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COMPARISON THEOREMS AND HYPERSURFACES

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We compare the second fundamental forms of a family of parallel hypersurfaces in different Riemannian manifolds. This leads to new proofs for the distance and volume comparison theorems in Riemannian geometry. In particular, we get a new result on the volume of the set of points with distance $\leq r$ from a totally geodesic submanifold, for any r. The analytic prerequisite is the investigation of the Riccati type ODE which is satisfied by the second fundamental form of a parallel hypersurface family.

Contents

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0. Introduction

Comparison theorems in Riemannian geometry usually are derived by means of the index form and the minimizing property of the Jacobi fields (cf. [1,3,9,12]). In the present paper, we want to show a different approach: Sectional curvature controls the principal curvatures in a family of parallel hypersurfaces. The bigger the sectional curvatures, the smaller are these principal curvatures. E.g. the distance spheres in a space of positive curvature get concave for big radii, while they stay convex if the sectional curvature is nonpositive. This is expressed by a comparison theorem for solutions of the Riccati equation (ch.2) which among others implies the Rauch comparison theorems (ch.3). Similarly, the Ricci curvature controls to some extend the *mean* curvature of parallel hypersurfaces. So the Ricci curvature comparison theorems can be derived in the same fashion, in particular the Bishop-Gromov inequality for the volume of balls (ch.4). One advantage of our approach is that equality discussions become very easy since we estimate logarithmic derivatives. So we get an even simpler version of Shiohama's proof of Cheng's rigidity theorem [2,21]. In ch. 5 and 6 we discuss comparison theorems for tubes around submanifolds of higher codimension in our framework and prove the corresponding Bishop-Gromov type inequality, extending results of Heintze and Karcher [15].

The idea of using the matrix valued Riccati equation for comparison arguments was common in Sturmian theory for ODE's since long time (cf. [19,20]). In Riemannian geometry, it has been applied by L.W.Green [8] and later by several other authors [4,5, 14,17,10,11]. It was used in General Relativity by Hawking and Ellis [13]. Especially M. Gromov [11] has emphasized that the evolution of the principal curvatures of a parallel hypersurface family is the source of the comparison theorems. However, a systematic treatment of the Riemannian comparison theory via Riccati equation was still missing. The ODE theorems of ch.2 also apply to spacelike hypersurfaces in a Lorentzian manifold.

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1. Parallel hypersurface families

1.1 Let $(M, \langle\ ,\ \rangle)$ be a complete Riemannian manifold and let D be its Levi-Civita connection. For an open subset $M' \subset M$ let $f : M' \to \mathbb{R}$ be a C^2-function whose gradient $V = \nabla f$ has unit length. Then for any $x \in TM'$ we have

$$0 = x(\langle V,V\rangle) = 2\langle D_x V, V\rangle = 2\langle D_V V, x\rangle ,$$

thus

$$(1) \qquad D_V V = 0 .$$

So the gradient lines are unit speed geodesics. The function f can be considered as a Riemannian submersion onto an open subset of $\mathbb{R}$ or else, at least locally, it is the distance function of each of its level sets $S_t = \{f = t\}$, up to a constant and a sign. In particular, close level hypersurfaces have constant distance from each other Therefore $\{S_t ;\ t \in f(M')\}$ is called a parallel hypersurface family.

1.2 Let $B = DV$ be the Hessean tensor field of f. By (1), V is in the kernel of B, so we will often restrict ourselves to $B|V^{\perp}$. Since V is a unit normal field for each level hypersurface, $B|V^{\perp}$ can be viewed as the 2^{nd} fundamental tensor or Weingarten map of the level hypersurfaces.

For any vector field J on M' with $[J,V] = 0$ we have

(2) $$D_V J = B \cdot J \quad ,$$

and moreover, by (1) and (2),

$$(D_V B)J = D_V D_J V - B(D_V J) = R(V,J)V - B(B \cdot J) \quad ,$$

where R denotes the Riemannian curvature tensor. Thus putting $R_V = R(\ ,V)V$, we get the so called Riccati equation

(3) $$D_V B + B^2 + R_V = 0 \quad .$$

Thus differentiation of (2) yields

(4) $$D_V D_V J + R_V J = 0 \quad .$$

This shows that J is a Jacobi field along any integral curve of V which is clear also from (1) and $[J,V] = 0$. Hence the Jacobi equation (4) is broken up into the two first order equations (2) and (3). Though (3) is no longer linear, it is very useful for the comparison theory since B is a self adjoint tensor field.

1.3 Fix an integral curve $\gamma : I \to M'$ of V. Using parallel transport along γ, we may identify the normal bundle of γ with $I \times E$ where E is some fixed normal space $(\gamma'(t_0))^{\perp}$. Then $B(t) := B|_{\gamma(t)}$ and $R(t) := R_V|_{\gamma(t)}$ are considered as self adjoint endomorphisms of E satisfying

(3) $$B' + B^2 + R = 0 \quad .$$

<u>1.4</u> *Examples:* (a) Let $S \subset M$ be an oriented hypersurface with unit normal vector field N along S. This gives a trivialisation of the normal bundle $\phi : NS \to S \times \mathbb{R}$. Let M' be a neighborhood of S where $e := \exp|NS$ has a smooth inverse e^{-1}. Put $f = pr_2 \circ \phi \circ e^{-1}$. Then f is called <u>signed distance</u> of S on M'. The level set $S_t = f^{-1}(t)$ is a hypersurface with constant distance $|t|$ from S.

(b) Let $p \in M$ with cut locus $C(p)$. Let $U \subset T_pM$ be a neighborhood of the origin where the exponential map $e = \exp_p$ is a diffeomorphism. Let $M' = e(U) \setminus \{p\}$ and put $f(q) = \|e^{-1}(q)\|$. In particular, we may choose $e(U) = M \setminus C(p)$ where $C(p)$ is the cut locus of p. Then $f = d(\ ,p)$ and the level sets are the distance spheres centered at p. In this case, the vector field $X = f \cdot V$ extends smoothly to p with $(DX)_p = Id$. Thus, if γ is a unit speed geodesic with $\gamma(0) = p$ and $B(t) \in S(E)$ with $E = (\gamma'(0))^{\perp}$ as in 1.3, then $t \cdot B(t) \to Id$ as $t \to 0$.

(c) More generally, the point p may be replaced with a submanifold L of codimension ≥ 2. Then the level hypersurfaces are the tubes around L.

(d) Let M be a simply connected manifold without focal points and $f : M \to \mathbb{R}$ the Busemann function of a ray in M (cf. [5,14]). The level hypersurfaces are the horospheres corresponding to the ray.

