Branch Points of Conformal Mappings of Surfaces

J. Eschenburg[1,*] and R. Tribuzy[2,**]

[1] Mathematisches Institut der WWU, Einsteinstrasse 62, D-4400 Münster, Federal Republic of Germany
[2] Universidade do Amazonas, Departamento de Matemática, ICE, 69000 Manaus, AM., Brasil

Contents

0. Introduction

Let S be a 2-dimensional differentiable manifold ("surface") and M an arbitrary differentiable manifold of dimension $n \geqq 2$. Let $f: S \to M$ be a differentiable map. A point $p \in S$ is a *critical point* if df_p has rank < 2. A critical point p is called a *branch point of f of order k* [6] if there are coordinate charts $z = x + iy$ around p in S (with values in $\mathbb{C}$) and $u = (u_1, \ldots, u_n)$ around $f(p)$ in M such that

$$u_1(f) + iu_2(f) = z^{k+1} + O(|z|^{k+2}) \ ,$$
$$u_\alpha(f) = O(|z|^{k+2})$$

for $\alpha = 3, \ldots, n$. This type of critical points occurs in connection with minimal surfaces and related problems. E.g., the solution of Plateau's problem for the closed curve $\gamma: S^1 \to \mathbb{R}^4 = \mathbb{C}^2$, $\gamma(\tau) = (\tau^2, \tau^3)$ is a C^1-mapping $f: D \to \mathbb{C}^2$ defined on the unit disk D with $f|\partial D = \gamma$ which minimizes the Dirichlet integral $\int_D (\|f_x\|^2 + \|f_y\|^2) dx dy$, and it follows from Wirtinger's inequality that $f(z) = (z^2, z^3)$ is the only solution [9]. Here, $z = 0$ is a branch point of order 1. However, it has been shown by Osserman [10] (see also [6]) that solutions of Plateau's problem in $\mathbb{R}^3$ do not admit branch points. In this paper, we want to study branch points of *conformal* mappings of

* Partially supported by DFG
** Supported by CNPq and GMD

surfaces which satisfy a certain type of PDE. These contain minimal surfaces and surfaces of prescribed mean curvature in 3-manifolds. We will relate the order and the number of branch points to geometric quantities like curvature and draw topological conclusions in case that S is a closed surface. In particular, we can exclude the existence of branch points in certain cases if S has genus 1. This can be considered as a theorem of Osserman type. However, the arguments are very different from those in Osserman's case but resemble those of Hopf [8] and Chern [1, 2].

This work was originally intended as a continuation of [4] and [5]. In particular, we extend results of these papers to the case where branch points may occur.

1. Smooth Critical Points and Conformal Maps

Let S be a surface, M an n-manifold and $f: S \to M$ a smooth mapping as above. To simplify notation, we assume that the target manifold M is a submanifold of some euclidean space $\mathbb{R}^N$, but this assumption is not essential in this chapter. For every non-critical point $q \in S$, there is a tangent plane $df_q(T_q S)$ of the image $f(S)$ at $f(p)$. On the other hand, if p is a critical point, a tangent plane at $f(p)$ does not exist in general. However, if p is a branch point in the sense of the definition above, the plane spanned by $\partial/\partial u_1$ and $\partial/\partial u_2$ at $f(p)$ plays a similar role. More precisely, consider the Gauss map

$$G: S_0 \to G_{2,N} \ , \quad G(q) = df_q(T_q M) \ ,$$

where S_0 denotes the set of non-critical ("*regular*") points in S and $G_{2,N}$ the Grassmannian of 2-planes in $\mathbb{R}^n$. If p is a branch point of f, then G can be continuously extended to p. (Note that f is regular around p.) We would like to know when this extended Gauss map is smooth (i.e. C^∞).

More generally, a critical point p is called *smooth* if there exists a neighborhood U of p in S and a smooth map $G: U \to G_{2,N}$ with $G(q) \subset T_{f(q)} M \subset \mathbb{R}^N$ for all $q \in U$ such that G agrees with the Gauss map on $U \cap S_0$. If all critical points of f are smooth, the Gauss map extends to a smooth mapping $G: S \to G_{2,N}$, thus defining a 2-plane bundle Tf over S with fibres $T_p f := G(p)$, $p \in S$. This is a subbundle of the pull back bundle f^*TM, and df is considered as a vector bundle homomorphism $df: TS \to Tf$ over S. If f is an immersion (i.e. $S_0 = S$) then df is an isomorphism.

If M is considered as a Riemannian manifold (with metric induced from the ambient space $\mathbb{R}^N$) then Tf inherits a metric $\langle , \rangle$ and a connection D from f^*TM. Let Nf be the orthogonal complement of Tf in f^*TM. Then a second fundamental form $\alpha: TS \otimes TS \to Nf$ is defined as follows:

$$\alpha(v, w) = (D_v f_*(W))^N \ ,$$

where $v, w \in T_p S$ and W is a vector field on S with $W_p = w$. Here, $f_* W$ is considered as a section of Tf. The 2-form α is symmetric and extends the usual 2^{nd} fundamental form on $f|S_0$.

