Riemannian Manifolds with Flat Ends

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Contents

1. Introduction

Let M be a complete noncompact Riemannian manifold of dimension $n \geqq 2$. An *end* of M is a mapping E, which assigns to each compact subset Ω of M a connected component $E(\Omega)$ of $M \backslash \Omega$ such that $E(\Omega') \subset E(\Omega)$ if $\Omega \subset \Omega'$ (cf. [E]). An end E is called *flat*, if $E(\Omega)$ is flat for some compact subset Ω of M, i.e. the sectional curvature vanishes on $E(\Omega)$. The main purpose of this paper is to classify flat ends up to isometry, i.e. we determine the isometry type of $E(\Omega)$ for a suitable set Ω. An essential step in the classification is the following result (Sect. 7):

Theorem. *Let E be a flat end of a manifold M. Then there exists a compact subset Ω in M such that $E(\Omega)$ is isometric to the interior of $(Y \times \mathbb{R}^k)/\Gamma$ where*

(a) *either $k=n-1$ and $Y=\mathbb{R}_+=[0, \infty)$, and Γ is a Bieberbach group on $\mathbb{R}^{n-1}$,*

(b) *or $k=n-2$ and Y is diffeomorphic to $\mathbb{R} \times \mathbb{R}_+$, and Γ is a Bieberbach group on $\mathbb{R} \times \mathbb{R}^{n-2}$ preserving the product structure,*

(c) *or $k \leqq n-3$ and Y is the complement of a distance ball in $\mathbb{R}^{n-k}$, and Γ is a finite extension of a Bieberbach group on $\mathbb{R}^k$.*

* Supported by DFG

This result reduces the classification essentially to a 2-dimensional situation. The 2-dimensional case itself is nontrivial. Namely, consider a complete regular planar curve $\sigma\colon \mathbb{R} \to \mathbb{R}^2$ with unit normal vector field n and nonnegative curvature function $\kappa = \langle \sigma'', n \rangle \geqq 0$. Then $g\colon \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^2$ with $g(s, t) = \sigma(s) - tn(s)$ is an immersion. Let $Y(\sigma)$ be the surface $\mathbb{R} \times \mathbb{R}_+$ with the pull back metric. We will show in Sect. 7 that case (b) of the theorem can be reduced further to the case $Y = Y(\sigma_{\beta, R})$ where $\sigma_{\beta, R}$ is the cycloide

$$\sigma_{\beta, R}(s) = (\beta s - R \cdot \sin s, R \cdot \cos s).$$

Furthermore, we study flat ends in manifolds which satisfy additional curvature conditions. In particular, we consider the case that M is a complete manifold of nonpositive sectional curvature with a flat end E. We represent M as H/Γ, where H is simply connected and $\Gamma \approx \pi_1(M)$. We prove that, besides in a few exceptional cases (where the global structure of the manifold is determined), there exists a compact flat totally geodesic embedded hypersurface T and M dividing M into the pieces $E(T)$ and $M \setminus E(T)$. The piece $E(T)$ is diffeomorphic (under the normal exponential map of T) to $T \times (0, \infty)$ and $M \setminus E(T)$ is totally convex.

The proof of this result and a discussion of the exceptional cases is contained in Sect. 8. As a corollary of this description, we see that flat ends in nonpositively curved manifolds are of type (a) or (b) as described in the Theorem and the type (c) cannot occur. It is interesting to remark that this is also true in the case of nonnegatively curved manifolds by a result in [SZ].

For hints and discussion we wish to thank Martin Strake.

2. Convex Cutting

A *ray* in a complete Riemannian manifold M is a unit speed geodesic $\gamma\colon [a, \infty) \to M$ such that $\gamma \mid [a, b]$ is shortest for any $b > a$. The ray $\gamma \mid [a', \infty)$ is called a *subray* of γ for any $a' \geqq a$.

Let E be a flat end of M and choose Ω such that $F := E(\Omega)$ is flat. Let γ be a ray in M with subray in F and let

$$b_\gamma(x) = \lim_{t \to \infty} (t - d(x, g(t)))$$

be its Busemann function (e.g. cf. [EH]). Since Ω is compact, b is bounded on Ω and so, for sufficiently large $t_\gamma > 0$, the "horo-ball"

$$B(\gamma) = \{x \in M;\, b_\gamma(x) > t_\gamma\} = \bigcup_{r > 0} B_r(\gamma(t_\gamma + r))$$

lies outside a neighborhood of Ω. So $B(\gamma) \subset F$ because $B(\gamma)$ is connected and contains a subray of g. Since F has sectional curvature $K \geqq 0$, the function b_γ is convex on F (cf. [CG, EH]) and therefore, $C(\gamma) := M \setminus B(\gamma)$ is a closed totally convex subset of M. Let $R(F)$ be the set of all rays with a subray in F. Put

$$C_0 = \bigcap_{\gamma \in R(F)} C(\gamma).$$

Then C_0 is a closed and totally convex subset containing Ω, and the closure of $M \setminus C_0$ is contained in F.