2. The Riccati equation

2.1 Let E be a real n-dimensional vector space with euclidean inner product $\langle\ ,\ \rangle$. The space $S(E)$ of self adjoint endomorphisms inherits the inner product

$$\langle A,B\rangle = \operatorname{trace} A\cdot B$$

for $A,B \in S(E)$. We get a partial ordering $<$ ($\le$) on $S(E)$ by putting $A < B$ ($A \le B$) if $\langle Ax,x\rangle < \langle Bx,x\rangle$ ($\langle Ax,x\rangle \le \langle Bx,x\rangle$) for every $x \in E \setminus \{0\}$.

Now let $I \subset \mathbb{R}$ be an open real interval and $R : I \to S(E)$ a smooth curve. We consider the corresponding Riccati equation on $S(E)$:

$$(3) \qquad B' + B^2 + R = 0 .$$

Due to the non-linearity of (3), a solution B may have poles on I, more precisely, some eigenvalue of $B(t)$ may tend to $-\infty$ as $t \to t_1$, $t < t_1$, for some $t_1 \in I$, but not to $+\infty$, since $-B^2$ is negative semi-definite.

If B is non-singular, one may pass to the inverse $C = B^{-1}$ to treat poles as in example 1.4(b). Since $C' = -B^{-1}\cdot B'\cdot B^{-1}$, (3) is equivalent to the ODE

$$C' = Id + C\cdot R\cdot C .$$

Together with a solution B of (3), we investigate solutions $J : I \to E$ of the equation

$$(2) \qquad J' = B\cdot J .$$

Though B may have poles, J can be smoothly extended to the whole interval I since from (2) and (3) we get the linear ODE

$$(4) \qquad J'' + R\cdot J = 0 \quad .$$

E.g. if B satisfies $t\cdot B(t) \to Id$ as $t \to 0$ as in 1.4(b), the solutions of (2) are exactly the solutions J of (4) with $J(0) = 0$.

The case $n=1$ is of particular interest since the general case is reduced to this by taking traces as follows: Let us put

$$b = (\text{trace } B)/n \quad , \quad r = (\text{trace } R)/n \quad .$$

Then $S := B - b\cdot Id$ is the trace free part of B , and $\|S\|^2 = \|B\|^2 - n\cdot b^2$. Taking the trace of (3) we get

$$(3a) \qquad b' + b^2 + r_+ = 0 \quad ,$$

$$r_+ := r + \|S\|^2/n \quad .$$

Observe that r_+ remains bounded if B has a pole at $t_0=0$ with $t\cdot B(t) \to Id$ as $t \to 0$ as in 1.4(b)

Further, let $J_1,\dots,J_n$ be a basis of solutions of (2) and put $j = \|J_1 \wedge\dots\wedge J_n\|^{1/n}$. Since

$$(J_1 \wedge\dots\wedge J_n)' = \sum_{k=1}^{n} J_1 \wedge\dots\wedge B\cdot J_k \wedge\dots\wedge J_n$$

$$= (\text{trace } B)\cdot J_1 \wedge\dots\wedge J_n \quad ,$$

we get

$$(2a) \qquad j' = b\cdot j \quad .$$

<u>2.2</u> Consider first the case $n = 1$ and $R = k = \text{const}$, i.e. the equation

$$(3)_k \qquad b_k' + b_k^2 + k = 0 \quad .$$

Putting $\sigma = |k|^{1/2}$, we get the following families of solutions $b_k(t)$:

$k > 0$:		$\sigma\cdot\cot(\sigma(t-c))$	for $c < t < c + \pi/\sigma$, $c \in R$,
$k = 0$:	(i)	$1/(t-c)$	for $t > c$, $c \in R$,
	(ii)	0	for $t \in R$,
	(iii)	$1/(t-c)$	for $t < c$, $c \in R$,
$k < 0$:	(i)	$\sigma\cdot\coth(\sigma(t-c))$	for $t > c$, $c \in R$,
	(ii)	σ	for $t \in R$,
	(iii)	$\sigma\cdot\tanh(\sigma(t-c))$	for $t \in R$, $c \in R$,
	(iv)	$-\sigma$	for $t \in R$,
	(v)	$\sigma\cdot\coth(\sigma(t-c))$	for $t < c$, $c \in R$.

In particular, for any $k \in R$ the only solution with a pole at 0 is c_k/s_k where (s_k, c_k) is the solution of

$$s_k' = c_k , \quad c_k' = -k\cdot s_k , \quad s_k(0) = 0 , \quad c_k(0) = 1 .$$

If n is arbitrary and $R = k\cdot Id$ where Id denotes the identity on E , then $B = b_k\cdot Id$ are solutions of (3). By ch.1, these correspond to a family $\{S_t\}$ of umbilic parallel hypersurfaces in a simply connected Riemannian space Q_k of constant curvature k , and $b_k(t)$ is the mean curvature of S_t . For $k > 0$, this is the family of concentric spheres. For $k = 0$ there are three such families: concentric spheres with outer and inner normal vector and parallel hyperplanes. For $k < 0$, we have five families: concentric spheres and horospheres, both with outer and inner normal vector, and parallel hypersurfaces of a totally geodesic hypersurface.

2.3 Now let $I = (t_-, t_+)$ with $-\infty \le t_- < t_+ \le +\infty$. We consider two smooth curves R_1, $R_2 : \mathbb{R} \to S(E)$ and solutions B_j $(j=1,2)$ of the corresponding Riccati equations $(3)_j$. Fix an initial point $t_0 \in I$ and let $t_j > t_0$ be the first pole of B_j if there is some, otherwise put $t_j = t_+$.

Proposition 2.3 Suppose

(a) $R_1(t) \ge R_2(t)$ for all $t \in I$,

(b) $B_1(t_0) \le B_2(t_0)$.

Then we get

(c) $t_1 \le t_2$,

(d) $B_1(t) \le B_2(t)$ for $t_0 < t < t_1$.

If strong inequality holds in (a), then it holds also in (d).

Proof. Let us first assume strong inequality in (a). Suppose there exists $s > t_0$ such that $B_1(t) \le B_2(t)$ for $t_0 < t < s$. Then we certainly have $B_1(s) \le B_2(s)$.

Claim: $B_1(s) < B_2(s)$. Namely, otherwise $B_2(s) - B_1(s)$ is positive semi-definite with a nonzero kernel. Choose $x \in E \setminus \{0\}$ such that $B_2(s)x = B_1(s)x$. Then the function

$$g(t) := \langle (B_2 - B_1)(t)x,\ x\rangle$$

is nonnegative on $[t_0, s]$ with a zero at s. By the Riccati equations we have

$$g'(s) = \langle (B_1{}^2 - B_2{}^2)(s)x,\ x\rangle + \langle (R_1 - R_2)(s)x,\ x\rangle .$$

The second term is positive, and the first term vanishes since

$$\langle (B_1{}^2 - B_2{}^2)(s)x,\ x\rangle = \langle (B_1 - B_2)(s)(B_1(s)x),\ x\rangle$$
$$= \langle B_1(s)x,\ (B_2 - B_1)(s)x\rangle = 0 .$$

So $g'(s) > 0$ which is a contradiction to $g|[t_0,s] > 0$.

