Stability of Hypersurfaces of Constant Mean Curvature in Riemannian Manifolds

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1. Introduction

(1.1) Hypersurfaces M^n with constant mean curvature in a Riemannian manifold $\bar{M}^{n+1}$ display many similarities with minimal hypersurfaces of $\bar{M}^{n+1}$. They are both solutions to the variational problem of minimizing the area function for certain variations. In the first case, however, the admissible variations are only those that leave a certain volume function fixed (for precise definitions, see Sect. 2). This isoperimetric character of the variational problem associated to hypersurfaces of constant mean curvature introduces additional complications in the treatment of stability of such hypersurfaces.

In [BdC] a definition of stability for hypersurfaces of constant mean curvature in the euclidean space R^{n+1} was given, and it was proved that the round spheres are the only compact hypersurfaces with constant mean curvature in R^{n+1} that are stable. This is interesting in view of the fact that there exist compact nonspherical hypersurfaces with constant mean curvature in R^{n+1} (cf. Hsiang et al. [HTY] for $n>2$, Wente [W] and Abresch [A], for $n=2$).

In the present paper we extend these investigations to hypersurfaces of Riemannian manifolds and prove the following result.

Let $\bar{M}^{n+1}(c)$ be a simply-connected complete Riemannian manifold with constant sectional curvature c and let $x\colon M^n \to \bar{M}^{n+1}(c)$ be an immersion of a differentiable manifold M^n (superscripts will, in general, denote dimensions). Recall that a (geodesic) sphere of a Riemannian manifold $\bar{M}$ is the set of points of $\bar{M}$ at a fixed distance (the radius of the geodesic sphere) from a given point $p \in \bar{M}$.

(1.2) **Theorem.** *Assume that M^n is compact without boundary and that $x\colon M^n \to \bar{M}^{n+1}(c)$ has constant mean curvature. Then x is stable* (see definition in Sect. 2) *if and only if $x(M^n) \subset \bar{M}^{n+1}(c)$ is a geodesic sphere.*

The case where M^n, $n=2$, is complete and noncompact has been treated in the recent thesis at IMPA of A.M. da Silveira [S]. It turns out *that when $x\colon M^2 \to \bar{M}^3(c)$ has constant mean curvature H, M^2 is complete and noncompact,* *and x is stable*: a) *if* $c=0$, *then* $x(M)\subset R^3$ *is a plane*; b) *if* $c=-1$ *and* $H\geqq 1$, *then* $x(M)\subset H^3(-1)$ *is a horosphere*; c) *if* $c=-1$ *and* $0\leqq H<1$, *then there are many such examples.*

Except for the case where $H\equiv 0$ very little is known about stability of complete and noncompact hypersurfaces of $\bar{M}^{n+1}(c)$ with constant mean curvature, when $n>2$.

For a general Riemannian manifold, the situation seems too complicated. An interesting special example is given by the symmetric spaces of rank one. It is known that the group of isometries of such spaces acts transitively on their geodesic spheres which have, therefore, constant mean curvature. However, not all spheres are stable, and not all stable hypersurfaces are spheres. It turns out that certain tubes around projective subspaces are also stable. Let $\bar{M}=P^{r-1}\mathbb{K}$ be the projective space over the field $\mathbb{K}$, with metric of diameter $\pi/2$ and curvature between 1 and 4, where $\mathbb{K}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$. For $q<r$ let $U_\rho(P^{q-1}\mathbb{K})$ denote the tubular neighborhood of radius ρ around the totally geodesic subspace $P^{q-1}\mathbb{K}$ of $P^{r-1}\mathbb{K}$, and put $T_\rho(q)=\partial U_\rho(P^{q-1}\mathbb{K})$. Note that $T_\rho(q)$ is congruent to $T_{\pi/2-\rho}(p)$ if $p=r-q$, and that $T_\rho(1)$ is the geodesic sphere of radius ρ. Set $d=\dim_{\mathbb{R}}\mathbb{K}$ and, for reasons of orientability, assume that r is even if $\mathbb{K}=\mathbb{R}$.

(1.3) **Theorem.** *For* $2\leqq q\leqq r-2$, $T_\rho(q)$ *is stable in* $P^{r-1}\mathbb{K}$ *if and only if*

$$\frac{pd-1}{qd+1}\leqq \tan^2\rho\leqq\frac{pd+1}{qd-1}.$$

For $q=1$ $(q=r-1)$, *the lower (upper) bound is not present: A sphere of radius* ρ *is stable if and only if* $\tan^2\rho\leqq((r-1)d+1)/(d-1)$.

We will prove the theorem in the context of group-invariant stability (see Sect. 4).

On the other hand for the complex hyperbolic space $H^n\mathbb{C}$, the noncompact dual of $P^n\mathbb{C}$, we will prove:

(1.4) **Theorem.** *Every geodesic sphere in* $H^n\mathbb{C}$ *is stable.*

The stability problem is closely related to eigenvalue estimates of the Laplacian of a submanifold. This will be further developed in an forthcoming paper of E. Heintze [H].

The paper is organized as follows. In Sect. 2 we fix our notation and extend the basic definitions of [BdC] to the case where the ambient space is a Riemannian manifold. Most of the proofs in [BdC] apply to this new situation and we are very sketchy as far as proofs are concerned. For future reference, however, we have stated all facts in full. In Sect. 3 we prove Theorem (1.2). After giving the equivariant setup in Sect. 4, we prove Theorem (1.3) in Sect. 5 and Theorem (1.4) in Sect. 6. In an appendix, we compare the areas of the various stable hypersurfaces of $P^{r-1}\mathbb{K}$ which we obtained in Theorem (1.3).

2. Preliminaries

2.1. Let $\bar{M}^{n+1}$ be an oriented Riemannian manifold and let $x\colon M^n \to \bar{M}^{n+1}$ be an immersion of a compact, connected, orientable differentiable manifold with boundary ∂M (possibly $=\phi$) into $\bar{M}^{n+1}$. We choose the orientation of M to be compatible with the orientation of $\bar{M}$. More explicitly, let $e_1, \ldots, e_n, e_{n+1}$ be an orthonormal moving frame in a neighborhood $U \subset \bar{M}$ of $x(p)$, $p \in M$, that is adapted (i.e., $e_1, \ldots, e_n$ are tangent to $x(M)$) and positive (i.e., $d\bar{M}(e_1, \ldots, e_{n+1}) > 0$, where $d\bar{M}$ is the volume form of $\bar{M}$). Since M is orientable, $e_{n+1} = N$ is a globally defined unit normal vector field and we choose it to be the *orientation* of M.