Lemma. $C_0 \cap \mathrm{Clos}(F)$ *is compact.*

Proof. Suppose not. Then there exists a sequence $(p_i)_{i \geqq 1}$ in $C_0 \cap \mathrm{Clos}(F)$ with $d_i := d(p_0, p_i) \to \infty$ for any $p_0 \in \Omega$. Let γ_i: $[0, d_i] \to M$ be a shortest geodesic segment joining p_0 to p_i. By convexity, γ_i lies in C_0. Let d be the diameter of Ω. Then $\gamma_i(t) \in M \setminus \Omega$ for $t > d$. Since $\gamma_i(d_i) = p_i \in F$, we get $\gamma_i((d, d_i]) \subset F$. Passing to a subsequence, we may assume that (γ_i) converges to a ray γ which lies in C_0 and has a subray in $\mathrm{Clos}(F)$, hence in F, which is a contradiction to the construction of C_0.

3. Smoothing

We may reparametrize each ray $\gamma \in R(F)$ such that t_γ becomes 0. Then the function $f = \sup_{\gamma \in R(F)} b_\gamma$ is convex on F, and we get $C_0 = \{f \leqq 0\}$. By Sect. 2, the set $C_0 \cap \mathrm{Clos}(F)$ is compact, and similarly $C_t \cap \mathrm{Clos}(F)$ is compact for all $t \geqq 0$, where $C_t = \{f \leqq t\}$. The function b is nonexpanding, i.e. $|f(x) - f(y)| \leqq d(x, y)$ for all x, y, since the same property holds for the functions b_γ. Therefore ∂C_t has distance at least t from ∂C_0. Choose $t > 0$ and $\varepsilon < t/2$ so that for some $\delta \in (0, \varepsilon)$, the ball $B_\delta(p)$ is isometric to a euclidean ball for any point p in the compact set

$$T = \{t - \varepsilon \leqq f \leqq t + \varepsilon\} \subset F.$$

Let $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$ be a smooth function with $\varphi = \mathrm{const}$ near 0 and support in $[0, \delta]$ such that

$$\int_{\mathbb{R}^n} \varphi(\|x\|)\, dx = 1,$$

and put for $p \in T$

$$f_\delta(p) = \int_{T_p M} f(\exp_p v)\, \varphi(\|v\|)\, dv$$

where dv is the euclidean volume element on $T_p M$. It is well known that f_δ defines a smooth and convex function on T with $|f_\delta - f| < \delta$. In particular, $S' := \{f_\delta = t\}$ does not meet ∂T, and S' is a smooth hypersurface, since t is not a minimum value of f_δ. Moreover, S' bounds the set

$$C = \{x \in T;\, f_\delta(x) \leqq t\} \cup C_{t-\varepsilon} \subset C_{t+\varepsilon},$$

which is totally convex, by convexity of f_δ on T and of f on $M \setminus C_{t-\varepsilon} \subset F$.

Let $\Omega' = \Omega \cup (C \cap \mathrm{Clos}(F))$. Then $G := E(\Omega') \subset F$ is a connected component of $M \setminus C$, and since C is connected, $S := \partial G$ is a connected component of S'.

Let N be the unit normal vector field on S which points into G. Let e: $S \times \mathbb{R}_+ \to \mathrm{Clos}(G)$ given by $e(p, t) = \exp_p(tN_p)$. This map is onto, and by convexity of S and nonpositive curvature on G, it is a local diffeomorphism. We claim that e is in fact a diffeomorphism. Namely, for small $t > 0$, the immersion $e_t \colon S \to G$ with $e_t(p) = e(p, t)$ is an embedding. Let s be the largest value such that e_t is one-to-one for every $t \in (0, s)$. If $s < \infty$, then $e_s(x) = e_s(y)$ for different points $x, y \in S$, and the tangent hyperplanes at $e_s(x)$ and $e_s(y)$ agree. Then the geodesics $g_x(t) = e(x, t)$ and $g_y(t) = e(y, t)$ join at $e_s(x)$ and form an unbroken geodesic from x to y which lies outside C. This is a contradiction since C was totally convex.

We may sum up the preceding results by using the following notion: A complete connected Riemannian manifold-with-boundary is called *concave* if the distance function from the boundary is convex. E.g. the complement of an open convex subset with smooth boundary in $\mathbb{R}$ is concave. In general, for a concave manifold M the normal exponential map $e \colon \partial M \times \mathbb{R}_+ \to M$ is a diffeomorphism. Namely, let $g \colon \mathbb{R}_+ \to M$ be a unit speed geodesic starting orthogonally from the boundary. Then for small $t > 0$ we have $d(g(t), \partial M) = t$. By convexity, this holds for all $t \in \mathbb{R}_+$. Thus ∂M has no cut locus which shows that e is a diffeomorphism. The subsets $M_t = e(\partial M \times [t, \infty))$ are also concave manifolds, for all $t \geqq 0$. Now we obtain from the preceding considerations:

Proposition 1. *Let M be a complete Riemannian manifold and E a flat end. Then there is a compact subset Ω in M such that $E(\Omega)$ is concave.*

So from now on we may forget $M \setminus E(\Omega)$ and assume that M is already a flat concave manifold with compact boundary which consequently has only one end E.

4. Isometric Flat Ends

Let M and M' be flat concave manifolds with boundaries S and S'. An isometric embedding $g \colon M \to M'$ is called *simple* if $g(S)$ projects diffeomorphically onto S'. We claim that this is equivalent to saying that $f := g \mid S$ is an isometric embedding preserving the second fundamental form such that $\pi' \circ f \colon S \to S'$ is a diffeomorphism, where π' is the projection of M' onto S'. Namely, if such an embedding $f \colon S \to M'$ is given, we may define $g \colon M \to M'$ by $g(e(x, t)) = \exp(tN_x)$ where N is the unit normal field along f which has positive scalar product with the gradient of $d' = d(\ , S')$. Since by convexity d' is increasing along the rays $r_x(t) = \exp(tN_x)$, the mapping g is everywhere defined and one-to-one, by a similar argument as in Sect. 3. By flatness, g is isometric, since f preserves the second fundamental forms with respect to the normal fields $\nabla d \mid S$ and N.