Thus the strong inequality (d) holds up to the first pole of B_1 or B_2. But since the eigenvalues at a pole can tend only to $-\infty$, (d) shows that $t_1 \le t_2$.

The initial assumption $s > t_0$ is clearly satisfied if strong inequality holds in (b). So this case is proved. By continuity, we get (d) also if only the weak inequality holds in (b). Thus in this case we also have such $s > t_0$, and the argument above shows the *strong* inequality in (d).

The assertion for the weak inequality in (a) follows by continuity.

<u>Remark 1.</u> If $R_1 = R_2$ and strong inequality holds in (b), then strong inequality holds also in (c) (cf. [6], 3.1). We do not know whether this is also true if $R_1 \ge R_2$.

<u>Remark 2.</u> If $n = 1$, we can discuss the equality case in the previous proposition: If $B_1(s) = B_2(s)$ for some $s \in (t_0, t_1)$, then on $[0,s]$ we have $B_1 = B_2$ and hence $R_1 = R_2$.

Namely, if $B_1 < B_2$ on (t_0,s) for some $s \in (t_0,t_1)$, then the Riccati equation implies

$$(\log(B_2-B_1))' \ge -(B_2 + B_1)$$

on (t_0,s). Hence $\log(B_2-B_1)$ is bounded from below on $[r,s]$ for every $r \in (t_0,s)$ which implies $B_1(s) < B_2(s)$.

2.4 Proposition. *Let $R_1 \geq R_2$ and B_1, B_2 solutions of the corresponding Riccati equations which are invertible near 0 with $B_j(t)^{-1} \to 0$ as $t \to 0$, $j = 1,2$. Let $t_j > 0$ be the first pole of B_j. Then $t_1 \leq t_2$ and $B_1 \leq B_2$ on $(0,t_1)$. If $R_1 > R_2$, the strong inequality holds.*

Proof. Let $C_j = B_j^{-1}$. Then C_j solves $C_j' = Id + C_j \cdot R_j \cdot C_j$ with initial condition $C_j(0) = 0$. Differentiating, we get

$$C_j'(0) = Id, \quad C_j^{(2)}(0) = 0, \quad C_j^{(3)}(0) = 2 R_j .$$

Suppose first $R_1 > R_2$. Then the leading terms of the Taylor expansions of C_2 and of $C_1 - C_2$ near 0 are positive definite, hence $C_1(t) > C_2(t) > 0$ for $t > 0$ small enough. Therefore $B_1(t) < B_2(t)$ by the subsequent lemma. Thus choosing $t_0 > 0$ small enough, we may apply Prop. 2.3 and get the result. The case $R_1 \geq R_2$ follows by continuity.

Lemma. *Let $F, G \in S(E)$ with $F > G > 0$. Then $F^{-1} < G^{-1}$.*

Proof. The positive definite self adjoint endomorphisms form a convex cone $P \subset S(E)$. Therefore, the endomorphisms

$$F_t := t \cdot F + (1-t) \cdot 2 \cdot Id \quad ,$$

$$G_t := t \cdot G + (1-t) \cdot Id \quad ,$$

$$F_t - G_t = t \cdot (F - G) + (1-t) \cdot Id$$

are positive definite for any $t \in [0,1]$. In particular,

$$D_t := G_t^{-1} - F_t^{-1} = G_t^{-1} \cdot (F_t - G_t) \cdot F_t^{-1}$$

is self adjoint and invertible. Since $D_0 = \frac{1}{2} Id$ is in P, the same is true for $D_1 = G^{-1} - F^{-1}$ since ∂P contains no invertible endomorphisms. So $F^{-1} < G^{-1}$ which finishes the proof.

2.5 For $A \in S(E)$ let $\lambda_+(A)$ denote the highest and $\lambda_-(A)$ the lowest eigenvalue. For $A_1, A_2 \in S(E)$ we have $\lambda_+(A_1) \le \lambda_-(A_2)$ if and only if $A_1 \le D^{-1} \cdot A_2 \cdot D$ for every rotation $D \in O(E)$.

Proposition 2.5 Let $B_1, B_2 : I \to S(E)$ such that $\lambda_+(B_1) \le \lambda_-(B_2)$ everywhere. Let $J_1, J_2 : I \to E$ be nonzero solutions of $J_j' = B_j \cdot J_j$ $(j = 1,2)$. Then $\|J_1\|/\|J_2\|$ is monotonously decreasing. Moreover, if $\|J_1\|/\|J_2\|$ is constant on a sub-interval $I' \subset I$, then on I' we have $\lambda_+(B_1) = \lambda_-(B_2)$, and the corresponding eigenspaces contain $J_1/\|J_1\|$ resp. $J_2/\|J_2\|$ which are constant on I'.

Proof. Since J_j satisfies a 1st order equation, $\|J_j\|$ is nowhere zero and hence smooth. Now

$$(\log \|J_1\|)' = \langle J_1', J_1\rangle/\|J_1\|^2 = \langle B_1 \cdot J_1, J_1\rangle/\|J_1\|^2$$

$$\le \lambda_+(B_1) \le \lambda_-(B_2) \le \langle B_2 \cdot J_2, J_2\rangle/\|J_2\|^2 = (\log \|J_2\|)' .$$

Hence $(\log(\|J_1\|/\|J_2\|))' \le 0$ which implies that $\|J_1\|/\|J_2\|$ is monotonously decreasing. If equality holds on a subinterval I', the computation above shows that $\lambda_+(B_1) = \lambda_-(B_2)$ and that $J_j(t)$ are corresponding eigenvectors of $B_j(t)$ for $t \in I'$. Since $J_j' = B_j \cdot J_j$, J_j' and J_j are linearly dependent on I' which shows that $J_j/\|J_j\|$ is constant along I'.

3. Sectional curvature comparison theorems

Let S be a hypersurface in a Riemannian manifold with given unit normal vector field N along S. To fix signs, the eigenvalues of the Weingarten map DN will be called principal curvatures of S, and the mean value of those at any point the mean curvature of S.

Throughout this chapter, let M_1 and M_2 be complete Riemannian manifolds of dimension $n+1$ whose sectional curvature functions K_1 and K_2 satisfy

$$\inf K_1 \geq \sup K_2$$

on the subsets which we consider. The index j will always refer to $\{1,2\}$.