A *variation* of x is a differentiable map $X\colon (-\varepsilon, \varepsilon) \times M \to \bar{M}$ such that $X_t\colon M \to \bar{M}$, $t \in (-\varepsilon, \varepsilon)$, defined by $X_t(p) = X(t, p)$, $p \in M$, is an immersion, $X_0 = x$, and $X_t | \partial M = x | \partial M$, for all t. We define the *area function* $A\colon (-\varepsilon, \varepsilon) \to \mathbb{R}$ by

$$A(t) = \int_M dM_t,$$

where dM_t is the volume element of M in the metric induced by X_t, and the *volume function* $V\colon (-\varepsilon, \varepsilon) \to \mathbb{R}$ by

$$V(t) = \int_{[0,t] \times M} X^* d\bar{M}.$$

Let $W(p) = \left.\dfrac{\partial X}{\partial t}\right|_{t=0}$ be the variation vector field of X and set $f = \langle W, N \rangle$.

(2.1) **Lemma**

(i) $\dfrac{dA}{dt}(0) = -\displaystyle\int_M nHf\, dM,$

(ii) $\dfrac{dV}{dt}(0) = \displaystyle\int_M f\, dM$

where H is the mean curvature of the immersion x.

Proof. (i) is well known. To prove (ii), fix a point $p \in M$ and choose a positive adapted orthonormal frame $e_1, \ldots, e_n, e_{n+1} = N$ around $x(p)$. Then

$$X^*(d\bar{M}) = a(t, p)\, dt \wedge dM,$$

where

$$a(t, p) = X^*(d\bar{M})\left(\frac{\partial}{\partial t}, e_1, \ldots, e_n\right) = d\bar{M}\left(\frac{\partial X}{\partial t}, dX_t(e_1), \ldots, dX_t(e_n)\right)$$

$$= \mathrm{vol}\left(\frac{\partial X}{\partial t}, dX_t(e_1), \ldots, dX_t(e_n)\right) = \left\langle \frac{\partial X}{\partial t}, N_t \right\rangle,$$

and N_t is a unit normal vector of the immersion X_t. It follows that

$$\frac{dV}{dt}(0)=\frac{d}{dt}\Big(\int_{[0,t]\times M} a(t,p)\,dt\wedge dM\Big)_{t=0}=\int_M a(0,p)\,dM$$

$$=\int_M \left\langle \frac{\partial X}{\partial t}(0), N\right\rangle dM=\int_M f\,dM,$$

as we wished. q.e.d.

A variation is *normal* if W is parallel to N, and *volume-preserving* if $V(t)=V(0)$, for all t.

(2.2) **Lemma.** *Given a smooth function $f\colon M\to R$ with $f\,|\,\partial M=0$ and $\int_M f\,dM=0$, there exists a volume-preserving normal variation whose variation vector is fN.*

Proof. Let $\varphi\colon(-\varepsilon,\varepsilon)\times M\to\mathbb{R}$ be a differentiable function and define a variation $X\colon(-\varepsilon,\varepsilon)\times M\to\bar{M}$ by

$$X(t,p)=\exp_{x(p)}\varphi(t,p)N,\qquad t\in(-\varepsilon,\varepsilon),\ p\in M.$$

X is a normal variation with $(\partial X/\partial t)_0=(\partial\varphi/\partial t)_0 N$. We want to show that φ can be so chosen that X satisfies the conditions of the lemma.

For that, we compute the volume function $V(t)$ of X. Notice that $X=e\circ\psi$, where $\psi=(-\varepsilon,\varepsilon)\times M\to\mathbb{R}\times M$ is the map $\psi(t,p)=(\varphi(t,p),p)$ and $e(u,p)=\exp_{x(p)}uN$, $u\in\mathbb{R}$. By setting $E(u,p)=\det(de_{(p,u)})$, we obtain

$$V(t)=\int_{[0,t]\times M} X^*d\bar{M}=\int_{[0,t]\times M}\psi^*e^*d\bar{M}$$

$$=\int_{[0,t]\times M} E(\varphi(t,p),p)\frac{\partial\varphi}{\partial t}(dM\wedge dt)=\int_M\left(\int_0^t E(\varphi(t,p),p)\frac{\partial\varphi}{\partial t}\,dt\right)dM.$$

Now let $f\colon M\to\mathbb{R}$ be as in the statement of the lemma. and let φ be the solution of the initial value problem:

$$\frac{\partial\varphi}{\partial t}=\frac{f(p)}{E(\varphi(t),p)},\qquad \varphi(0,p)=0.$$

From the above expression for $V(t)$, and the fact that $\int_M f\,dM=0$, it follows that $V(t)\equiv 0$ for such a variation. Since $E(\varphi(0),p)=1$, this is a normal, volume-preserving variation whose variation vector is fN. q.e.d.

For a given variation X of an immersion $x\colon M^n\to\bar{M}^{n+1}$ we set

$$H_0=A^{-1}\int_M H\,dM,\qquad A=A(0),$$

and define $J\colon(-\varepsilon,\varepsilon)\to\mathbb{R}$ by $J(t)=A(t)+nH_0V(t)$.

(2.3) **Proposition.** *Let $x\colon M^n\to\bar{M}^{n+1}$ be an immersion. The following statements are equivalent:*

(i) *X has constant mean curvature H_0.*
(ii) *For all volume-preserving variations, $A'(0)=0$.*
(iii) *For all (arbitrary) variations, $J'(0)=0$.*

The proof is essentially the same as in Proposition (2.7) of [BdC].

(2.4) *Remark.* Notice that we do not have to assume $H_0 \neq 0$ (this corrects a mistake in [BdC]). Thus minimal hypersurfaces are also included among the critical points of area for volume-preserving variations.