Recall that ends E, E' of two manifolds M, M' are called *isometric* if $E(\Omega)$ is isometric to $E(\Omega')$ for suitable compact subsets Ω and Ω' of M and M'.

Proposition 2. *Let M, M' be flat concave manifolds with compact boundaries S and S'. Then M and M' have isometric ends if and only if there exists a simple isometric embedding of M_t into M' for some $t \geqq 0$.*

Proof. Assume that M and M' have isometric ends. Since $d=d(\ ,\partial M)$ is an exhaustion function on M, there is an isometric embedding $g: M_t \to M'$ for some $t \geqq 0$ such that $M' \backslash g(M_t)$ is relatively compact. Using g, we may consider M_t as a subset of M'. Each ray in M' starting orthogonally from $\partial M'$ intersects M_t, and once entered, it can never leave M_t again since the function d on M_t is convex. Moreover, by concavity of M_t, the unique intersection with $S_t = \partial M_t$ is transversal. So the embedding g is simple. Conversely, if a simple embedding $g: M_t \to M'$ is given, then $d' = d(\ ,\partial M')$ takes a maximum on $g(\partial M_t)$ which shows that $M \backslash g(M_t)$ is relatively compact. Thus M and M' have isometric ends.

Observe that $g: M_t \to M'$ is a homotopy equivalence. So it defines an isomorphism between the fundamental groups, and we get:

Corollary. *Let M, M' as above and X, X' the universal covers with deck transformation groups Γ and Γ'. Then M and M' have isometric ends if and only if there exists a group isomorphism $\vartheta: \Gamma \to \Gamma'$ and a simple isometric embedding $g: X \to X'$ such that $g \circ \gamma = \vartheta(\gamma) \circ g$ for all $\gamma \in \Gamma$.*

5. The Developing Map

Let X be an n-dimensional flat Riemannian manifold (possibly with boundary) which is simply connected. For any small open subset of X, there is an isometry into the euclidean space $\mathbb{R}^n$, and this map can be analytically extended to an isometric immersion $D: X \to \mathbb{R}^n$, the so called *developing map* [Th] which is uniquely determined up to an isometry of $\mathbb{R}^n$. Moreover, there exists a group homomorphism $\varphi: I(X) \to E(n)$ of the isometry group of X into the euclidean group, i.e. the isometry group of $\mathbb{R}^n$, *such that* D is equivariant. The subgroup $I^+(X)$ of orientation preserving isometries is mapped under φ into the group $E^+(n)$ of proper (i.e. orientation preserving) motions of $\mathbb{R}^n$.

In particular, if M is an arbitrary flat manifold and X its universal cover, then $D: X \to \mathbb{R}^n$ is Γ-equivariant, where $\Gamma \cong \pi_1(M)$ is the deck transformation group of the covering $X \to M$.

6. Simply Connected Concave Flat Manifolds

Let M be a concave flat manifold with compact boundary. Let X be the universal covering of M. Then X is also a concave flat manifold. The boundary ∂X is not necessarily compact, but since ∂X covers ∂M, the isometry group of X acts uniformly on ∂X. In other words, there is a compact subset $B \subset \partial X$ such that every point $x \in \partial X$ can be mapped into B by an isometry of X.

Proposition 3. *Let X be a simply connected concave flat manifold such that the isometry group of X acts uniformly on ∂X. Then X is a Riemannian product $X = Y \times \mathbb{R}^k$, where Y is either $\mathbb{R}_+$ or 2-dimensional or isometric to $\mathbb{R}^m \backslash C$ where C is an open, relatively compact subset of $\mathbb{R}^m$, and $m \geqq 3$.*

The proposition a consequence of the following two lemmas.

Lemma 1. *Let Y be a simply connected flat concave manifold with compact boundary of dimension $m \geqq 3$. Then Y is isometric to $\mathbb{R}^m \backslash C$, where C is an open convex relatively compact subset of $\mathbb{R}^m$.*

Lemma 2. *Let X be a simply connected flat concave manifold with noncompact boundary such that the isometry group acts uniformly on ∂X. If $\dim(X) \geqq 3$, then X splits isometrically as $X = X' \times \mathbb{R}$.*

Before we prove these statements, we show that they imply Proposition 3: Let k be the maximal integer such that X splits isometrically as $X = Y \times \mathbb{R}^k$. This decomposition is unique by the de Rham splitting theorem and is respected by all isometries of X. It follows that the isometry group of Y acts uniformly on ∂Y. If $\dim(Y) \geqq 3$, then ∂Y is compact by Lemma 2 and the maximality of k. Lemma 1 implies, that Y is isometric to $\mathbb{R}^m \backslash C$.

Proof of Lemma 1. We consider the developing map $D: Y \to \mathbb{R}^m$. let $S = \partial Y$. Then $D | S: S \to \mathbb{R}^m$ is an immersed convex hypersurface. By Sacksteder's theorem [S], this is an embedding and $D(S)$ is the boundary of a convex body in $\mathbb{R}^m$. Thus D maps Y bijectively onto the complement of an open convex subset.