We want to assume from now on that our surface S is orientable and provided with a fixed complex structure as a Riemann surface. A smooth nonconstant mapping $f: S \to M \subset \mathbb{R}^N$ is called *conformal* if for any holomorphic chart $z = x + iy$

on S we have

$$\langle f_x, f_y\rangle = 0 \ , \qquad \langle f_x, f_x\rangle = \langle f_y, f_y\rangle \ ,$$

or equivalently, f_z is *isotropic*, i.e.

$$\langle f_z, f_z\rangle = 0 \ . \tag{1}$$

Here as usual, f_x and f_y are the partial derivatives and $f_z = \frac{1}{2}(f_x - if_y)$, while $\langle\,,\rangle$ denotes the scalar product on $\mathbb{R}^N$ and its complex bilinear extension to C^N. The function

$$\mu := \|f_x\| = \|f_y\| = \tfrac{1}{2}\langle f_z, \overline{f_z}\rangle^{1/2}$$

is called the *conformal factor* with respect to the chart z.

Theorem 1. *Let S be a Riemann surface and $f: S \to M \subset \mathbb{R}^N$ a conformal map. Then each smooth critical point $p \in S$ is a branch point of some order $k \geqq 1$ and for any holomorphic chart z around p with $z(p) = 0$ we have*

$$\mu(z) = |z|^k \mu_0(z) \ ,$$

where μ_0 is smooth with $\mu_0(0) \neq 0$.

Proof. Replacing S with an open subset if necessary, we may assume that the Gauss map extends to a smooth mapping $G: S \to G_{2,N}$. So the bundle $Tf \subset f^*TM \subset M \times \mathbb{R}^N$ is defined and inherits a metric $\langle,\rangle$ and a connection D from f^*TM. Let $\{e_1, e_2\}$ be a local orthonormal basis of Tf, defined on some open subset S' of S and $z = x + iy: S' \to \mathbb{C}$ a holomorphic chart. Then f_z is an isotropic section of $Tf \otimes \mathbb{C}$ [by conformality, (1)], and therefore,

$$f_z = \alpha(e_1 - ie_2) + \beta(e_1 + ie_2)$$

for complex valued functions α, β with $\alpha \cdot \beta = 0$. Dually, let θ_j be the 1-forms on S' defined by

$$\theta_j(v) = \langle df_p(v), e_j(p)\rangle$$

for $p \in S'$, $v \in T_pS$, $j \in \{1, 2\}$, and put

$$\phi = \theta_1 + i\theta_2 = \langle df, e_1 + ie_2\rangle \ ,$$

then

$$\phi = 2\alpha dz + 2\beta d\bar{z} \ .$$

Moreover, put $\theta_{12} = \langle De_1, e_2\rangle = -\theta_{21}$, then we have $d\theta_j = \theta_{jk} \wedge \theta_k$ for $j \neq k$, and so

$$d\phi = -i\theta_{12} \wedge \phi \ . \tag{2}$$

Let $Z_\alpha \subset S'$ be the zero set of α. Outside Z_α we have $\beta = 0$ and so $\phi = \alpha dz$, hence from (2)

$$(d\alpha + 2i\alpha\theta_{12}) \wedge dz = 0 \ .$$

This means that the $d\bar{z}$-part of $d\alpha + 2i\alpha\theta_{12}$ vanishes, i.e. on Z_α we have

$$\partial\alpha/\partial\bar{z} = b \cdot \alpha \tag{3}$$

where $b \cdot d\bar{z}$ is the $d\bar{z}$-part of $-2i \cdot \theta_{12}$. Also, (3) holds trivially on the interior of Z_α and so it holds everywhere on S'. A particular solution of (3) is $\alpha_0 = e^u$ where u solves $\partial u/\partial \bar{z} = b$, and thus the general solution is

$$\alpha = \alpha_0 \cdot \alpha_1 \tag{4}$$

for some holomorphic function α_1, [4].

In particular, the zeros of α are isolated unless $\alpha \equiv 0$. So either α or β vanish identically on all of S', and by choice of the frame $\{e_1, e_2\}$, we may assume $\beta \equiv 0$. Thus,

$$f_z = \alpha(e_1 - ie_2) \ . \tag{5}$$

Now suppose that $z = 0$ is a (smooth) critical point of f, i.e. a zero of α. Thus by (4), we have a smooth decomposition

$$\alpha = z^k \cdot \alpha_0 \tag{4'}$$

with $\alpha_0(0) \neq 0$. Now take coordinates $(u_1, \dots, u_n)$ on M around $f(0)$ such that the coordinate vector fields $U_j = \partial/\partial u_j$ satisfy $U_j(f(0)) = e_j(0)$ for $j \in \{1, 2\}$. Then we have