To compute the second variation of J, we observe that

$$\frac{dJ}{dt} = \int_M (-nH_t + nH_0) f_t \, dM.$$

Here H_t is the mean curvature of X_t, and $f_t = \left\langle \frac{\partial X}{\partial t}, N_t \right\rangle$, where N_t is the unit normal vector of X_t. Thus

$$J''(0) = -\int_M \left(\frac{\partial H_t}{\partial t}\right)(0) f \, dM,$$

and the computation is the same as the computation of the second variation for the area function. It turns out that

(2.5) **Proposition.** *Let $x\colon M^n \to \bar{M}^{n+1}$ be an immersion with constant mean curvature H and let X be a variation of x. Then $J''(0)$ depends only on f and is given by*

$$J''(0)(f) = \int_M (-f\Delta f - (\bar{R} + \|B\|^2) f^2) \, dM.$$

Here Δ is the Laplacian in the induced metric, $\|B\|$ is the norm of the second fundamental form of x, and $\bar{R} = n\,\overline{\mathrm{Ricc}}(N)$, where $\overline{\mathrm{Ricc}}(N)$ is the (normalized) Ricci curvature of $\bar{M}$ in the direction N.

(2.6) **Definition.** Let $x\colon M^n \to \bar{M}^{n+1}$ have constant mean curvature. The immersion x is *stable* if $A''(0) \geqq 0$ for all volume-preserving variations of x. If M is noncompact, we say that x is stable if for every submanifold with boundary $\tilde{M} \subset M$, the restriction $x|\tilde{M}$ is stable.

Just as in [BdC], one can prove, using (2.2) and (2.5), the following criterion for stability. Let $\mathscr{F}$ be the set of differentiable functions $f\colon M \to \mathbb{R}$ with $f|\partial D = 0$ and $\int_M f \, dM = 0$.

(2.7) **Proposition.** $x\colon M^n \to \bar{M}^{n+1}$ *is stable if and only if $J''(0)(f) \geqq 0$ for all $f \in \mathscr{F}$.*

Similarly to the case of minimal surfaces, we have a notion of Jacobi field for hypersurfaces with constant mean curvature. With a view towards a Morse index theorem for such a situation, we choose the following definition.

Define a bilinear form $I\colon \mathscr{F} \to R$ by

$$I(f,g)=\int_M g(-\Delta f-(\bar{R}+\|B\|^2)f)\,dM.$$

(2.8) **Definition.** A normal vector field $V=fN$, $f\in\mathscr{F}$, to an immersion $x\colon M^n \to \bar{M}^{n+1}$ with constant mean curvature is a *Jacobi field* if $f\in\operatorname{Ker} I$, i.e., if $I(f,g)=0$, for all $g\in\mathscr{F}$.

(2.9) **Proposition.** *Let $f\in\mathscr{F}$. Then fN is a Jacobi field if and only if*

$$\Delta f+(\bar{R}+\|B\|^2)f=\text{const.} \tag{2.10}$$

Proof. Clearly if (2.10) holds, $f\in\operatorname{Ker} I$, since $g\in\mathscr{F}$. To show the converse, let F_0 be the mean value of $F=\Delta f+(\bar{R}+\|B\|^2)f$ in M. Since $f\in\operatorname{Ker} I$,

$$\int_M g(F-F_0)\,dM=0, \qquad \text{for all } g\in\mathscr{F}.$$

We want to show that $F\equiv F_0$ and the argument now is entirely similar to that of Proposition (2.7) in [BdC]. q.e.d.

(2.11) *Remark.* By using the above Jacobi fields a Morse index theorem can be proved for hypersurfaces $x\colon M^n\to\bar{M}^{n+1}$ with constant mean curvature. The statement is entirely analogues to the case of minimal surfaces (cf. Lawson [L], pp. 51–53). This was incorrectly stated in [BdC] where only those Jacobi fields were considered that satisfied (2.10) with zero in the right hand side.

A particular kind of Jacobi fields is obtained from Killing vector fields on $\bar{M}$:

(2.12) **Proposition.** *Let W be a Killing vector field on $\bar{M}$. Then $f:=\langle W,N\rangle$ satisfies*

$$\Delta f+(\bar{R}+\|B\|^2)f=0.$$

Proof. This is a straightforward computation and we shall omit it.

If M^n is without boundary, a hypersurface immersion $x\colon M\to\bar{M}$ is called *bounding* if x extends to an immersion of some $(n+1)$-manifold with boundary M into $\bar{M}$. In this case we have $\int_M f\,dM=0$ by the divergence theorem (note that $\operatorname{div} W=0$).

If, in addition, M is a hypersurface with constant mean curvature such that $\bar{R}+\|B\|^2=\lambda=\text{const.}$, then λ is an eigenvalue of the Laplacian of M, provided that there is a Killing field on $\bar{M}$ which is not everywhere tangent to M. In this case we have a nice criterion for stability:

(2.13) **Proposition.** *Let $x\colon M\to\bar{M}$ be a bounding hypersurface with constant mean curvature such that $\bar{R}+\|B\|^2=\lambda=$const. M is stable if and only if $\lambda=\lambda_1$, the first eigenvalue of the Laplacian on M.*

Proof. Since λ is an eigenvalue of Δ, we have either $\lambda=\lambda_1$ or $\lambda>\lambda_1$. In the first case, for any $f\in\mathscr{F}$,

$$I(f,f)=\int_M(-f\Delta f-\lambda f^2)\,dM\geqq(\lambda_1-\lambda)\int_M f^2\,dM=0,$$

hence M is stable. In the latter case, choose f to be a first eigenfunction of the Laplacian. Then $f\in\mathscr{F}$ and

$$I(f,f)=(\lambda_1-\lambda)\int_M f^2\,dM<0$$

and therefore M is not stable.

3. Proof of Theorem (1.2)

(3.1) Let $\bar{M}^{n+1}(c)$ be as in the statement of Theorem (1.2).

When $c=0$, $\bar{M}^{n+1}(c)$ is the euclidean space R^{n+1} and the theorem has already been proved in [BdC]. When $c\neq 0$, we will need the following model of $\bar{M}^{n+1}(c)$.