Remark. Since S is compact and oriented, we can prove Sacksteder's theorem for a locally convex immersion $x: S \to \mathbb{R}^m$ easily as follows: We replace x with another immersion $y: S \to \mathbb{R}^m$ which is arbitrarily C^∞-close to x and all of whose principal curvatures are strictly positive. Then the Gauß map of y is a local diffeomorphism onto S^{m-1} and hence a diffeomorphism, since S^{m-1} is simply connected. So the maximum value of each height function $\langle y, v \rangle$ for $v \in S^{m-1}$ is taken at exactly one point of S which immediately implies that y is an embedding (Hadamard's theorem, see [H, Es]). Therefore, x was already an embedding.

To prove the existence of y, we note that by compactness, the immersion x is strictly convex near some point. By standard convolution methods like in Sect. 3, one can replace x with a convex immersion y_1 being arbitrarily C^∞-close to x such that the domain of strict convexity is larger. By iterating this process one obtains y.

Proof of Lemma 2. Since X is flat and simply connected, we have a basis of globally defined parallel vector fields on X. For $x, y \in X$ and $v \in T_y X$ let $P_x v \in T_x X$ be the parallel transport of v to x. This allows us to define a global Gauß map $v: \partial X \to S^{n-1}$ as follows. Let $N := \nabla d$, where d is the distance function of ∂X. We fix a point $o \in X$ and put $v(x) = P_o N_x$, identifying S^{n-1} with the unit sphere in $T_o X$.

Claim (1). Either, the image of v is open or there exists a complete geodesic $g: \mathbb{R} \to X$ in $X \backslash \partial X$ such that d is constant along g.

Proof of (1). Let $P_o N_y$ be a point in $v(\partial X)$ for $y \in S$. Let $x = \exp_y N_y$, thus $d(x) = 1$. Let $Q^T = \{v \in T_x X \mid \langle v, N_x \rangle = 0\}$. If h is a geodesic with $h(0) = x$, $h'(0) \in Q^T$, then $(d \circ h)'(0) = 0$ and by the convexity of d, $d \circ h(s) \geqq d \circ h(0) = 0$ for all $s \in \mathbb{R}$ and h does not reach the boundary of X. Thus $\exp_x$ is defined on Q^T; let $Q = \exp_x Q^T$.

We claim that Q is isometric to $\mathbb{R}^{n-1}$ and totally convex, i.e. every geodesic in X joining two points of Q is contained in Q. Note that $\exp_x: Q^T \to X$ is a totally geodesic immersion. Let $D: X \to \mathbb{R}^n$ be a developing map, then also $D \circ \exp_x: Q^T \to \mathbb{R}^n$ is a totally geodesic isometric immersion. Thus $D(Q)$ is a hyperplane in $\mathbb{R}^n$ and thus isometric to $\mathbb{R}^{n-1}$ and totally convex. We distinguish the following cases:

(a) there exists a geodesic ray $h: \mathbb{R}_+ \to Q$ with $h(0)=x$ such that $d \circ h$ is bounded. Since $(d \circ h)'(0)=0$ and d is convex, this implies that d is constant on h. Thus the points $h(i)$, $i \in \mathbb{N}$, have bounded distance 1 from ∂X. Since the isometry group of X acts uniformly on ∂X, there are isometries γ_i such that $\gamma_i h(i)$ are contained in a fixed compact set. Since d is invariant under γ_i, the sequence $\gamma_i h$ has an accumulation geodesic g with d constant on g.

(b) If $d \circ h$ is unbounded for all such rays h, then, due to the convexity of d, there exists a constant $r>0$ such that $d(y) \geqq 2$ for all $y \in Q$ with $\operatorname{dist}(x, y) \geqq r$. By continuity there exists a neighborhood U of N_x in the unit sphere of $T_x X$ such that for $u \in U$ and a unit speed geodesic h with $h(0)=x$, $h'(0) \perp u$ we have $d \circ h(s) \geqq 1/2$ for $0 \leqq s \leqq r$ and $d \circ h(r) \geqq 3/2$.

By convexity, $h(s)$ cannot reach the boundary for $s \geqq 0$. For $u \in U$ let $u^\perp$ be the hyperplane in $T_x X$ orthogonal to u. As above, $Q(u)=\exp_x u^\perp$ is a totally geodesic hyperplane isometric to $\mathbb{R}^{n-1}$ with $d(x)=1$ and $d(y) \geqq 3/2$ if $y \in Q(u)$ and $\operatorname{dist}(x, y) \geqq r$, thus d assumes a minimum in say $z \in Q(u)$. It follows that N_z is normal to $Q(u)$ which means $u=P_x N_z$. But $N_z = P_z N_{\pi(z)}$ where $\pi: X \to \partial X$ is the orthogonal projection. Therefore $u = P_x N_{\pi(z)}$ and thus $P_0 u = P_0 N_{\pi(z)}$ is in the image of v for all $u \in U$.

Claim (2). If $\dim(X) \geqq 3$, then $v(\partial X) \subset S^{n-1}$ has measure 0.

Proof of (2). Note that ∂X is a convex hypersurface in the flat manifold X. The Gauß equations imply that the intrinsic sectional curvature of ∂X is nonnegative. Since ∂X is noncompact but the group of isometries operates uniformly, ∂X contains a globally minimizing intrinsic geodesic line. By Toponogov's splitting theorem, ∂X splits isometrically as a product $Y \times \mathbb{R}$. We claim that the Gauß-Kronecker curvature of ∂X, i.e. the determinant of the second fundamental form, vanishes everywhere.