$$e_j = U_j(f) + O(|z|) \ ,$$

thus

$$\begin{aligned} f_z &= z^k \cdot \alpha_0(z) \cdot ((U_1 - iU_2)(f) + O(|z|)) \\ &= c \cdot z^k \cdot (U_1 - iU_2)(f) + O(|z|^{k+1}) \ , \end{aligned}$$

where $c = \alpha_0(0)$. Consequently,

$$\begin{aligned} \partial u_1(f)/\partial z &= c \cdot z^k + O(|z|^{k+1}) \ , \\ \partial u_2(f)/\partial z &= -ic \cdot z^k + O(|z|^{k+1}) \ , \\ \partial u_\alpha(f)/\partial z &= O(|z|^{k+1}) \ , \end{aligned}$$

for $\alpha = 3, \dots, n$. Putting $c' = 2c/(k+1)$ and

$$w_1 = u_1(f) - \operatorname{Re}(c' z^{k+1}) \ , \qquad w_2 = u_2(f) - \operatorname{Im}(c' z^{k+1}) \ ,$$

we have $\partial w_j/\partial z = O(|z|^{k+1})$. Since $w := (w_1, w_2, u_3, \dots, u_n)$ is real, we have $dw = 2 \operatorname{Re}(w_z \cdot dz) = O(|z|^{k+1})$, hence $w = O(|z|^{k+2})$. Thus

$$\begin{aligned} u_1(f) + i \cdot u_2(f) &= c' z^{k+1} + O(|z|^{k+2}) \ , \\ u_\alpha(f) &= O(|z|^{k+2}) \end{aligned}$$

for $\alpha = 3, \dots, n$, which shows that $z = 0$ is a branch point of order k. For the conformal factor μ, we get from (5) and (4'):

$$\mu = \tfrac{1}{2} \|f_z\| = |z^k| \cdot \|\alpha_0\| \ ,$$

and $\mu_0 := \|\alpha_0\|$ is smooth and positive near $z = 0$. This completes the proof of Theorem 1.

A conformal map $f: S \to M$ all of whose critical points are smooth will be called a *conformal smoothly branched immersion*. Theorem 1 asserts that all critical points of such a map are branch points.

2. A Certain PDE

We do not know in general whether a smooth conformal mapping $f: S \to \mathbb{R}^N$ can have non-smooth critical points at all. However, in minimal surface and related problems, f satisfies a PDE which excludes non-smooth critical points:

Theorem 2. *Let U be an open domain in $\mathbb{C}$ and $f: U \to \mathbb{R}^N$ a smooth conformal map. Assume that $g := f_z$ satisfies*

$$\partial g / \partial \bar{z} = A \cdot g \ , \tag{*}$$

where $A(z)$ is a complex $N \times N$-matrix depending smoothly on $z \in U$. Then f is a conformal smoothly branched immersion.

Proof. Let U_0 be the set of regular points of f and $G: U_0 \to G_{2,N}$ its Gauss map. We may express G in terms of g as follows: Let $CQ = \{u \in \mathbb{C}^N | \langle u, u \rangle = 0\}$ be the set of isotropic vectors and Q its projection to $\mathbb{C}P^{N-1}$. The quadric Q is diffeomorphic to the Grassmannian $G_{2,N}^+$ of oriented 2-planes in $\mathbb{R}^N$, via the map β sending a homogeneous vector $[u]$ to the oriented plane spanned by $\mathrm{Re}(u)$ and $\mathrm{Im}(u)$, and hence we have $G = \beta([g])$ where $[g]$ denotes the projection of $g | U_0$ to $\mathbb{C}P^{N-1}$. So G has a smooth extension to U if and only if $[g]$ has. The latter is true if near each zero z_0 of g, we have a smooth decomposition $g = \alpha \cdot g_0$ where α is a complex valued scalar function and $g_0(z_0) \neq 0$.

We may assume $z_0 = 0$. Since g satisfies (*), we know from a theorem of Chern [1] which uses the inhomogeneous Cauchy formula that the Taylor expansion of g around 0 does not vanish entirely. Suppose that $m \geqq 1$ is the order of the lowest nonvanishing Taylor polynomial. We claim that $\alpha = z^m$ divides g and $g_0 = g/z^m$ is smooth with $g_0(0) \neq 0$. This is shown by the following two lemmas which complete the proof of Theorem 2:

Lemma 2.1. *Let g be a solution of (*) with a zero of order m at $z_0 = 0$. Let g_r, $r \geqq m$ be the Taylor polynomial of order r around $z_0 = 0$. Then z^m divides g_r for every $r \geqq m$.*

Proof. We show by induction over k that $\partial^j \bar{\partial}^k g(0) = 0$ for $0 \leqq j \leqq m-1$, $k \geqq 1$ where $\partial := \partial/\partial z$, $\bar{\partial} := \partial/\partial \bar{z}$. In fact, $\partial^j \bar{\partial} g = \partial^j (A \cdot g)$ vanishes at 0 for $j \leqq m-1$ since $A \cdot g$ has order $\geqq m$ at 0. Moreover,

$$\partial^j \bar{\partial}^k g = \partial^j \bar{\partial}^{k-1} (A \cdot g)$$

vanishes at 0 by induction hypothesis since every term of the right hand side contains a factor $\partial^\alpha \bar{\partial}^\beta g$ with $\alpha \leqq j$, $\beta \leqq k-1$.