Let L^{n+2} be the euclidean space R^{n+2} with canonical basis $a_A=(0,\ldots,0,1,0,\ldots,0)$, $A=0,1,\ldots,n+1$, and inner product $\langle\,,\rangle$ given by:

$$\langle a_0,a_0\rangle=\frac{c}{|c|}\qquad\langle a_\alpha,a_\beta\rangle=\delta_{\alpha\beta},\ \alpha,\beta=1,\ldots,n+1,\qquad\langle a_0,a_\beta\rangle=0.$$

Let $S^{n+1}(c)\subset L^{n+2}$ be a connected component of

$$\{y\in L^{n+2};\langle y,y\rangle=1/c\}.$$

It is well known that $S^{n+1}(c)$ with the induced metric is isometric to $\bar{M}^{n+1}(c)$, $c\neq 0$, and this is the model we are going to use in the proof of Theorem (1.2).

Now let $x\colon M^n\to S^{n+1}(c)\subset L^{n+2}$ be an immersion with mean curvature H. Let N be a unit normal vector field along x that defines the orientation of M and fix a vector $v\in L^{n+2}$. Define functions $g\colon M\to\mathbb{R}$ and $f\colon M\to\mathbb{R}$ by

$$g(p)=\langle x(p),v\rangle,\qquad f(p)=\langle N(p),v\rangle,\qquad p\in M.\tag{3.2}$$

(3.3) **Lemma.** *Let Δ denote the Laplacian of M in the metric induced by x. Then*

(a) $\Delta g=-nHf-cng$.

If, in addition, $H=$const.,

(b) $\Delta f=-\|B\|^2f-cnHg$.

Proof. The proof is a simple computation and we will omit it.

(3.4) **Lemma.** *The geodesic spheres $S^n\subset\bar{M}^{n+1}(c)$ are stable.*

Proof. It is well known that geodesic spheres are umbilic hypersurfaces of $\bar{M}^{n+1}(c)$. Thus $\|B\|^2=nH^2$. Furthermore $\mathrm{Ricc}(N)=c$, so

$$\lambda:=\bar{R}+\|B\|^2=n(c+H^2).$$

On the other hand, since S^n has constant curvature $c+H^2$, the first eigenvalue λ_1 of the Laplacian on the sphere S^n is given by $\lambda_1 = n(c+H^2) = \lambda$. The result now follows from (2.13).

(3.5) *Remark.* If $c>0$, the equator $S^n(c) \subset S^{n+1}(c)$ is a minimal hypersurface of $S^{n+1}(c)$ and, as such, is not stable. However as a hypersurface of constant mean curvature, i.e., considering only those variations that leave fixed the volume, $S^n(c)$ is stable.

(3.6) *Proof of Theorem* (1.2). We will use the fact that $\|B\|^2 \geqq nH^2$, and equality holds at a point $p \in M$ if and only if p is umbilic. Since the only compact umbilic hypersurfaces of $\bar{M}^{n+1}(c)$ are the geodesic spheres, the proof will be complete once we show that stability of $x\colon M^n \to \bar{M}^{n+1}(c)$ implies that $\|B\|^2 = nH^2$.

To do that, we first observe that the function $u = Hf + cg$ satisfies, by Lemma (3.2)(a), the condition

$$\int_M u\,dM = \int_M Hf + cg = -\frac{1}{n}\int_M \Delta g = 0,$$

since M is compact without boundary. A straightforward computation shows that

$$\begin{aligned} J''(0)(u) &= -\int_M (u\Delta u + (\|B\|^2 + nc)u^2)\,dM \\ &= -\int_M (\|B\|^2 - nH^2)(c^2 g^2 + cHfg)\,dM. \end{aligned}$$

For notational convenience, let us write $J''(0)(u) = I(u)$; notice that u depends on the fixed vector $v \in L^{n+2}$. It is somewhat surprising that only the variations given by u suffice for the proof. In fact, we will prove that $I(u) \geqq 0$ implies that $\|B\|^2 = nH^2$, thereby concluding the proof.

We will treat separately the cases $c>0$ and $c<0$. There will be no loss of generality if we assume $c=1$ in the first case and $c=-1$ in the second case.

First Case. $c=1$. Choose v as an element of a canonical orthonormal basis $a_0, a_1, \ldots, a_{n+1}$ of L^{n+2} (in this case, $L^{n+2} = R^{n+2}$ with the standard metric), and let f_A and g_A be the functions in (3.2) that correspond to $v = a_A$, $A = 0, 1, \ldots, n+1$. Set $u_A = Hf_A + cg_A$. Since x is stable,

$$I(u_A) = -\int_M (\|B\|^2 - nH^2)(g_A^2 + Hf_A g_A)\,dM \geqq 0.$$

Thus

$$0 \geqq \sum I(u_A) = -\int_M (\|B\|^2 - nH^2)\left(\sum g_A^2 - H\sum f_A g_A\right) dM.$$

Since $x(M)$ is contained in a unit sphere of R^{n+2}, we obtain

$$\begin{aligned} \sum g_A^2 &= \sum \langle x, a_A\rangle \langle x, a_A\rangle = \langle x, x\rangle = 1, \\ \sum f_A g_A &= \sum \langle N, a_A\rangle \langle x, a_A\rangle = \langle N, x\rangle = 0. \end{aligned}$$

It follows that

$$0 \leqq -\int_M (\|B\|^2 - nH^2)\, dM,$$

and since $\|B\|^2 \geqq nH^2$, we obtain that $\|B\|^2 = nH^2$.