Let $x \in \partial X$. Since ∂X splits intrinsically and $\dim(\partial X) \geqq 2$, there is a 2-plane $\sigma \subset T_x \partial X$ such that the intrinsic curvature $K(\sigma)$ vanishes. The Gauß equations then imply $\det(b \mid \sigma)=0$ for the second fundamental form of ∂X. Thus there exists a vector $v \in \sigma$ with $b(v, v)=0$. Since b is positive semidefinite, it follows that $\det(b)=0$. The Gauß-Kronecker curvature is the determinant of the differential of v. It follows that the image $v(\partial X) \subset S^{n-1}$ has zero measure.

Lemma 2 now follows from (1) and (2) by the following result:

Claim (3). If there exists a complete geodesic $g: \mathbb{R} \to X$ in the interior of X such that d is constant on g, then X is isometric to $X' \times \mathbb{R}$.

Proof of (3). Let $v=g'(0)$, then we consider the parallel vector field V on X defined by $V(x)=P_x v$. For $x \in X$ let g_x be the geodesic $g_x(s)=\exp_x sV(x)$, thus the geodesics g_x are the integral curves of V. Let

$$A = \{x \in X \setminus \partial X \mid g_x \text{ is defined on } \mathbb{R} \text{ and } d \text{ is constant on } g_x\}.$$

Then A is not empty by assumption and A is clearly closed. We claim that A is open. Let $x \in A$ and g_x the corresponding integral curve of V with $d \circ g_x \equiv a > 0$. Let $J(s)$ be any normal parallel vector field along g_x with $\|J\| < a$. Since X is flat, $h(s) = \exp J(s)$ is a complete geodesic in $X \setminus \partial X$, which is an integral curve of V. Since $\|\nabla d\| \leqq 1$, we have $d \circ h(s) < 2a$ and thus d is constant on h as a bounded convex function. It follows that A is open. Thus $A = X \setminus \partial X$ and $X \setminus \partial X$ splits an euclidean de Rham factor. This splitting extends to X.

7. The Classification

As mentioned in the introduction, any regular complete curve $\sigma: \mathbb{R} \to \mathbb{R}^2$ with unit normal vector n and curvature $\kappa = \langle \sigma'', n \rangle / \|\sigma'\|^2 \geqq 0$ defines a flat concave metric on $\mathbb{R} \times \mathbb{R}_+$, induced by the immersion

$$e: \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^2, \qquad e(s, t) = c(s) - t\,n(s).$$

Let $Y(\sigma)$ be the manifold $\mathbb{R} \times \mathbb{R}_+$ with this metric. The isometries of Y are those parameter transformations $\psi: \mathbb{R} \to \mathbb{R}$ which preserve length and curvature of σ. If Y is an arbitrary simply connected flat concave 2-dimensional manifold, then ∂Y is isometric to $\mathbb{R}$. Let $D: Y \to \mathbb{R}^2$ be the developing map. Then $\sigma = D \mid \partial Y$ is a complete curve and $Y = Y(\sigma)$.

In particular consider the cycloide $\sigma_\beta: \mathbb{R} \to \mathbb{R}^2$,

$$\sigma_\beta(s) = \sigma_{\beta, R}(s) = (\beta s - R \sin s, R \cos s),$$

for some $\beta \geqq 0$ and $R \gg \beta$. Let $Y_\beta = Y_{\beta, R} = Y(\sigma_{\beta, R})$ and Y_∞ the upper halfplane. Reparametrizing σ_β as $\tilde{\sigma}_\beta(u) = \sigma_\beta(u/\beta)$ we see that $Y_\beta \to Y_\infty$ as $\beta \to \infty$.

The isometry group of Y_0 and Y_∞ consists of all translations and reflections of the first coordinate while $I(Y_\beta)$ for $0 < \beta < \infty$ contains only those which preserve the lattice $2\pi\mathbb{Z}$. In the latter case, the fixed points of any reflection is πk for some integer k, and the reflections with even and with odd k belong to two different conjugacy classes.

Theorem 1. *Let M be a complete manifold with a flat end E. Then there exists a compact set Ω in M such that $E(\Omega)$ is isometric to the "standard end" $(Y \times \mathbb{R}^k)/\Gamma$, where*

(a) *either Y is the complement of a closed metric ball of radius R around 0 in $\mathbb{R}^m$, and $m = n - k \geqq 3$, and Γ is a discrete uniform subgroup of $O(m) \times E(k)$, in particuliar Γ is a finite extension of a Bieberbach group of rank k,*

(b) *or $Y = Y_{\beta, R}$ for $0 \leqq \beta < \infty$ and Γ is a uniform discrete subgroup of $I(Y_\beta) \times E(n-2)$; in particular Γ is a Bieberbach group of rank $n-1$,*

(c) *or $Y = [0, \infty)$ and Γ is a Bieberbach group of rank $n-1$.*

Moreover, two ends are isometric if and only if these standard ends (for suitable R) are isometric manifolds.

Proof. By Proposition 1, we may assume that M is flat and concave. Let X be its universal cover and Γ the deck transformation group. By Proposition

3, we have an isometric splitting $X=Y\times\mathbb{R}^k$ which is preserved by Γ, and three cases may occur:

Case 1. $Y=\mathbb{R}_+$. Then we get Case (c) of the theorem.