Lemma 2.2 *Let $g: U \to \mathbb{C}^N$ be a smooth map and assume that z^m divides each Taylor polynomial of g around $z_0 = 0$. Then z^m divides g, i.e. there is a smooth decomposition $g = z^m \cdot g_0$.*

Proof. Let g_r denote the Taylor polynomial of order r. Then $g - g_r$ is of order $\geqq (r+1)$ at 0, and

$$h_{jk} := \partial^j \bar{\partial}^k (z^{-m}(g - g_r))$$

is a sum of multiples of $z^{-m-i}\cdot\partial^{j-i}\bar{\partial}^k(g-g_r)$ with $0\leqq i\leqq j$, thus

$$\|h_{jk}(z)\|\leqq C\cdot|z|^{r+1-(m+j+k)} .$$

So the function h with $h(0)=0$, $h(z)=(g(z)-g_r(z))/z^m$ for $z\neq 0$ is $(r-m)$-times differentiable. Since g_r/z^m is smooth for arbitrary $r>m$, the function g/z^m has smooth extension to $z=0$.

3. Local Applications

We will apply Theorem 2 as follows: Let $M\subset\mathbb{R}^N$ be a submanifold with induced Riemannian metric $\langle,\rangle=ds^2$. A conformal mapping $f:S\to M\subset\mathbb{R}^N$ is called *harmonic* if the component of Δf tangent to M vanishes, where Δ denotes the Laplacian of some metric on S which is compatible to the complex structure. Thus a conformal harmonic map is a minimal surface outside its critical points. For any holomorphic coordinate $z=x+iy$ with conformal factor μ for the metric chosen, we have $\mu^2\cdot\Delta f=f_{xx}+f_{yy}=4f_{z\bar{z}}$. Thus harmonicity implies

$$\mu^2\cdot\Delta f=\mu^2(\Delta f)^{\perp}=\alpha(f_x,f_x)+\alpha(f_y,f_y)=4\alpha(\overline{f_z},f_z) ,$$

where $\alpha: TM\otimes TM\to\perp M$ denotes the 2[nd] fundamental form of $M\subset R^N$, extended complex linearly to complex tangent vectors. Thus, a conformal harmonic mapping is given by the equation

$$f_{z\bar{z}}=\alpha(\overline{f_z},f_z) . \tag{6}$$

Similarly, let $M\subset\mathbb{R}^N$ be a 3-dimensional submanifold and $f:S\to M\subset\mathbb{R}^N$ a conformal map such that f^*TM is oriented. So $T_{f(p)}M$, $p\in S$, is an oriented 3-dimensional vector space with euclidean inner product which defines a unique vector product $\times$ on $T_{f(p)}M$. On the subset S_0 of regular points, the component of Δf tangent to M is twice the mean curvature vector of f, i.e.

$$\mu^2(\Delta f)^T=2Hf_x\times f_y=4iH\overline{f_z}\times f_z ,$$

where H is the mean curvature of $f|S_0$. Thus on S_0

$$f_{z\bar{z}}=iH\overline{f_z}\times f_z+\alpha(\overline{f_z},f_z) . \tag{7}$$

More generally, if $H:S\to R$ is any smooth function, a conformal map f solving (7) everywhere on S is called a *generalized surface of mean curvature H*.

Obviously, (6) and (7) have the shape of (*) in Theorem 2 for a suitable matrix valued function A. Since we may embed each Riemannian manifold isometrically into some $\mathbb{R}^N$ (in fact we only need a local embedding theorem), we get from Theorem 2:

Theorem 3. *Let S be a Riemann surface, M a Riemannian manifold and $f:S\to M$ a conformal map. If either f is harmonic or* $\dim M=3$ *and f is a generalized surface of mean curvature H for some smooth function H on S, then f is a conformal smoothly branched immersion.*

This is a generalization of the results of Chap. 2 in [6].

As an important example, consider a complete 3-manifold (M, ds^2) of constant sectional cirvature c and a generalized surface $f: S \to M$ of *constant* mean curvature $H \geqq 0$. Let N be the oriented unit section of the normal bundle Nf. Then $H \cdot N$ is the mean curvature vector on the regular set S_0. Let

$$f_t: S \to M \quad , \quad f_t(p) = \exp_{f(p)}(tN_p)$$

for $t \in \mathbb{R}$ be the parallel surface. Let $(\sin_c, \cos_c)$ be the solution of the initial value problem

$$\sin_c' = \cos_c \ , \quad \cos_c' = -c \cdot \sin_c \ , \quad \sin_c(0) = 0 \ , \quad \cos_c(0) = 1$$

and let t be the first positive solution of

$$\cos_c(t)/\sin_c(t) = H \ .$$

Then f_t is also a conformal map and $f_t^*(ds^2) = a^2 f^*(ds^2)$ with

$$a = \tfrac{1}{2}\,|k_1 - k_2| \cdot \sin_c(t)$$

on S_0, where k_1 and k_2 are the principal curvatures of the immersion $f|S_0$. Moreover, f_t has constant mean curvature $-H$ and thus is also a conformal smoothly branched immersion. The umbilic points of f are the zeros of a and correspond to the branch points of f_t, and vice versa, since the rôles of f and f_t can be interchanged.