3.1 Theorem. *Let $S_j \subset M_j$ be hypersurfaces with unit normal fields N_j and let $p_j \in S_j$. Suppose that the largest principal curvature of S_1 at p_1 is not bigger than the least principal curvature of S_2 at p_2. Then the same is true for the parallel hypersurfaces $S_{j,t}$ at $\exp(t \cdot N_j(p_j))$ for $0 \leq t \leq t_1$ where t_1 denotes the focal distance of S_1 at p_1.*

Proof. Let $f_j : M_j' \to \mathbb{R}$ be the signed distance of S_j near p_j with level hypersurfaces $S_{j,t}$ (cf. 1.4(a)). Fix the geodesics $\gamma_j(t) = \exp(t \cdot N_j(p_j))$. Put $E_j = T_{p_j} S_j = (\gamma_j'(0))^{\perp}$ and let $B_j(t) \in S(E_j)$ as in 1.3. Then $B_j(t)$ is the Weingarten map of $S_{j,t}$ at $\gamma_j(t)$, up to parallel transport along γ_j By assumption,

$$B_1(0) \leq \iota^{-1} \circ B_2(0) \circ \iota$$

for every linear isometry $\iota : E_1 \to E_2$. From the curvature assumption and 2.3 we get

$$B_1(t) \le \iota^{-1} \circ B_2(t) \circ \iota$$

for $t \ge 0$ (up to the first focal point of S_1 along γ_1) and therefore $\lambda_+(B_1(t)) \le \lambda_-(B_2(t))$ since ι was arbitrary (cf. 2.5). This proves the theorem.

__3.2__ __Theorem.__ __The principal curvatures of any smooth part of a distance sphere in__ M_1 __with outer normal vector are not bigger than those of a distance sphere with the same radius in__ M_2.

__Proof.__ Let $p_j \in M_j$ and $f_j = d(\ ,p_j)$ as in 1.4(b). Fix an arbitrary unit speed geodesic γ_j with $\gamma_j(0) = p_j$. Put $E_j = (\gamma_j'(0))^{\perp}$ and for $t > 0$ let $B_j(t) \in S(E_j)$ as in 1.3. As long as $\gamma_j|[0,t]$ contains no cut points of p_j, $B_j(t)$ is the Weingarten map of the distance sphere $\Sigma_t(p_j)$ at $\gamma_j(t)$, up to parallel displacement. By the curvature assumption and 2.4,

$$B_1(t) \le \iota^{-1} \circ B_2(t) \circ \iota$$

for every isometry $\iota : E_1 \to E_2$ and thus $\lambda_+(B_1(t)) \le \lambda_-(B_2(t))$ (see 2.5) which finishes the proof.

__3.3__ __Theorem.__ (Rauch/Berger) __Let__ γ_j __be unit speed geodesics in__ M_j __and__ J_j __Jacobi fields along__ γ_j __with__ $J_j \perp \gamma_j$ __and__

(a) $J_j(0) = 0$, $\|J_1'(0)\| = \|J_2'(0)\| \ne 0$

__or__

(b) $J_j'(0) = 0$, $\|J_1(0)\| = \|J_2(0)\| \ne 0$.

Then $\|J_1(t)\| \le \|J_2(t)\|$ for $t \ge 0$ up to the first conjugate point (case (a)) or focal point (case (b) of γ_1. If equality holds for some $t_0 > 0$, it holds on $[0,t_0]$ and $J_j/\|J_j\|$ is parallel along $\gamma_j|[0,t_0]$ for $j = 1,2$.

Proof. *Case (a):* Let t_j be the first conjugate point of γ_j and put $p_j = \gamma_j(0)$, $v_j = \gamma_j'(0)$. Then in $T_{p_j}M_j$ there is an open neighborhood U_j of 0 containing $t \cdot v_j$ for $0 \le t < t_j$ where $\exp_{p_j}$ has a smooth inverse. Let f_j be as in 1.4(b). We put $E_j = (v_j)^\perp$ and $B_j(t) \in S(E_j)$ as in 1.3. Now J_j (considered as a curve in E_j) solves $(2)_j$ since it satisfies $(4)_j$ with $J_j(0) = 0$ and $t \cdot B_j(t) \to Id$ as $t \to 0$ (cf. 2.1). By de l'Hopital's rule, $\|J_1(t)\|/\|J_2(t)\| \to 1$ as $t \to 0$. Now the result follows from 2.4 and 2.5.

Case (b): This is immediate from the following general comparison theorem for Jacobi fields:

3.4 Theorem. Let γ_j be a unit speed geodesic in M_j and $J_j \perp \gamma_j'$ a Jacobi field along γ_j with

$$\|J_1(0)\| = \|J_2(0)\| \neq 0 \quad , \quad J_j'(0) = A_j \cdot J_j(0)$$

for some $A_j \in S(E_j)$, $E_j = (\gamma_j'(0))^\perp$ with $\lambda_+(A_1) \le \lambda_-(A_2)$. Then $\|J_1(t)\| \le \|J_2(t)\|$ for $0 \le t \le t_1$ where t_1 is the smallest positive zero of all Jacobi fields J along γ_1 with $J'(0) = A_1 \cdot J(0) \neq 0$. Equality at $t_0 \in (0,t_1)$ implies equality along $[0,t_0]$ and $J_j/\|J_j\|$ is parallel along $\gamma_j|[0,t_0]$.

<u>Proof.</u> Let $p_j = \gamma_j(0)$ and $S_j' \subset T_{p_j}M_j$ a hypersurface through the origin with $T_0S_j' = E_j$ and Weingarten map A_j at 0 with respect to the normal vector $v_j = \gamma_j'(0)$. E.g. we may choose $S_j' = \{x \in T_{p_j}M_j \,;\, \langle A_j \cdot x, x\rangle = 2\,\langle v_j, x\rangle\}$. Let $U_j \subset T_{p_j}M_j$ be an open neighborhood of the origin where the exponential map is diffeomorphic. Then $S_j := \exp(S_j' \cap U_j)$ is a smooth hypersurface with Weingarten map A_j with respect to v_j. Let $f_j : M_j' \to R$ be the signed distance function of S_j as in 1.4(a). If U_j is small enough, M_j' contains $\gamma_j([0,t_j))$ where t_j is the first focal point of S_j along γ_j. Put $B_j(t) \in S(E_j)$ as in 1.3. Then $B_j(0) = A_j$ and from the initial values we see that J_j solves $J_j' = B_j \cdot J_j$ (up to parallel displacement along γ_j). Now the result follows from 2.3 and 2.5.

<u>Remark.</u> The Rauch type theorem 8.13 in [11] is incorrectly stated. The theorem above is the corrected version.