Case 2. $Y=\mathbb{R}^m\setminus C$, where C is an open, relatively compact, convex subset of $\mathbb{R}^m$ with $m\geqq 3$. Let B be the smallest open distance ball in $\mathbb{R}^m$ containing C. Since B is uniquely determined, it is invariant under all isometries of Y, i.e. under all isometries of $\mathbb{R}^m$ leaving C invariant. So $X':=(\mathbb{R}^m\setminus B)\times\mathbb{R}^k$ is a Γ-invariant subset of X contained in X_t for some $t>0$, and $(X\setminus X')/\Gamma$ is relatively compact. If Ω is the closure of this set, then $E(\Omega)$ is as claimed in Case (a) of the theorem.

Case 3. Y is a 2-dimensional concave, flat, simply connected manifold. Hence $Y=Y(\sigma)$ for some complete, locally convex curve $\sigma\colon\mathbb{R}\to\mathbb{R}^2$. Moreover, Γ acts as a Bieberbach group on $\partial Y\times\mathbb{R}^{n-2}$ and preserves the splitting, i.e. $\Gamma\subset I(Y)\times E(n-2)$. Let Γ_1 be the projection of Γ into $I(Y)$; this acts uniformly on ∂Y. If Γ_1 is not discrete, then σ must have constant curvature, hence $Y=Y_0$ or $Y=Y_\infty$, and we get cases (b) or (c) of the theorem. So we may assume that Γ_1 is discrete and hence generated by a translation γ_0 and possibly a reflection γ_1 of $\partial Y=\mathbb{R}$.

Let $D\colon Y\to\mathbb{R}^2$ be an orientation preserving developing map such that $\sigma=D\,|\,\partial Y$. Let $\varphi\colon\Gamma_1\to E(2)$ be the corresponding homomorphism. Put $A_0=\varphi(\gamma_0)$ and $A_1=\varphi(\gamma_1)$ if γ_1 is present. A_0 is a proper motion and A_1 a reflection. Now we have

$$\sigma\circ\gamma_1=A_1\circ\sigma$$

for $i=0$ and possibly $i=1$. We may assume that σ is parametrized by arc length in such a way that γ_0 is given by the parameter shift $u\to u+\omega$ for some $\omega>0$ and γ_1 (if present) by the reflection $u\to -u$.

Consider the following two invariants of the curve σ and the translation γ_0: first the total curvature per period

$$\tau=\tau(\sigma,\gamma_0)=\int_0^\omega \kappa(u)\,du=\lim_{r\to\infty}(\omega/2r)\int_{-r}^{r}\kappa(u)\,du$$

where $\kappa=\langle\sigma'',n\rangle$ is the nonnegative curvature function of σ, and second the displacement per period "on the long run":

$$\delta=\delta(\sigma,\gamma_0)=\lim_{r\to\infty}(\omega/2r)\,\|\sigma(-r)-\sigma(r)\|.$$

Let $\alpha\in[0,2\pi)$ be the angle between $\sigma'(0)$ and $\sigma'(\omega)$. Then $\tau=2\pi k+\alpha$ for some integer $k\geqq 0$. If $\tau=0$, then σ is a straight line and we get case (c) of the theorem. So we may assume $\tau>0$. We will distinguish between the cases $\delta=0$ and $\delta>0$.

Case (i). $\delta=0$. Then A_0 is a rotation by the angle α. We may choose the developing map D in such a way that the fixed point of A_0 is the origin. Then the whole curve $\sigma(\mathbb{R})$ has finite distance $R>0$ from the origin. If A_1 is present, it leaves invariant the ball $B = B_R(0)$. Thus its axis passes through 0 and we may assume that A_1 is the reflection at the x_2-axis.

Case (ii). $\delta > 0$. Then A_0 is a translation, and δ its displacement. So we have $\alpha = 0$. By choice of D we may assume that $A_0(x) = x + \delta e_1$. If A_1 is present, it satisfies $A_1 A_0 A_1 = A_0^{-1}$, and so its axis is parallel to the x_2-axis. Thus we may assume that A_1 is the reflection at the x_2-axis.

In both cases, if A_1 is present, then $\sigma(0)$ lies on the x_2-axis and $\sigma'(0) = \pm e_1$. If A_1 is not present, we may assume $\sigma'(0) = -e_1$ by choice of the parameter of σ. Moreover, in case (ii) we may assume $\sigma(0) = 0$, since the developing map can still be changed by a translation of $\mathbb{R}^2$.