Second Case. $c = -1$. Here we first notice that

$$\begin{aligned}
&\tfrac{1}{2}\Delta g^2 = g\Delta g + |\nabla g|^2 = -nHfg + ng^2 + |\nabla g|^2,\\
&\tfrac{1}{2}\Delta f^2 = f\Delta f + |\nabla f|^2 = -\|B\|^2 f^2 + nHgf + |\nabla f|^2,\\
&\Delta(fg) = -\|B\|^2 gf + nHg^2 - nHf^2 + ngf + 2\langle \nabla f, \nabla g\rangle.
\end{aligned}$$

A straightforward computation shows that

$$\tfrac{1}{2}H^2\Delta g^2 - H\Delta(fg) + \tfrac{1}{2}\Delta f^2 = (H|\nabla g| - |\nabla f|)^2 - (\|B\|^2 - nH^2)(f^2 - Hfg).$$

By integrating the above expression over M, we obtain

$$\int_M (\|B\|^2 - nH^2)(Hfg - f^2)\, dM = -\int_M (H|\nabla g| - |\nabla f|)^2\, dM.$$

It follows that

$$\begin{aligned}
I(u) &= \int_M (\|B\|^2 - nH^2)(Hfg - g^2)\, dM\\
&= \int_M (\|B\|^2 - nH^2)(f^2 - g^2)\, dM - \int_M (H|\nabla g| - |\nabla f|)^2\, dM.
\end{aligned}$$

So far, we have made no choice of the vector $v \in L^{n+2}$. Now choose v so that $\langle v, v\rangle = -1$. To compute ∇g, choose a moving frame $\{e_A\}$ around a point $x(p)$, $p \in M$, such that $e_0 = x$, $e_{n+1} = N$ and $e_1, \ldots, e_n$ are orthonormal and tangent to $x(M)$. Thus

$$v = -\langle v, x\rangle x + \langle v, N\rangle N + \sum_i \langle v, e_i\rangle e_i, \qquad i = 1, \ldots, n$$

and

$$\nabla g = \sum \langle v, e_i\rangle e_i.$$

It follows that

$$\begin{aligned}
-1 = \langle v, v\rangle &= -\langle v, x\rangle^2 + \langle v, N\rangle^2 + \sum_i \langle v, e_i\rangle^2\\
&= -g^2 + f^2 + |\nabla g|^2.
\end{aligned}$$

Therefore $f^2 - g^2 = -(1 + |\nabla g|^2)$, hence,

$$0 \leqq I(u) = -\int_M \{(\|B\|^2 - nH^2)(1 + |\nabla g|^2) + (H|\nabla g| - |\nabla f|)^2\}\, dM.$$

The above implies that $\|B\|^2 = nH^2$. This concludes the proof of Case 2 and of the theorem.

4. G-stability

(4.1) Let $\bar{M}^{n+1}$ be an oriented Riemannian manifold and let G a compact group of orientation-preserving isometries on $\bar{M}$. An (immersed) orientable G-invariant hypersurface M^n in $\bar{M}^{n+1}$ is called *G-stable* if M has constant mean curvature and $I(f,f) \geqq 0$ for all $f \in \mathscr{F}^G := \{f \in \mathscr{F}; f(g(p)) = f(p) - g \in G, p \in M\}$ (see Sect. 2 for notation). There is an obvious G-invariant version of Proposition 2.13: If M is a G-invariant bounding immersed hypersurface of constant mean curvature with $\bar{R} + \|B\|^2 = \lambda = \text{const.}$ and if there is a G-invariant Killing field on $\bar{M}$, not everywhere tangent to M, then M is G-stable if and only if λ is the first G-invariant eigenvalue of the Laplacian on M, i.e. the first nonzero eigenvalue which belongs to a G-invariant eigenfunction.

(4.2) If G acts freely, the orbit space $\bar{M}_0 := \bar{M}/G$ is a manifold and the metric on $\bar{M}$ induces a Riemannian metric on $\bar{M}_0$ such that the projection $\pi: \bar{M} \to \bar{M}_0$ is a Riemannian submersion. If M is a G-invariant (immersed) orientable hypersurface in $\bar{M}$, then M projects down to a hypersurface $M_0 = M/G$ (also immersed) in $\bar{M}_0$. Assume that M_0 is orientable and let N_0 be a unit normal vector field on M_0. Then the horizontal lift N of N_0 is a unit normal vector field on M. Denote the corresponding second fundamental tensors by $B := DN$ and $B_0 := DN_0$. By O'Neill's formulas [O'N] we have for all tangent vectors $u, v \in T_p M$ which are horizontal (i.e. orthogonal to the orbit Gp)

$$(*) \qquad \langle B(u), v \rangle = \langle B_0(\pi_* u), \pi_* v \rangle.$$

(4.3) **Proposition.** *Let G act freely on $\bar{M}$ with totally geodesic orbits and let M be an orientable G-invariant hypersurface in $\bar{M}$ such that M/G is orientable. Then M is G-stable if and only if $M_0 = M/G$ is stable in $\bar{M}_0 = \bar{M}/G$.*

Proof. Since all orbits are totally geodesic, they have all the same volume, say c. (A smooth family of minimal submanifolds has constant volume since the volume function has zero derivative everywhere.) Let $x: M \to \bar{M}$ be the given G-equivariant immersion and $x_0: M_0 \to \bar{M}_0$ the induced immersion. Any G-equivariant variation $X: (-\varepsilon, \varepsilon) \times M \to \bar{M}$ of x induces a variation $X_0: (-\varepsilon, \varepsilon) \times M_0 \to \bar{M}_0$ of x_0, and Fubini's theorem for Riemannian submersions yields for the area and volume functions defined in Sect. 2

$$A(t) = c A_0(t), \qquad V(t) = c V_0(t).$$

Moreover, from $(*)$ we derive

$$\text{trace } B = \text{trace } B_0,$$

since the orbits are totally geodesic and do not contribute to trace B. It follows that $J(t) = c J_0(t)$ and thus $J''(0) = c J_0''(0)$. This completes the proof.

Remark. We can also show directly that the index forms of M and M_0 are equal up to the factor c. In fact, if f_0 is a smooth function on M_0 and $f = f_0 \circ \pi$, then $\Delta f = \Delta f_0 \circ \pi$ since the fibres are minimal; in particular, the first eigenvalue of M_0 is the first G-invariant eigenvalue of M. Moreover, the O'Neill formulas

[O'N] show that $\bar{R}+\|B\|^2=(\bar{R}_0+\|B_0\|^2)\circ\pi$. The easiest way to see this is as follows: Extend the normal field N of M locally to a vector field near M with $D_N N=0$. Then the tensor field $B:=DN$ satisfies the Riccati equation

$$D_N B+B^2+R(\cdot,N)N=0.$$

Taking the trace, we obtain $\|B\|^2+\bar{R}=-N(\text{trace } B)$, and a similar formula holds for B_0. As earlier, we have $\text{trace } B=(\text{trace } B_0)\circ\pi$, hence $N(\text{trace } B)=N_0(\text{trace } B_0)\circ\pi$.