Remark. In case $c=0$, $H=0$, the previous discussion does not apply directly since $\cos_0(t)/\sin_0(t) = 1/t$ has no zero. However, it applies in the limit as $t \to \infty$ if we renormalize and put $g_t = f_t/t$ for $t > 0$. Then g_t converges to the Gauss map as $t \to \infty$. Thus we recover the conformality of the Gauss map for conformal harmonic maps in $\mathbb{R}^3$.

4. Gauss-Bonnet Theorem

From now on, let S be a *compact* Riemann surface and $f: S \to M \subset \mathbb{R}^N$ a conformal smoothly branched immersion. Then the induced 2-form $f^*(ds^2)$ on the regular set S_0 is a Riemannian metric compatible to the complex structure.

Let $E \to S$ be an oriented real 2-plane bundle with metric $\langle\,,\rangle$, metric connection D and curvature tensor R_E. If $\{s_1, s_2\}$ is an oriented local orthonormal basis of E, the curvature 2-form $\Omega_E = \langle R_E(\,,)s_2, s_1\rangle$ is independent of the choice of the basis and hence globally defined. By the generalized Gauss-Bonnet Theorem we have

$$\int_S \Omega_E = 2\pi \cdot \chi_E \ , \tag{8}$$

where χ_E denotes the Euler number of E. On S_0, let dv denote the volume element of the metric $f^*(ds^2)$. Then $\Omega_E = K_E \cdot dv$ where $K_E = \Omega_E(e_1, e_2)$ for an arbitrary oriented local orthonormal basis $\{e_1, e_2\}$ of $(TS_0, f^*(ds^2))$. E.g. we may choose $e_1 = f_x/\mu$, $e_2 = f_y/\mu$ for some holomorphic chart $z = x + iy$ with conformal factor μ. The local formula

$$K_E = \langle R_E(f_x, f_y)s_1, s_2\rangle / \mu^2$$

shows that $K_E = K_{E,0} \cdot |z|^{-2k}$ for some smooth function $K_{E,0}$ defined in a neighborhood of $z=0$, if $z=0$ is a branch point of order k.

If $E = Tf$, then $K_E = :K$ is the Gaussian curvature of the metric $f^*(ds^2)$ on S_0. Choose any Riemannian metric ds_0^2 on S which is compatible to the complex structure. Then

$$f^*(ds^2) = m^2\, ds_0^2$$

for some smooth nonnegative function m^2, and on S_0, the curvatures K of $f^*(ds^2)$ and K_0 of ds_0^2 satisfy the well known relation

$$m^2 \cdot K = K_0 - \Delta_0 \log m \ , \tag{9}$$

where Δ_0 denotes the Laplacian of ds_0^2. Near a branch point $z=0$ of order k, we have $m = \mu/\mu_0$ where μ_0 is the conformal factor with respect to ds_0^2. Thus by Theorem 1, $m = m_0 \cdot |z|^k$ for some smooth positive function m_0 near $z=0$.

More generally, a function $u: S \to [0, \infty]$ is called of *absolute value type* if for all $p \in S$ and any holomorphic chart z around p there is an integer $k = \operatorname{ord}_p(u)$ and a smooth positive function u_0 on a neighborhood of p such that

$$u = |z - z(p)|^k \cdot u_0 \ .$$

If $k > 0$ ($k < 0$) then p is called a zero (pole) of order $|k|$. Poles and zeros are isolated. Let $n(u)$ ($p(u)$) be the number of zeros (poles), counted with multiplicities, and put

$$N(u) := \sum_{p \in S} \operatorname{ord}_p(u) = n(u) - p(u) \ .$$

The following lemma is an easy consequence of the divergence theorem [4,5]:

Lemma 4. *If u is an absolute value type function and ds_0^2 a compatible metric on the Riemann surface S with Laplacian Δ_0 and volume element dv_0, then*

$$\int_S \Delta_0 \log u\, dv_0 = -2\pi \cdot N(u) \ .$$

We may apply this lemma to the absolute value type function m which has no poles, hence $N(m) = n(m)$ equals the number of branch points, counted with multiplicities. Hence, multiplying (9) by dv_0 and integrating, from Lemma 4 and the usual Gauss-Bonnet Theorem for (S, ds_0^2) we get a Gauss-Bonnet formula which essentially was already proved by Heinz and Hildebrandt [7]:

Theorem 4. *Let $f: S \to M \subset \mathbb{R}^N$ be a conformal smoothly branched immersion. Then*

$$(1/2\pi) \int_S K\, dv = \chi_{Tf} = \chi(S) + b \ ,$$

where $\chi(S)$ is the Euler number of S and b the number of branch points of S, counted with multiplicities.