Let $\vartheta(\gamma_0)$ be the translation $s \to s + \tau$ in $\mathbb{R}$ and $\vartheta(\gamma_1)$ the reflection $s \to -s$. This defines a homomorphism $\vartheta: \Gamma_1 \to E(1)$ which we extend to a homomorphism $\vartheta: \Gamma \to E(1) \times E(n-2)$ by putting $\vartheta(\gamma) = (\vartheta(\gamma_1), \gamma_2)$ for any $\gamma = (\gamma_1, \gamma_2) \in \Gamma$. Now the following lemma shows that we end up with Case (b) of the theorem:

Lemma 1. *The end of M is isometric to $(Y_\beta \times \mathbb{R}^{n-2})/\vartheta(\Gamma)$, where $\beta = \delta/\tau$.*

Proof of Lemma 1. Let $R' > 0$ be so big that $\sigma([0, \omega])$ is contained in $B_{R'}(0)$ and put $R = R' + 2\tau\beta$. Then $\sigma_\beta | [0, 2\tau]$ and its tangents lie outside $B_{R'}(0)$ where $\sigma_\beta = \sigma_{\beta, R}$ is the cycloid defined above. In the following, we will assume $\sigma'(0) = -e_1$. In case that $\sigma'(0) = +e_1$, we must replace $\sigma_\beta(s)$ with $\sigma_\beta(s - \pi)$ in the subsequent argument. Recall that the developing map of $Y = Y(\sigma)$ was given in coordinates by the immersion $e: \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}^2$ with

$$e(u, t) = \sigma(u) - t\, n(u).$$

We will show that the rays r_u with $r_u(t) = e(u, t)$ intersect σ_β transversally at $r_u(t(u))$ for some smooth, ω-periodic function $t: \mathbb{R} \to \mathbb{R}$, and we have $\bar{\sigma}(u) := e(u, t(u)) = \sigma_\beta(s(u))$ for some parameter transformation $s: \mathbb{R} \to \mathbb{R}$ with $s(\gamma_i(u)) = \vartheta(\gamma_i)(s(u))$ for $i = 0$ and $i = 1$ (if necessary).

Consider first the case $\delta = 0$. Then all rays r_u start inside $B = B_R(0)$ and so they intersect ∂B exactly once and transversally, say at $r_u(t(u))$ with t as above. Thus $\bar{\sigma}(u) = \sigma(u) - t(u)\, n(u)$ is a parametrization of the circle ∂B and hence $\bar{\sigma}(u) = \sigma_\beta(s(u))$ for some parameter transformation $s: \mathbb{R} \to \mathbb{R}$. Observe that $\bar{\sigma}' = (1 + \kappa t)\, \sigma' - t' n$ and therefore $\langle \bar{\sigma}', \sigma' \rangle \geqq 1$. Thus on each finite interval, the total curvatures of σ and $\bar{\sigma}$ differ by less than π which implies $\tau(\bar{\sigma}, \gamma_0) = \tau(\sigma, \gamma_0) = \tau$. Since σ_β is parametrized by total curvature, we have $s(u + \omega) = s(u) + \tau$ and moreover $s(-u) = -s(u)$.

Now we assume $\delta > 0$ and hence $\tau = 2\pi k$ for some integer $k > 0$. Then the rays r_u for $0 \leqq u \leqq \omega$ have only transversal intersections with $\sigma_\beta | [-2\tau, 2\tau]$. Recall that we have assumed $\sigma(0) = 0$ and $\sigma'(0) = -e_1$. Then $e(0, R) = \sigma_\beta(0)$. By the implicit function theorem, there is a maximal value $u_1 \in [0, \omega]$ such that there are smooth functions $s: [0, u_1] \to [0, 2\tau]$ with $s' > 0$ and $t: [0, u_1] \to \mathbb{R}_+$ such that $\bar{\sigma}(u) := e(u, t(u)) = \sigma_\beta(s(u))$ for $0 \leqq u \leqq u_1$.

If $u_1 < \omega$, then $s(u_1) = 2\tau$, by maximality of u_1. But this is impossible since the total curvature of $\sigma | [0, u_1]$ is not bigger than τ and differs from that of $\bar{\sigma}$ by less than π (see above), but on the other hand, the total curvatures, of $\sigma_\beta | [0, 2\tau]$ and hence of $\bar{\sigma} | [0, u_1]$ are 2τ, a contradiction.

Thus we have $u_1=\omega$. Let $s_1=s(\omega)$. As above, the total curvatures of $\bar{\sigma}\,|\,[0,\omega]$ and $\sigma_\beta\,|\,[0,s_1]$ differs from $\tau=2\pi k$ by less than π. Therefore, the number of intersections of $\sigma_\beta\,|\,[0,s_1]$ with the x_1-axis lies between $2k-1$ and $2k+1$. Moreover,

$$\sigma_\beta(s_1)=\bar{\sigma}(\omega)=\delta\, e_1+t(\omega)\, e_2,$$

and since $t(\omega)>0$, the number of these intersections is exactly $2k$. So we get that $\tau-\pi/2\leqq s_1\leqq\tau+\pi/2$. Since $\langle\sigma_\beta(s_1), e_1\rangle=\delta=\beta\tau$, we must have $s_1=\tau$ and $t(\omega)=t(0)$. Thus t can be extended to a smooth ω-periodic function on $\mathbb{R}$ and s gets a parameter transformation with $s(u+\omega)=s(u)+\tau$ and $s(-u)=s(u)$.

To sum up, the map $f\colon\mathbb{R}\times\mathbb{R}^{n-2}\to\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}^{n-2}$ with

$$f(s,v)=(u(s),t(u(s)),v)$$

gives a ϑ^{-1}-equivariant isometric embedding of $\partial(Y_\beta\times\mathbb{R}^{n-2})$ into $X=Y(\sigma)\times\mathbb{R}^{n-2}$ preserving the second fundamental form, where $u(s)$ denotes the inverse function of $s(u)$. Thus we finish the proof of Lemma 1 using the corollary of Proposition 2.