5. Proof of Theorem (1.3)

(5.1) The projective spaces $P^{r-1}\mathbb{K}$, $\mathbb{K}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$ can be treated by the methods of Sect. 4, since they are all quotients of euclidean spheres by groups acting isometrically with totally geodesic orbits. Let S_R^N denote the euclidean N-sphere of radius R. Let $d=\dim_{\mathbb{R}}\mathbb{K}$ and put $n=rd-2$. The group $S_1^{d-1}\subset\mathbb{K}^*$ acts freely on $S_1^{n+1}\subset\mathbb{R}^{n+2}=\mathbb{K}^r$ by left scalar multiplication and $P^{r-1}\mathbb{K}$ is the orbit space. We have thus a Riemannian submersion

$$\pi\colon\ S_1^{n+1}\to P^{r-1}\mathbb{K},$$

the Hopf fibration. Let $U_\rho(\cdot)$ denote the tubular neighborhood of radius ρ. Then

$$\pi^{-1}(T_\rho(q))=\pi^{-1}(\partial U_\rho(P^{q-1}\mathbb{K})=\partial U_\rho(\pi^{-1}(P^{q-1}\mathbb{K}))=\partial U_\rho(S_1^m)$$

where $m=qd-1$. Putting $p=r-q$ and $k=pd-1=n-m$, we get

$$\partial U_\rho(S_1^m)=S_c^m\times S_s^k\subset(\mathbb{R}^{m+1}\times\mathbb{R}^{k+1})\cap S_1^{n+1},$$

where we use the abbreviations

$$c=\cos\rho,\qquad s=\sin\rho.$$

Let us put $G=S_1^{d-1}$, $M=S_c^m\times S_s^k$, $\bar{M}=S_1^{n+1}$ and $M_0=T_\rho(q)$, $\bar{M}_0=P^{r-1}\mathbb{K}$. Then we have $M_0=M/G$, and by Proposition 4.3, stability of M_0 in $\bar{M}_0$ is equivalent to G-stability of M in $\bar{M}$.

(5.2) We may apply the G-invariant version of Proposition 2.12 (see (4.1)): M is a homogeneous bounding hypersurface and there are 1-parameter-groups of isometries of $\bar{M}$ which commute with G and do not leave M invariant. Thus we have to check whether $\lambda:=\bar{R}+\|B\|^2$ is the first G-invariant eigenvalue of M. The 2$^{\text{nd}}$ fundamental tensor B of M has constant eigenvalues $-s/c$ of multiplicity m and c/s of multiplicity k, and the Ricci curvature of $\bar{M}=S_1^{n+1}$ is $\bar{R}=n=m+k$, so $\lambda=m(1+s^2/c^2)+k(1+c^2/s^2)=m/c^2+k/s^2$.

(5.3) Recall that the eigenvalues of the Laplacian of S_R^N are $\lambda_0=0$, $\lambda_1=N/R^2$, $\lambda_2=2(N+1)/R^2, \ldots, \lambda_l=l(N+l-1)/R^2, \ldots$ and the eigenfunctions corresponding to λ_l are the restrictions to S_R^N of the homogeneous harmonic polynomials of degree l in $\mathbb{R}^{N+1}$ [BGM].

If M_1, M_2 are compact Riemannian manifolds with eigenfunctions f_i, g_j corresponding to eigenvalues λ_i on M_1, μ_j on M_2, then the functions $f_i \otimes g_j$ on $M_1 \times M_2$ form a basis of eigenfunctions of the Riemannian product $M := M_1 \times M_2$, and the corresponding eigenvalues are $\lambda_i + \mu_j$.

Put $M_1 = S_c^m$, $M_2 = S_s^k$. The eigenfunctions corresponding to $\lambda_1 + \mu_0$ (resp. $\lambda_0 + \mu_1$) are linear functions on $\mathbb{R}^{m+1}$ ($\mathbb{R}^{k+1}$) which are never G-invariant since $-\mathrm{id} \in G$. The eigenvalue $\lambda_1 + \mu_1 = m/c^2 + k/s^2 = \lambda$ has an invariant eigenfunction: Let $w_1, \ldots, w_q, w_{q+1}, \ldots, w_r \colon \mathbb{K}^r \to \mathbb{K}$ denote the coordinate functions on $\mathbb{K}^r$. Then $f := \mathrm{Re}(\bar{w}_1 w_{q+1})$ is a G-invariant eigenfunction corresponding to λ (where $w \to \bar{w}$ denotes the conjugation in $\mathbb{K}$). The eigenvalues $\lambda_2 + \mu_0 = 2(m+1)/c^2$ (resp. $\lambda_0 + \mu_2 = 2(k+1)/s^2$) have the invariant eigenfunctions $\mathrm{Re}(\bar{w}_1 w_2)$ (resp. $\mathrm{Re}(\bar{w}_{q+1} w_{q+2})$) provided that $q \geqq 2$ (resp. $p \geqq 2$). If $q = 1$ then $m = d-1$ and $G = S_1^{d-1}$ acts transitively on S_c^m, hence S_c^m has no G-invariant functions other than constants. So, in this case, all eigenvalues $\lambda_l + \mu_0$, $l \geqq 1$ are noninvariant, and a similar argument holds for $\lambda_0 + \mu_l$ if $p = 1$. Thus we get as a result that λ is the smallest G-invariant eigenvalue if and only if

(a) $\lambda \leqq 2(m+1)/c^2$, i.e. $s^2/c^2 \geqq k/(m+2)$

and

(b) $\lambda \leqq 2(k+1)/s^2$, i.e. $s^2/c^2 \leqq (k+2)/m$

provided that $2 \leqq q \leqq r-2$, and in the case $q = 1$ ($q = r-1$) Inequality (a) (Inequality (b)) is not necessary. This proves Theorem (1.3).