5. Estimating the Number of Branch Points

We are now able to generalize the results of [4,5] to the case of branched immersions. In all these cases, the geometry is governed by a holomorphic differential Φ on the Riemann surface S, i.e. a holomorphic symmetric r-form which is locally $\Phi = h(z)dz^r$ for a holomorphic chart z and a holomorphic function h. Let $N(\Phi)$ be the number of zeros of Φ, counted with multiplicities. The following fact can be considered as an easy special case of the Riemann-Roch theorem:

Lemma 5. *Let Φ be a holomorphic symmetric r-form on a closed Riemann surface S. Then either $\Phi \equiv 0$ or $N(\Phi) = -r \cdot \chi(S)$.*

Proof. Choose any compatible metric ds_0^2 on S. Let z be a holomorphic chart and μ_0 the corresponding conformal factor. Let $\Phi = h(z)dz^r$. Then $|\Phi| := |h(z)| \cdot \mu_0^{-r}$ does not depend on the choice of z. Therefore, $|\Phi|$ is a globally defined absolute value type function without poles, unless $\Phi \equiv 0$. Moreover, on the domain of z we have

$$\Delta_0 \log|\Phi| = \Delta_0 \log \mu_0^{-r} = r \cdot K_0 \ ,$$

where K_0 is the Gaussian curvature of ds_0^2. Thus, by Lemma 4 and the Gauss-Bonnet theorem,

$$\begin{aligned} -N(\Phi) = -N(|\Phi|) &= (1/2\pi) \int_S \Delta_0 \log|\Phi| dv_0 \\ &= (1/2\pi) \cdot r \cdot \int_S K_0 dv_0 = r \cdot \chi(S) \end{aligned}$$

which completes the proof.

Now let S be a closed Riemann surface, (M, ds^2) a Riemannian manifold and $f: S \to M$ a conformal smoothly branched immersion such that f^*TM is an oriented bundle. Let S_0 be the set of regular points and b the number of branch points, counted with multiplicities given by their orders. We consider three different cases:

(a) M is 3-dimensional with constant sectional curvature c and f is of constant mean curvature H,

(b) M is 4-dimensional with constant sectional curvature c and f is harmonic,

(c) M is a (real) 4-dimensional Kähler manifold of constant holomorphic sectional curvature 4σ and f is harmonic.

In all cases, a holomorphic differential Φ (of degree 2 resp. 4 resp. 3) was constructed using local oriented orthonormal frames of the tangent bundle TS and the normal bundle NS if f was an immersion [4,5]. The construction carries over to branched surfaces if we replace TS with Tf and NS with Nf. Since branch points are isolated, the orientation of TS determines an orientation of Tf via df and consequently also of Nf. Let $\{e_1, e_2\}$ be a local oriented orthonormal basis of Tf, defined on some open subset U of S. As before, put $\theta_j = \langle df(\ ), e_j \rangle$ and $\phi = \theta_1 + i\theta_2$. Then ϕ is a $(1,0)$-form whose zeros are precisely the branch points and

$$d\phi = -i\theta_{12} \wedge \phi \ ,$$

where $\theta_{12} = \langle De_1, e_2 \rangle$ (see proof of Theorem 1).

Case a (cf. [5]). There exists a unique unit section N of Nf such that (e_1, e_2, N) is an oriented orthonormal basis of $(f^*TM)|U$. Put

$$\psi = \langle De_1, N \rangle - i \langle De_2, N \rangle - H \cdot \bar{\phi} \ .$$

On $U \cap S_0$ we have $\psi = k \cdot \phi$ for some smooth complex valued function k with

$$|k|^2 = c + H^2 - K \ .$$

In particular, ψ is a (1,0)-form on U, and the zeros of ψ on $U \cap S_0$ are precisely the umbilic points. Moreover, since the mean curvature H is constant, we get

$$d\psi = i\theta_{12} \wedge \psi \ .$$

Now $\Phi := \phi \cdot \psi$ is independent of the choice of frame and holomorphic: If $\phi = \mu \cdot dz$ and $\psi = \sigma \cdot dz$, then the exterior differential equations imply

$$(d\mu + i \cdot \mu \cdot \theta_{12}) \wedge dz = 0 = (d\sigma - i \cdot \sigma \cdot \theta_{12}) \wedge dz \ ,$$

thus $d(\mu \cdot \sigma) \wedge dz = 0$ which shows that $\mu \cdot \sigma$ is holomorphic. Therefore we get from Lemma 5:

Theorem 5.1a. *Let M be a 3-spaceform of constant sectional curvature and $f: S \to M$ a conformal map of constant mean curvature. Then either f is totally umbilic or*

$$-2\chi(S) \geqq u + b \ ,$$

where u is the number of umbilic points on S_0, counted with multiplicities.