4. Lower Ricci curvature bounds

Throughout this section, let M be a complete Riemannian manifold of dimension n+1 with Ricci curvature bounded from below, more precisely $\mathrm{Ric}(v) \geq k \cdot n$ for some constant $k \in R$ and any unit tangent vector v. As in 2.2, let b_k be a solution of

$$(3)_k \qquad b_k' + b_k^2 + k = 0 .$$

4.1 Theorem. *Let $S \subset M$ be a hypersurface with unit normal field N and $p \in S$. Let $b(t)$ be the mean curvature of the parallel hypersurface S_t of S at $\gamma(t) = \exp(t \cdot N(p))$. Let b_k be the solution of $(3)_k$ with $b_k(0) = b(0)$. Then*

$$b(t) \le b_k(t)$$

for all $t > 0$ up to the first focal point of S along γ. For $t < 0$, the opposite inequality holds.

Proof (cf. [7]). Let f be the signed distance of S as in 1.4(a) and $B(t)$ as in 1.3. Then $b(t) = \text{trace } B(t)/n$ and hence b solves equation (3a) in 2.1. So the inequality for $t > 0$ follows from 2.3 (one-dimensional case) since $r_+ \ge r \ge k$. The opposite inequality for $t < 0$ follows by reversing the normal field N.

4.2 Theorem (cf [7]). *The mean curvature of any smooth part of a distance sphere of radius $t > 0$ in M is bounded from above by $c_k(t)/s_k(t)$.*

Proof. Fix $p \in M$ and some unit speed geodesic γ with $\gamma(0) = p$. Let $f = d(\ ,p)$ as in 1.4(b) and $B(t)$ as in 1.3. Then $b(t) := \text{trace } B(t)/n$ is the mean curvature of the distance sphere $\Sigma_t(p)$ at $\gamma(t)$ if $\gamma|(0,t]$ contains no cut points of p. So b solves (3a) in 2.1 with a pole at 0. So the result follows from the one-dimensional case of 2.4 and 2.2.

4.3 For $p \in M$ let $B_r(p)$ denote the open ball of radius r centered at p and $V_p(r)$ the volume of $B_r(p)$. Moreover, let $V(r)$ be the volume of a ball of radius r in the simply connected space of constant sectional curvature k which we denote by Q_k.

Theorem 4.3 (Bishop-Gromov inequality, cf. [12]) *For any* $p \in M$ *and* $0 < r < R$ *we have*

$$V_p(r)/V(r) \geq V_p(R)/V(R) \quad .$$

Equality holds for some $0 < r < R \leq \mathrm{diam}(M)$ *if and only if* $B_R(p)$ *is isometric to a ball of radius* R *in* Q_k.

Proof. Fix some $p \in M$ and $r \in (0,d)$ where d is the diameter of M. Put $E = T_pM$ and $e = \exp_p : E \to M$. Let S be the unit sphere in E. For every $v \in S$ let γ_v be the geodesic with $\gamma_v'(0) = v$. Put

$$\mathrm{cut}(v) = \sup \{t < 0 \; ; \; \gamma_v|[0,t] \text{ is shortest}\}$$

and

$$C = \{t \cdot v \; ; \; v \in S \; , \; 0 \leq t \leq \mathrm{cut}(v)\} \quad .$$

Let $B_r \subset E$ be the ball of radius r with center 0. Then

$$V_p(r) = \int_{B_r \cap C} |\det De_u| \, du = \int_S \int_0^{r(v)} |\det De_{t\cdot v}| \, t^n \, dt \, dv$$

where du and dv denote the volume elements of E and S, and $r(v) := \min(r, \mathrm{cut}(v))$. Let $(v, e_1, \ldots, e_n)$ be an orthonormal basis of E. Put $J_i(t) = De_{t\cdot v}(t \cdot e_i)$ and

$$j_v(t) = \| J_1(t) \wedge \ldots \wedge J_n(t) \|^{1/n}$$

for $0 \leq t \leq \mathrm{cut}(v)$. Then

$$|\det De_{t\cdot v}| = j_v(t)^n / t^n \quad .$$

Thus putting $j_v(t) = 0$ for $t > \mathrm{cut}(v)$, we get

$$V_p(r) = \int_0^r \int_S j_v(t)^n \, dv \, dt \quad .$$

On the other hand, if we put $f = d(\ ,p)$ and $B(t)$ as in 1.3 with $\gamma = \gamma_v$, then $j_v|[0, \mathrm{cut}(v)]$ solves (2a) in 2.1. Thus by 2.4 and 2.5 (one-dimensional case), j_v/j is monotonously decreasing on $[0, \mathrm{cut}(v)]$ for every j with $j' = b_k \cdot j$ where $b_k = c_k/s_k$. If we choose $j = s_k$, then $j_v(t)/j(t) \to 1$ as $t \to 0$ since $j_v(0) = 0$, $j_v'(0) = 1$. Now the function

$$q_v = (j_v/s_k)^n$$

is (weakly) monotonously decreasing on $[0,r)$ with $q_v(t) \to 1$ as $t \to 0$. Putting $w = (s_k)^n$ we get

$$V_p(r) = \int_S \int_0^r q_v(t) \cdot w(t) \, dt \, dv \, ,$$

$$V(r) = \int_S \int_0^r w(t) \, dt \, dv \, .$$

Hence

$$V_p(r)/V(r) = \int_S m_v(r) \, dv \, / \, \mathrm{vol}(S)$$

where

$$m_v(r) = \int_0^r q_v(t) w(t) dt \, / \int_0^r w(t) dt$$

is a weighted mean of q_v on the interval $[0,r)$. Therefore, m_v is also monotonously decreasing for $0 \le r \le d$. Moreover, $m_v(r) = m_v(R)$ for some $r < R \le d$ implies that $q_v \equiv 1$ on $[0,R)$ since otherwise, due to the monotonicity, the mean over $[0,r]$ would be strictly larger than the mean over $[r,R)$. So V_p/V is monotonously decreasing which shows the inequality. Further, $V_p(r)/V(r) = V_p(R)/V(R)$ for some $0 < r < R \le d$

implies that $j_v = s_k$ on $[0,R)$ for any fixed $v \in S$, hence $b = b_k$ and $r_+ = r = k$ in (3a), ch. 2.1. So $B(t) = b(t)\cdot Id$ and by (3), $R(t) = k\cdot Id$ on $(v)^\perp$. Thus $J_i = s_k \cdot e_i$, up to parallel displacement along γ_v. This shows that for any $\bar{p} \in Q_k$ the map $\exp_{\bar{p}} \circ e^{-1}$ is an isometry from $B_R(p)$ onto $B_R(\bar{p}) \subset Q_k$.