6. Stability of Spheres in $H^n\mathbb{C}$: Proof of Theorem (1.4)

(6.1) On $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$ consider the indefinite scalar product

$$\langle x, y \rangle = \mathrm{Re}\left(-x_0 \bar{y}_0 + \sum_{i=1}^{n} x_i \bar{y}_i\right).$$

The submanifold

$$P = \{x \in \mathbb{C}^{n+1}; \langle x, x \rangle = -1\}$$

inherits a Lorentzian metric from $\mathbb{C}^{n+1}$ (of type $(-1, 1, \ldots, 1)$). The group $G = S_1^1 \subset \mathbb{C}^*$ acts on P isometrically by scalar multiplication, and the quotient manifold $P/G = H^n\mathbb{C}$ is Riemannian, namely the complex hyperbolic space with curvature between -4 and -1, by O'Neill's formulas [O'N], since the projection $\pi \colon P \to H^n\mathbb{C}$ is a (pseudo-) Riemannian submersion.

Let $B_\rho(0)$ be the ball of radius ρ around $o := [1, 0, \ldots, 0] \in P/G = H^n\mathbb{C}$. Then

$$\pi^{-1}(B_\rho(o)) = U_\rho(S_1^1),$$

where S_1^1 denotes the unit circle in $\mathbb{C} \cdot e_0 \subset \mathbb{C}^{n+1}$ and $U_\rho(S_1^1)$ again means the tubular neighborhood of radius ρ around S_1^1 (note that S_1^1 is timelike while its normal vectors are spacelike). Furthermore,

$$\partial U_\rho(S_1^1) = S_0 \times S_1 \subset \mathbb{C} \times \mathbb{C}^n,$$

where $S_0 = S_c^1$, $S_1 = S_s^{2n+1}$, with $c = \cosh \rho$, $s = \sinh \rho$, and the metric is the Lorentzian product metric $-ds_0^2 + ds_1^2$ where ds_0^2, ds_1^2 are the standard metrics on S_0 and S_1. So, $M := \partial B_\rho(0)$ is the quotient space $(S_0 \times S_1)/G$ with its induced metric. This space is diffeomorphic to S_1 and its metric equals to that of S_1 for vectors perpendicular to the Hopf fibres, while the Hopf fibres themselves are blown up by the factor $c = \cosh \rho$.

(6.2) If $f\colon M \to \mathbb{R}$ is an eigenfunction of the Laplacian of $M = \partial B_\rho(0)$, then $f \circ \pi\colon S_0 \times S_1 \to \mathbb{R}$ is a G-invariant eigenfunction of the Laplacian $\Delta(S_1) - \Delta(S_0) =: \Delta_1 - \Delta_0$ of the Lorentzian manifold $T = (S_0 \times S_1,\ -ds_0^2 + ds_1^2)$, since π is a Riemannian submersion with totally geodesic fibres. As in the definite case, if f, g are eigenfunctions of S_0, S_1 corresponding to eigenvalues μ, ν then $f \otimes g$ is an eigenfunction of $S_0 \times S_1$, but this time corresponding to the eigenvalue $\lambda = \nu - \mu$, and these functions form a basis of eigenfunctions on $S_0 \times S_1$.

(6.3) The eigenfunctions of $\Delta_j = \Delta(S_j)$ contain an orthonormal basis of the L^2-functions on S_j, for $j = 0, 1$. These are the homogeneous harmonic polynomials on E_j, restricted to S_j, where $E_0 = \mathbb{R}^2$ and $E_1 = \mathbb{R}^{2n}$. Thus each function $h \in L^2(S_0 \times S_1)$ can be represented as a possibly infinite sum of the form

$$(*) \qquad h = \sum_i f_i \otimes g_i$$

(convergent with respect to the L^2-norm) where f_i and g_i are eigenfunctions with respect to Δ_0 and Δ_1.

For $h \in L^2(S_0 \times S_1)$ and $\gamma \in G$ put $h^\gamma(x) = h(\gamma x)$. This defines a group action on $L^2(S_0 \times S_1)$ which preserves the subspace $H_0^p \otimes H_1^q$ for all nonnegative integers p, q, where H_j^k denotes the space of harmonic polynomials of degree k on E_j, restricted to S_j. Thus $H_0^p \otimes H_1^q$ is also preserved by the projection π_G which maps $L^2(S_0 \times S_1)$ onto the G-invariant functions, namely

$$\pi_G(h) = \int_G h^\gamma \, d\gamma$$

where $d\gamma$ is the invariant measure on G with volume 1. Therefore, a G-invariant L^2-function h on $S_0 \times S_1$ can be represented in the form $(*)$ where f_i and g_i now are G-invariant homogeneous harmonic polynomials, and we may assume that the components $f_i \otimes g_i$ are linearly independent.

(6.4) Now let h be a G-invariant eigenfunction of $\Delta = \Delta_1 - \Delta_0$, corresponding to a given eigenvalue λ. Since (by (6.2)) the components in $(*)$ are also Δ-eigenfunctions, corresponding to some eigenvalue $\lambda_i = \mu_i - \nu_i$, we get $\lambda_i = \lambda$ for all i. Thus $f_1 \otimes g_1$ (say) is also a G-invariant eigenfunction corresponding to λ, and f_1 and g_1 are homogeneous harmonic polynomials.

(6.5) Any harmonic real polynomial of degree p on $\mathbb{C}$ is the real part of $c \cdot z^p$ for some $c \in \mathbb{C}$. Hence we get from (6.4) that any G-invariant eigenvalue λ of $\Delta_1 - \Delta_0$ has a G-invariant eigenfunction

$$h(x) = \mathrm{Re}(x_0^p \cdot g(y))$$

where $y=(x_1, \ldots, x_n)$ and g is a complex valued homogeneous harmonic polynomial of degree q. We may write g as

$$g(y)=\sum_{\alpha,\beta} a_{\alpha\beta}\, y^\alpha\, \bar{y}^\beta$$

where $\alpha, \beta \in \mathbb{N}^n$ with $|\alpha|+|\beta|=q$, $a_{\alpha\beta}\in\mathbb{C}$ and $y^\alpha := x_1^{\alpha_1}\ldots x_n^{\alpha_n}$. Then for any $\gamma\in G = S_1^1$,

$$h(\gamma x)=\operatorname{Re}\Big(\sum_{\alpha,\beta} a_{\alpha\beta}\,\gamma^{p+|\alpha|-|\beta|}\, x_0^p\, y^\alpha\, \bar{y}^\beta\Big)$$

and so the G-invariance of h implies that $a_{\alpha\beta}\neq 0$ only if $p+|\alpha|-|\beta|=0$.