Case b (cf. [5, 11]). Here, Nf is a 2-plane bundle. Let K_{Nf} be its curvature as defined in Chap. 4; K_{Nf} may have poles at the branch points of f. Let (e_3, e_4) be an oriented orthonormal basis of $(Nf)|U$. We extend the metric of M bilinearly to the complexified bundle $f^*TM \otimes C$ and put

$$\psi_\pm = \langle D(e_1 - ie_2), e_3 \pm ie_4 \rangle \ .$$

Since $f|S_0$ is a conformal minimal immersion, we have $\psi_\pm = k_\pm \cdot \phi$ for complex valued functions $k_\pm$ with

$$|k_\pm|^2 = c - K \pm K_{Nf}$$

on $S_0 \cap U$. In particular, $\psi_\pm$ are (1,0)-forms on U whose zeros on $S_0 \cap U$ are precisely the so called *circular* or *pseudo-umbilical* points where the ellipse of curvature is a circle. Moreover,

$$d\psi_\pm = i \cdot (\theta_{12} - (\pm\theta_{34})) \wedge \psi_\pm \ ,$$

where $\theta_{34} = \langle De_3, e_4 \rangle$. As in Case a we conclude that the quartic form $\Phi := \psi_+ \cdot \psi_- \cdot \phi^2$ is independent of the chosen frames and holomorphic. Thus we get from Lemma 5:

Theorem 5.1b. *Let M be a 4-spaceform of constant sectional curvature and $f: S \to M$ a conformal harmonic map. Then either f is totally circular or*

$$-4\chi(S) \geqq c + 2b \ ,$$

where c is the number of circular points on S_0 (with multiplicities).

Case c (cf. [4]). Beneath the splitting $f^*TM = Tf \oplus Nf$ we get a second orthogonal decomposition $f^*TM = L_+ \oplus L_-$ as follows: Let $(,)$ be the hermitean inner product on TM given by

$$(v, w) = \langle v, w\rangle + i \cdot \langle v, Jw\rangle \ ,$$

where J is the almost complex structure on M. Then

$$s_\pm := \tfrac{1}{2}(e_1 - (\pm Je_2))$$

are sections of $(f^*TM)|U$ with $(s_+, s_-) = 0$ and $\|s_+\|^2 + \|s_-\|^2 = 1$, and $\mathbb{C} \cdot s_\pm$ is independent of the chosen frame, if we define the multiplication with complex scalars by

$$(\alpha + i\beta) \cdot v = \alpha \cdot v + \beta \cdot Jv$$

for any $v \in TM$. Thus $s_\pm$ span orthogonal complex line bundles $L_\pm$. Let E_1 and E_2 be unit sections of L_+ and L_- on U. Then $(E_1; E_2)$ is a unitary basis of $(f^*TM)|U$. Let

$$\omega_i = (df, E_i)$$

for $i \in \{1,2\}$. Then $\omega_1 = u_+ \cdot \phi$, $\omega_2 = u_- \cdot \phi$ for complex valued functions $u_\pm$ with

$$2\,|u_\pm|^2 = 1 \pm \langle Je_1, e_2\rangle \ .$$

Note that $C := \langle Je_1, e_2\rangle$ is independent of the chosen frame: If $\Omega(v, w) := \langle Jv, w\rangle$ is the Kähler form on M and dv the volume form of the metric $f^*(ds^2)$ on S, then we have $f^*\Omega = C \cdot dv$.

Moreover, put

$$\omega_{ij} = (DE_i, E_j)$$

for $i, j \in \{1,2\}$. From the fact that $f|S_0$ is a conformal minimal immersion we get $\omega_{12} = w \cdot \phi$ for some complex valued function w with

$$|w|^2 = 2\sigma - K + K_{Nf} \ .$$

Moreover, from the Cartan structure equations we get

$$d\omega_1 = -\omega_{11} \wedge \omega_1 \ ,$$

$$d\overline{\omega_2} = \quad \omega_{22} \wedge \overline{\omega_2} \ ,$$

$$d\omega_{12} = (\omega_{11} - \omega_{22}) \wedge \omega_{12} \ .$$

From this we conclude as above that $\Phi := \omega_1 \cdot \overline{\omega_2} \cdot \omega_{12}$ is independent of the chosen frames and defines a holomorphic cubic form on S. The zeros of Φ are called *isotropic* points (cf. [3]). Hence we get from Lemma 5:

Theorem 5.1c. *Let M be a 4-dimensional Kähler manifold with constant holomorphic sectional curvature and $f: S \to M$ harmonic. Then either f is totally isotropic or*

$$-3\,\chi(S) \geqq j + 2b \ ,$$

where j is the number of isotropic points on S_0 (with multiplicities).

In each of the three cases, we get

Corollary. *If S has genus 0, then $\Phi \equiv 0$. If S has genus 1, then f has no branch points and Φ has no zeros unless $\Phi \equiv 0$.*

In [4, 5] we got more refined results by looking closer to the local factors of Φ. All of these results carry over to the branched case. We want to discuss two of them.