4.4 **Theorem** (Myers - Cheng, cf. [2,21]) *If* $k > 0$, *then* M *is compact with diameter* $d \le \pi/\sqrt{k}$. *Equality holds if and only if* M *is isometric to* Q_k *which is the* (n+1)-*sphere of radius* $1/\sqrt{k}$.

Proof. Fix $p \in M$ and let $f = d(\ ,p)$ as in 1.4(b). For any unit speed geodesic γ with $\gamma(0) = p$ let $B(t)$ be as in 1.3. By 2.4, the first positive pole t_1 of $b := \text{trace } B/n$ comes not later than that of c_k/s_k which is $\pi/\sqrt{k}$. So B has a pole at $t_1 \le \pi/\sqrt{k}$ which implies that some nonzero solution of (2) vanishes at t_1, i.e. t_1 is a conjugate point. Therefore, no geodesic of length bigger than $\pi/\sqrt{k}$ can be shortest which shows the compactness and the inequality.

Now suppose equality and fix two points $p_1, p_2 \in M$ with maximal distance $R = \pi/\sqrt{k}$. Put $r = R/2$. Then $M = B_R(p_j)$ for $j = 1,2$ and $B_r(p_1) \cap B_r(p_2) = \phi$. On the other hand, by 4.3

$$\text{vol}(M)/\text{vol}(B_r(p_j)) \le V(R)/V(r) = 2 ,$$

thus

$$\text{vol}\,(B_r(p_1) \cup B_r(p_2)) \ge \text{vol}(M) .$$

This implies equality in 4.3, and therefore $B_R(p_1)$ is isometric to a ball of radius R in Q_k which is the complement of a point. This shows that M is isometric to Q_k.

5. Riccati equation with singular initial values.

5.1 As in ch.2, let E be a real n-dimensional inner product space and B a solution of

(3) $$B' + B^2 + R = 0$$

on $S(E)$ for given $R : \mathbb{R} \to S(E)$. Assume that B has a pole at $t = 0$. By (3), this can be only a pole of 1^{st} order. Thus we put

$$B(t) = t^{-1} \cdot F + G + t \cdot H + O(t^2)$$

with $F,G,H \in S(E)$. It follows from (3) that

(a) $$F^2 = F ,$$

(b) $$F \odot G = 0 ,$$

where $X \odot Y := X \cdot Y + Y \cdot X$ for $X,Y \in S(E)$. Thus by (a), F is the orthogonal projection P_N onto its image $N = \mathrm{im}(F)$. Let $T = N^\perp$. Then by (b), $\mathrm{im}(G) \subset T$ and $N \subset \ker(G)$, hence $G = A \circ P_T$ for some $A \in S(T)$ (we omit the inclusion $T \subset E$). Thus we get

(5) $$B(t) = t^{-1} \cdot P_N + A \circ P_T + O(t) .$$

(3)&(5) is a singular initial value problem. The first order term H is already determined by (3)&(5): We get

(6) $$H + P_N \odot H + A^2 \circ P_T + R(0) = 0 .$$

An easy case is $T = 0$. Then $B(t) = t^{-1} \cdot \mathrm{Id} - t \cdot R(0)/3 + O(t^2)$ (see 2.4).

5.2 We want to show that the solution of (3)&(5) is uniquely determined and depends continuously on A and R. The easiest way is to pass to the corresponding Jacobi equation. For any solution $B : I \to S(E)$ of (3) let $Y : I \to \mathrm{End}(E)$ be a solution of

(2) $\qquad Y' = B \cdot Y$.

Then Y is also a solution of the Jacobi equation

(4) $\qquad Y'' + R \cdot Y = 0$

and hence Y has a smooth extension to all of $\mathbb{R}$. Now B satisfies (5) if and only if $P_N \circ Y(0) = 0$, $P_T \circ Y'(0) = A \circ Y(0)$. Thus (2) is satisfied by the solution Y of

(7) $\qquad Y'' + R \cdot Y = 0 , \quad Y(0) = P_T , \quad Y'(0) = P_N + A \circ P_T ,$

and since $Y(t)$ is invertible for small $t > 0$, we have $B = Y' \cdot Y^{-1}$. So B is uniquely determined and depends continuously on A and R .

<u>5.3</u> Now let R_1 , R_2 : $\mathbb{R} \to S(E)$ and $A_1, A_2 \in S(T)$. Let B_j be a solution of (3)&(5) with $R = R_j$, $A = A_j$ for $j \in \{1,2\}$. Let $D = B_2 - B_1$.

<u>Lemma 5.3</u> <u>If</u> $R_1(0) > R_2(0)$ <u>and</u> $A_1 < A_2$ <u>then</u> $D(t) > 0$ <u>for small</u> $t > 0$.

<u>Proof.</u> By (5), D has no pole at 0 and $\langle D(0)x,x \rangle \geq 0$ with equality only if $x \in N$. On the other hand, for $x \in N$ we get from (6):

$$3 \langle D'(0)x,x \rangle = \langle (D'(0) + P_N \odot D'(0))x, x \rangle$$
$$= \langle (R_1(0)-R_2(0))x,x \rangle > 0 .$$

Thus for any $x \in E$ with $\|x\| = 1$ we have $\langle D(t)x,x \rangle > 0$ for sufficiently small $t > 0$. Since the unit sphere in E is compact, this finishes the proof.

5.4 As above let $t_j \in (0,\infty]$ be the first positive pole of B_j or ∞ if such a pole does not exist.

Proposition 5.4 *If* $R_1 \geq R_2$ *and* $A_1 \leq A_2$, *then* $t_1 \leq t_2$ *and* $B_1 \leq B_2$ *on* $(0,t_1)$.

Proof. Assume first the strong inequalities $R_1(0) > R_2(0)$ and $A_1 < A_2$. Then $B_1(t_0) < B_2(t_0)$ for small $t_0 > 0$, by 5.3. Now the result follows from 2.3. The general case is true by continuity (see 5.2).

6. Volume and distance of totally geodesic submanifolds

6.1 Let M be a Riemannian manifold and $L \subset M$ a submanifold of codimension ≥ 2 . Let $f = d(\ ,L)$ on $M' = M \setminus (C(L) \cup L)$ as in 1.4(c). Let γ be any unit speed geodesic with $p := \gamma(0) \in L$ and $v := \gamma'(0) \perp L$. Put $E = (\gamma'(0))^{\perp}$ and $B(t) \in S(E)$ as in 1.6. Then B satisfies (5) where $T = T_pL$, $N = (T_pL)^{\perp}$ and where $A = A_v$ is the 2^{nd} fundamental tensor of L with respect to v , i.e. $\langle A_v x,y\rangle = - \langle D_x y,v\rangle$ for $x,y \in T$. Thus immediately from 5.4 we get a generalization of Theorem 3.2 to tubes around totally geodesic submanifolds.