Hence

$$p=|\beta|-|\alpha|\leqq|\beta|+|\alpha|=q.$$

Therefore (see (5.3))

$$\lambda=\nu_q-\mu_p\geqq\nu_p-\mu_p\geqq\nu_1-\mu_1,$$

since

$$\begin{aligned}\nu_p-\mu_p&=(p^2+(2n-2)p)/s^2-p^2/c^2\\ &=(p^2+c^2(2n-2)p)/c^2 s^2\end{aligned}$$

grows monotonically with p. Thus we have shown

(6.6) **Proposition.** *The first eigenvalue of the Laplacian of a geodesic sphere of radius ρ in $H^n\mathbb{C}$ is*

$$\lambda_1=(1+2(n-1)\cosh^2\rho)/\cosh^2\rho\,\sinh^2\rho.$$

(6.7) Finally, to complete the proof of Theorem (1.4), we have to compute the value of $\lambda:=\bar{R}+\|B\|^2$ for the geodesic sphere $M=\partial B_\rho(o)$ in $\bar{M}=H^n\mathbb{C}$, for some $o\in\bar{M}$. Fix a unit vector $v\in T_o\bar{M}$. There exists an orthonormal basis $v=v_0, v_1, \ldots, v_{2n-1}$ of $T_o\bar{M}$ such that the curvatures of the planes $\sigma_j = \operatorname{Span}(v_0, v_j)$, $j=1, \ldots, 2n-1$ are

$$K(\sigma_1)=-4, \qquad K(\sigma_k)=-1 \qquad \text{for } k=2, \ldots, 2n-1.$$

Let J_i $(j=1, \ldots, 2n-1)$ be the Jacobi field along the geodesic $\gamma(t)=\exp_0 tv$ with $J_j(0)=0$, $J_j'(0)=v_j$. Then

$$\begin{aligned}&J_1(t)=\tfrac{1}{2}\sinh(2t)\, v_1(t),\\ &J_k(t)=\sinh(t)\, v_k(t) \qquad \text{for } k=2, \ldots, 2n-1,\end{aligned}$$

where $v_j(t)$ is the parallel field along γ with $v_j(0)=v_j$. Hence the 2nd fundamental tensor B of $M=\partial B_\rho(0)$ has eigenvalues

$$\partial(\log\sinh\rho)/\partial\rho=c/s$$

of multiplicity $2(n-1)$ and

$$\partial(\log\sinh 2\rho)/\partial\rho=(c^2+s^2)/cs$$

of multiplicity 1, where we again put $c=\cosh\rho$, $s=\sinh\rho$. Moreover, the Ricci curvature of $\bar{M}$ is

$$\bar{R}=-4-2(n-1),$$

thus

$$\begin{aligned}\lambda=\bar{R}+\|B\|^2&=2(n-1)(c^2/s^2-1)+(c^2+s^2)^2/c^2s^2-4\\&=2(n-1)/s^2+1/c^2s^2=\lambda_1,\end{aligned}$$

by Proposition (6.5). The result now follows from (2.12).

Appendix: The Isoperimetric Problem

The question of stability of constant mean curvature hypersurfaces M in $\bar{M}$ is closely related to the isoperimetric problem in $\bar{M}$: Find the hypersurfaces of least area bounding a domain of given volume. It follows from the first and second variation formula for the area of hypersurfaces that such a hypersurface is stable if it is smooth. For simply connected spaces of constant curvature, the solutions of the isoperimetric problem are precisely the spheres [Sc].

For $\bar{M}=P^{r-1}\mathbb{R}$, all spheres are stable, but those which bound a ball B with $\operatorname{vol} B=c\operatorname{vol}\bar{M}$ for some $c>1/2$ cannot be solutions of the isoperimetric problem: If B' is a ball with $\operatorname{vol} B'=(1-c)\operatorname{vol}\bar{M}$, then area $\partial B'<$ area ∂B, and $\partial B'$ bounds the domain $\bar{M}-B'$ with $\operatorname{vol}(\bar{M}-B')=\operatorname{vol} B$.

A similar fact is true for $\bar{M}=P^{r-1}\mathbb{K}$ with $\mathbb{K}=\mathbb{C}$ or $\mathbb{K}=\mathbb{H}$: Let ρ_1 be the radius of the sphere with largest area and ρ_2 the largest radius such that ∂B_{ρ_2} is stable. It is known that $\tan^2\rho_1=k/m$ and by Theorem (1.3) $\tan^2\rho_2=(k+2)/m$, where $m=d-1$, $k=d(r-1)-1$, hence $\rho_1<\rho_2$. Moreover, if ρ_0 denotes the radius of a ball of half the total volume, a computation yields that $\rho_0<\rho_1$. Now a similar argument as in the case $K=R$ shows that ∂B_ρ for $\rho\in(\rho_0,\rho_1)$ cannot be a solution of the isoperimetric problem though it is a stable hypersurface.

So one may ask whether all spheres which bound a ball of at most half the total volume can be solutions of the isoperimetric problem. But not even this is true in general. Take for example in $P^3\mathbb{R}$ the hypersurface $\partial U_{\pi/4}(P^1\mathbb{R})$, the projection of the Clifford torus in S^3. It has area π^2 and bounds a domain of half the total volume. On the other hand, the sphere bounding the ball B_{ρ_0} with the same volume has larger area. In fact, we have $\rho_0>\rho:=0.36\,\pi$ and area $\partial B_\rho>\pi^2$. An analogous fact is true in $P^3\mathbb{C}$. One might conjecture that these hypersurfaces are non-spherical solutions of the isoperimetric problem.

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