^∞ convex functions and manifolds of positive curvature. Acta Math. **137**, 209–245 (1976)
10. Gromoll, D., Klingenberg, W., Meyer, W.: Riemannsche Geometrie im Großen. Lect. Notes Math. **55** (1968)
11. Gromoll, D., Meyer, W.: On complete open manifolds of positive curvature. Ann. Math. **90**, 75–90 (1969)
12. Grove, K., Shiohama, K.: A generalized sphere theorem. Ann. Math. **106**, 201–211 (1977)
13. Guilleman, V., Pollack, A.: Differential Topology. Englewood Cliffs: Prentice Hall 1974
14. Hadamard, J.: Sur certaines propriétés des trajectories en dynamique. J. Math. Pures Appl. (5) **3**, 331–387 (1897)
15. Hopf, H.: Differential Geometry in the Large, Ch. IV: Hadamard's characterization of the ovaloids. Stanford Lectures 1956. Lect. Notes Math. **1000**, 1983
16. Jost, J., Karcher, H.: Geometrische Methoden zur Gewinnung von a-priori-Schranken für harmonische Abbildungen. Manuscr. Math. **40**, 27–77 (1982)
17. Klingenberg, W.: Über Riemannsche Mannigfaltigkeiten mit positiver Krümmung. Commun. Math. Helv. **35**, 47–54 (1961)
18. Kobayashi, S., Wu, H.: Complex differential geometry. DMV-Seminar Bd. 3. Basel: Birkhäuser 1983
19. Milnor, J.: Morse theory. Ann. Math. Stud. **51**, Princeton, N.J. 1963
20. Wu, H.: An elementary method in the study of nonnegative curvature. Acta Math. **142**, 57–78 (1979)

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