6.2 Moreover, from 5.4 we can derive a Bishop-Gromov inequality for tubes around a totally geodesic submanifold $L \subset M$ which extends a result of Heintze and Karcher [15]. To do this, we first must exhibit the model spaces which replace the standard spaces Q_k used in 4.3. Let $\pi : E \to L$ be a vector bundle with covariant derivative ∇. This determines a smooth decomposition $T_vE = \underline{V}_v \oplus \underline{H}_v$ where $\underline{V}_v$ contains the tangent vectors of $E_{\pi v}$ ("vertical vectors") and $\underline{H}_v$ those of parallel vector fields along curves in L ("horizontal vectors"). We will often identify $\underline{V}_v$ and $E_{\pi v}$. Let us denote vertical vector fields by A,B and let X,Y be horizontal vector fields which are "basic", i.e. there are vector fields X',Y' on L with $D\pi \circ X = X' \circ \pi$, $D\pi \circ Y = Y' \circ \pi$. Assume that $[X',Y'] = 0$. Since the integral curves of X and Y are parallel vector fields along the integral curves of X' and Y', we get that [X,Y] is vertical and

$$[X,Y](v) = R^E(X',Y')v$$

for any $v \in E$, where R^E is the curvature tensor corresponding to ∇ (e.g. compare [22], p. 5-41 and [9], p. 54). Moreover, if A is parallel along the integral curves of X', then $[X,A] = 0$. Further recall that [A,B] is always vertical since $\underline{V}$ is integrable.

<u>6.3</u> Now assume that L is Riemannian and complete with finite volume and E is equipped with a fibre metric (also denoted by $\| \ \|$) such that ∇ is a metric connection. Let $k \in \mathbb{R}$ and s_k, c_k as defined in 2.2. We define a Riemannian metric g_k on $E_k = \{v \in E;\ \|v\| < r_0\}$ with $r_0 = \pi/2\sqrt{k}$ for $k > 0$ and $r_0 = \infty$ otherwise as follows: On the fibres $E_{k,p}$ we take the metric $dr^2 + s_k(r)^2\|d\omega\|^2$ of constant curvature k where $r(v) = \|v\|$ and $\omega(v) = v/\|v\|$. Further, we declare $\underline{H}$ and $\underline{V}$ to be perpendicular and put $\|X_v\|_k = c_k(\|v\|)\cdot\|X'_{\pi v}\|$. Then the sphere bundle $S_{k,r} = \{v \in E_k;\ \|v\| = r\}$ with the induced metric becomes a Riemannian

submersion over $(L, c_k(r)\cdot\| \ \|)$. Putting $B_{k,r} = \{\|v\| \leq r\}$, we get

$$V(r) := \mathrm{Vol}\ B_{k,r} = \int_0^r \mathrm{vol}\ S_{k,r}\ dr = \omega_h \cdot \mathrm{vol}(L) \cdot \int_0^r s_k{}^h c_k{}^m(t)\ dt$$

where m is the dimension of L and $h+1$ the fibre dimension of E and ω_h the volume of the euclidean unit sphere of dimension h. Further, E_k has the following properties:

<u>Lemma 6.3</u> <u>The zero section, the fibres and the immersions</u> $F : (-r_0,r_0) \times R \to E_k$, $F(s,t) = s\cdot a(t)$ <u>for any parallel section</u> a <u>of</u> E <u>along any geodesic</u> $\gamma : R \to L$ <u>are totally geodesic.</u>

<u>Proof.</u> The zero section is clearly totally geodesic since it is the fixed point set of the isometry $v \to -v$ on E_k. Now choose A, B, X as above such that A, B are parallel along the integral curves of X'. Put $\langle\ ,\ \rangle_k = g_k$ and D the Levi-Civita connection of g_k. Then $\langle A,B\rangle_k$ is constant along the integral curves of X, and by 6.2 and the Levi-Civita formula we get $\langle D_A B,X\rangle_k = 0$. Thus the fibres are totally geodesic. Now choose A,X such that $A\circ f = \partial f/\partial s$ and $X\circ f = \partial f/\partial t$ and $B \perp A$, $Y \perp X$ with $[X',Y'] = 0$. Then $\langle D_A X,B\rangle_k = -\langle D_A B,X\rangle_k = 0$ as above, and $\langle D_A X,Y\rangle_k = 0$ by the Levi-Civita formula since

$$\langle A,[X,Y]\rangle_k \circ F = c_k(\|F\|)^2 \cdot \langle (1/s)F,\ R^E(X'_\gamma,Y'_\gamma)F\rangle = 0 ,$$

due to the skew symmetry of R^E. Moreover, $D_A A \circ F = 0$ since the fibres are totally geodesic. Further, since $[B,X] = 0$, we have

$$-\langle D_X X,B\rangle_k = \langle X,D_X B\rangle_k = \langle X,D_B X\rangle_k = \tfrac{1}{2}\ B\langle X,X\rangle_k .$$

But on any fibre, the gradient of the function $\langle X,X\rangle_k$ is radial, and $B \perp A$ is orthogonal to the radial direction along F . Thus we also get $\langle D_X X,B\rangle_k \circ F = 0$. Finally, the horizontal vector fields X and Y are tangent to the submanifolds $S_{k,r}$ which are Riemannian submersions over L. Thus by O'Neill's formula [18] we have $\langle D_X X, Y\rangle_k = c_k(\|\pi\|)^2 \cdot \langle D_{X'} X', Y'\rangle \circ \pi$ which vanishes along F since $X' \circ \gamma$ is the tangent field of the geodesic γ. So F is a totally geodesic immersion.

It follows that perpendicular to any geodesic in E_k which starts orthogonally from the zero section L, there is a basis of Jacobi fields $J_1, \ldots, J_{m+h}$ and a parallel orthonormal basis $E_1, \ldots, E_{m+h}$ with $J_i = c_k \cdot E_i$ for $1 \leq i \leq m$ and $J_\alpha = s_k \cdot E_\alpha$ for $m+1 \leq \alpha \leq m+h$. Thus all "radial curvatures" equal k, i.e. we have $K(\sigma) = k$ for any plane σ containing a tangent vector of a shortest geodesic from the base point of σ to L ("radial plane").

<u>6.4</u> Let M be a complete Riemannian manifold and $L \subset M$ a closed totally geodesic submanifold of finite volume with dimension m and codimension $h+1$. Put

$$B_r(L) = \{p \in M \; ; \; d(p,L) < r\} .$$

Let $V_L(r)$ and $V(r)$ denote the volumes of $B_r(L)$ and $B_{k,r}$ (as defined in 6.3). In the following theorem, the inequality for $k=0$ was already proved by Kasue [16].