Theorem 5.2. *Let M be 4-dimensional with constant sectional curvature c and S a closed Riemann surface. Let $f: S \to M$ be a conformal harmonic map such that $f^* TM$ is orientable, and let χ_{Nf} be the Euler number of Nf for some orientation. Then either f is totally circular or*

$$-2\chi(S) \geqq b + |\chi_{Nf}| \ .$$

Proof. The functions $|k_\pm| = (c - K \pm K_{Nf})^{1/2}$ introduced above are of absolute value type and satisfy

$$\Delta \log |k_\pm| = 2K - (\pm K_{Nf})$$

(cf. [5, Theorem 2] and [11, Theorem 1]), unless $\Phi \equiv 0$. Integrating this with respect to dv, we get from (8) and Lemma 4 (observe that $\Delta \log |k_\pm| dv = \Delta_0 \log |k_\pm| dv_0$):

$$2\chi_{Tf} = -N(|k_\pm|) \pm \chi_{Nf}$$

(compare (5) in [5]). The functions $|k_+|$ and $|k_-|$ may have poles of order not bigger than the branching order at that point. Hence, $N(|k_\pm|) \geqq n(|k_\pm|) - b$. Now from Theorem 4 we have $\chi_{Tf} = \chi(S) + b$, so we get the result.

Next, let (M, J, ds^2) be a compact (real) 4-dimensional Kähler manifold with constant holomorphic sectional curvature 4σ. As above, let Ω denote the Kähler 2-form on M. Then $3(\sigma/\pi)\Omega$ determines an integral cohomology class, the first Chern class $c_1(M)$. Thus for any smooth map $f: S \to M$, the number

$$d = (\sigma/\pi) \int_S f^*(\Omega)$$

is one-third integer, called the *degree* of f. If $M = \mathbb{C}P^2$, then $c_1(M)$ is 3-times the generator of $H^2(M; \mathbb{Z})$ and the degree is an integer. Our next result sharpens a theorem of Eells and Wood [3]:

Theorem 5.3. *Let M be a compact 4-dimensional Kähler manifold of constant holomorphic sectional curvature, S a compact Riemann surface and $f: S \to M$ a conformal harmonic map of degree d. Then either f is totally isotropic or*

$$-3\chi(S) \geqq |3d| + 2b \ .$$

Proof. If f is not totally isotropic, then the functions

$$b_\pm := |u_\pm^2 \cdot w| = (1 \pm C)(K_{Nf} - K + 2\sigma)^{1/2}$$

[see Case (c) above] are of absolute value type and satisfy

$$3\chi_{Tf} = -N(b_\pm) \pm 3d$$

(see (4.5) and (4.6) in [4]). The first factor of $b_{\pm}$ is bounded while the second one may have poles of order not bigger than the branching order at that point. Thus $N(b_{\pm}) \geqq n(b_{\pm}) - b$. Now the result follows from Theorem 4.

Acknowledgements. An essential part of the work was done while the second named author visited the University of Münster. We wish to express our thanks for hospitality and financial support from GMD, Germany, and CNPq, Brasil.

References

1. Chern, S.S.: On the minimal immersions of the two-sphere in a space of constant curvature. In: Gunning, R.C. (ed.) Problems in analysis. pp. 27–40. Princeton: Univ. Press 1970
2. Chern, S.S.: On surfaces of constant mean curvature in a three-dimensional space of constant curvature. Geometric dynamics. Lect. Notes Vol. 1007, 104–108. Berlin Heidelberg New York: Springer 1981
3. Eells, J., Wood, J.C.: Harmonic maps from surfaces to complex projective spaces. Adv. Math. **49**, 217–263 (1983)
4. Eschenburg, J.H., Guadalupe, I.V., Tribuzy, R.: The fundamental equations of minimal surfaces in $\mathbb{C}P^2$. Math. Ann. **270**, 571–598 (1985)
5. Eschenburg, J.H., Tribuzy, R.: Constant mean curvature surfaces in 4-space forms. Preprint Münster 1986. To appear in Rend. Semin. Mat. Univ. Padova
6. Gulliver, R.D., Osserman, R., Royden, H.L.: A theory of branched immersions of surfaces. Am. J. Math. **95**, 750–812 (1973)
7. Heinz, E., Hildebrandt, S.: The number of branch points of surfaces of bounded mean curvature. J. Differ. Geom. **4**, 227–235 (1970)
8. Hopf, H.: Differential geometry in the large. Stanford Lectures 1956. Lect. Notes Math. Vol. 1000. Berlin Heidelberg New York: Springer 1983
9. Lawson, B.: Lectures on minimal submanifolds, Vol. I. Boston: Publish or Perish 1980
10. Osserman, R.: A proof of the regularity everywhere of the classical solution to Plateau's problem. Ann. Math. **91**, 550–569 (1970)
11. Tribuzy, R., Valle Guadalupe, I.: Minimal immersions of surfaces into 4-dimensional space forms. Rend. Semin. Mat. Univ. Padova **73**, 1–13 (1985)