<u>Theorem 6.4</u> <u>Let</u> $L \subset M$ <u>be a closed totally geodesic submanifold of finite volume. Suppose that</u> $K(\sigma) \geq k$ <u>for any radial plane</u> σ. <u>Then for any radii</u> $0 < r < R$ <u>we have</u>

$$V_L(r)/V(r) \geq V_L(R)/V(R) .$$

Equality holds for some $0 < r < R < r_0(k)$ _if and only if_ $B_R(L)$ _is isometric to_ $B_{k,R}$.

Proof. We proceed as in 4.3. Let E be the normal bundle and S the unit normal bundle of L . Put $e = \exp|E : E \to M$. We equip E with the Riemannian metric g_0 (see 6.3). Let $\mathrm{cut}(v)$ for $v \in S$ and $C \subset E$ be as in 4.3, and let $B_r \subset E$ be the set of normal vectors of length smaller than r . For a fixed unit normal vector v at $p \in L$ choose an orthonormal basis $e_1,\dots,e_m$ of T_pL and extend it to an orthonormal basis $e_1,\dots,e_n,v$ of T_pM . For $a \in \{1,\dots,n\}$ let J_a be the Jacobi fields along $\gamma := \gamma_v$ with initial values

$$J_i(0) = e_i \ , \quad J_i'(0) = 0 \quad \text{for } i = 1,\dots,m \ ,$$

$$J_\alpha(0) = 0 \ , \quad J_\alpha'(0) = e_\alpha \quad \text{for } \alpha = m+1,\dots n \ .$$

Define $j_v = \|J_1 \wedge \dots \wedge J_n\|^{1/n}$ on $[0, \mathrm{cut}(v)]$ and $j_v = 0$ on $(\mathrm{cut}(v), \infty)$ as in 4.3. Then as before,

$$V_L(r) = \int_{B_r \cap C} |\det De_u| \, du = \int_0^r \int_S j_v(t)^n \, dv \, dt \ .$$

On the other hand, if we put $f = d(\ ,L)$ and let $B(t)$ as in 1.3 then $j_v' = b \cdot j_v$ on $[0, \mathrm{cut}(v)]$ with $b = \mathrm{trace}(B)/n$. Due to the curvature assumption and Proposition 5.4, we have $B \le B_k$ where

$$B_k(t)e_i = (\log c_k)'(t) \cdot e_i \quad \text{for } i = 1,\dots,m \ ,$$

$$B_k(t)e_\alpha = (\log s_k)'(t) \cdot e_\alpha \quad \text{for } \alpha = m+1,\dots,n \ .$$

Then from the 1-dimensional case of 2.5 we conclude that j_v/j is monotonously decreasing in $[0, \mathrm{cut}(v)]$ (and constant beyond $\mathrm{cut}(v)$), where $j = (c_k{}^m \cdot s_k{}^h)^{1/n}$. The remainder of the proof is strictly analogous to 4.3. The equality discussion leads to the case $B = B_v$ for any $v \in S$ for which the normal exponential map gives an isometry of $B_{k,R}$ onto $B_R(L)$. This finishes the proof.

__Remark.__ It follows from the Rauch-Berger Theorem (3.3(b)) that in the situation of Theorem 6.4 with $k = 1$ no point can have distance bigger than $\pi/2$ from L . This corresponds to Myers' Theorem (cf. 4.4). However, the rigidity part of 4.4 (Cheng's Theorem) does not extend to this case: We get equality if M is a rank-1 symmetric space with $K \geq 1$ and L a subspace of the same type. But unless M has constant curvature, it is not isometric to the model space E_1 as defined in 6.3. We do not know whether there are other cases in which equality holds.

__References.__

1. J.Cheeger, D.G.Ebin: *Comparison Theorems in Riemannian Geometry.* North-Holland 1975

2. S.Y.Cheng: *Eigenvalue comparison theorems and its geometric applications.* Math. Z. **143**, 289 - 297 (1975)

3. M.P. Do Carmo: *Geometria Riemanniana.* IMPA 1979

4. P.Eberlein: *When is a geodesic flow Anosow? I.* J. Diff. Geom. 8, 437 - 463 (1973)

5. J.-H.Eschenburg: *Stabilitätsverhalten des Geodätischen Flusses Riemannscher Mannigfaltigkeiten.* Bonner Math. Schr. **87** (1976)

6. J.-H.Eschenburg: *Local convexity and nonnegative curvature - Gromov's proof of the sphere theorem.* Invent. math. **84**, 507 - 522 (1986)

7. J.-H.Eschenburg, J.J. O'Sullivan: *Jacobi tensors and Ricci curvature.* Math. Ann. **252**, 1 - 26 (1980)

8. L.W.Green: *A theorem of E.Hopf.* Mich.Math.J. 5, 31 - 34 (1958)

9. D.Gromoll, W.Klingenberg, W.Meyer: *Riemannsche Geometrie im Großen.* Springer Lecture Notes in Math. 55 (1968)

10. D.Gromoll, W.Meyer: *Examples of complete manifolds with positive Ricci curvature.* J. Diff. Geom. 21, 195 - 211 (1985)

11. M.Gromov: *Structures metriques pour les varietes riemanniennes.* Cedic/Nathan 1981

12. K.Grove: *Metric Differential Geometry.* Differential Geometry, Proceedings Lyngby 1985 (ed. V.L.Hansen), Springer Lecture Notes in Math. 1263, 171 - 227

13. S.W.Hawking, G.F.R.Ellis: *The Large Scale Structure of Space-time.* Cambridge 1973

14. E. Heintze, H.C. Im Hof: *Geometry of horospheres.* J. Diff. Geom. 12, 481 - 491 (1977)

15. E. Heintze, H. Karcher: *A general comparison theorem with applications to volume estimates for submanifolds.* Ann. Scient. Ec. Norm. Sup., (4)11, 451 - 470 (1978)

16. A. Kasue: *On manifolds of asymptotically nonnegative curvature.* Preprint Berkeley, Ca., 1987

17. H. Karcher: *Riemannian center of mass and mollifier smoothing.* Commun. Pure Appl. Math. 30, 509 - 541 (1977)

18. B. O'Neill: *The fundammental equations of a submersion.* Mich. Math. J. 23, 459 - 469 (1966)

19. W.T.Reid: *Riccati Differential Equations.* Academic Press 1972

20. W.T.Reid: *Sturmian Theory for Ordinary Differential Equations.* Springer 1972

21. K.Shiohama: *A sphere theorem for manifolds of positive Ricci curvature.* Trans. Amer. Math. Soc. 275, 811 - 819 (1983)

22. Spivak, M.: *Differential Geometry,* vol. I. - Publish or Perish 1970

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