The proof of the theorem now is finished by the following lemma:

Lemma 2. *Let $M=X/\Gamma$ and $M'=X'/\Gamma'$ be standard ends (X, X' depending on constants $R, R'>0$) as defined in the theorem. If M and M' have isometric ends then they are isometric manifolds for suitable R and R'.*

Proof. If M and M' have isometric ends, then there is a simple isometric embedding $g\colon X\to X'$ (possibly after enlarging the constant R of X) which is equivariant with respect to an isomorphism $\vartheta\colon\Gamma\to\Gamma'$ (cf. the corollary of Prop. 2). We have to show that $X=X'$ and that ϑ is a conjugation in $I(X)$.

Case (c). $X'=[R',\infty)\times\mathbb{R}^{n-1}$. Then $g(\partial X)$ is the graph of a Γ'-periodic convex function on $\mathbb{R}^{n-1}$ which must be constant. Hence $X=[R,\infty)\times\mathbb{R}^{n-1}$ and $\Gamma=\Gamma'$.

Case (a). $X'=(\mathbb{R}^m\backslash B_{R'}(0))\times\mathbb{R}^k$. Then X is of the same type for topological reasons. Since X and X' are open subsets of $\mathbb{R}^n=\mathbb{R}^m\times\mathbb{R}^k$, the isometric embedding $g\colon X\to X'$ extends to an isometry of $\mathbb{R}^n$ which preserves the splitting, i.e. $g=(g_1,g_2)$ with $g_1\in E(m)$ and $g_2\in E(k)$, and we have $\vartheta(\gamma)=g\circ\gamma\circ g^{-1}$ for all $\gamma\in\Gamma$. if $g_1\in O(m)$, then ϑ is a conjugation in $I(X)$ and we are done. So suppose that $g_1^{-1}(0)=v\in\mathbb{R}^m$. Then v is a fixed vector of $\vartheta(\gamma)_1\in O(m)$. Hence $\vartheta(\gamma)$ commutes with the translation T_v sending x to $x+v$. Putting $g'=g\circ T_v\in O(m)\times E(k)=I(X)$, we have $\vartheta(\gamma)=g'\circ\gamma\circ g'^{-1}$ for all $\gamma\in\Gamma$.

Case (b). $X'=Y'\times\mathbb{R}^{n-2}$ where $Y'=Y_{\beta',R'}$. Then the only case which is left for X is $X=Y\times\mathbb{R}^{n-2}$ with $Y=Y_{\beta,R}$ for some $\beta\in[0,\infty)$. Since $\mathbb{R}^{n-2}$ is the euclidean de Rham factor in X and X', the isometric embedding g splits as $g=(g_1,g_2)$ where $g_1\colon Y_\beta\to Y_{\beta'}$ is an isometric embedding and $g_2\in E(n-2)$.

For an arbitrary planar curve $\sigma\colon\mathbb{R}\to\mathbb{R}^2$ with normal vector field n and for any parameter shift $\gamma(s)=s+\omega_\gamma$ preserving length and curvature of σ, we have defined the total curvature and the displacement "on the long run" per period, $\tau(\sigma,\gamma)$ and $\delta(\sigma,\gamma)$. Moreover, if $\bar{\sigma}=\sigma-tn$ for some ω_γ-periodic function $t\colon\mathbb{R}\to\mathbb{R}_+$, then $\tau(\sigma,\gamma)=\tau(\bar{\sigma},\gamma)$ and $\delta(\sigma,\gamma)=\delta(\bar{\sigma},\gamma)$. If γ is a translation of the cycloid σ_β, then $\tau(\sigma_\beta,\gamma)=\omega_\gamma$ and $\delta(\sigma_\beta,\gamma)=\beta\cdot\omega_\gamma$.

Now let D be the developing map of Y' such that $D|\partial Y'=\sigma_{\beta',R}=:\sigma$. By assumption, we have an isometric embedding $g: X\to X'$ which is equivariant with respect to an isomorphism $\vartheta: \Gamma\to\Gamma'$ and which splits as $g=(g_1, g_2)$ with $g_2\in O(n-2)$, where $g_1: Y\to Y'$ is an isometric embedding. Let $\tilde\sigma=D\circ g_1|\partial Y$. Then $\tilde\sigma=A\circ\sigma_{\beta,R}$ for some $A\in E(2)$. Moreover, $\tilde\sigma$ is a graph over σ, i.e. $\tilde\sigma$ is a reparametrization of $\bar\sigma=\sigma-tn$ where t is some positive function.

For $i=1, 2$, let Γ_i be the projection of $\Gamma\subset I(Y)\times E(n-2)$ into the i^{th} factor, and similarly Γ_i'. Since $\vartheta(\gamma)=g\circ\gamma\circ g^{-1}$ on the open subset $U=g(\mathrm{Int}(X))$ of X', we have the splitting

$$\vartheta(\gamma)_i=g_i\circ\gamma_i\circ g_i^{-1}=:\vartheta_i(\gamma_i)$$

on U. This defines a splitting $\vartheta=(\vartheta_1, \vartheta_2)$ with isomorphisms $\vartheta_i: \Gamma_i\to\Gamma_i'$. Since ϑ_2 is the conjugation with $g_2\in E(n-2)$, it suffices to prove that $\beta'=\beta$ and that ϑ_1 is a conjugation in $I(Y_\beta)$.

Note that $\Gamma_1'\subset I(Y')\subset E(1)$ contains an abelian translation subgroup of index at most 2. Let $\gamma\in\Gamma_1'$ be any translation $u\to u+\omega_\gamma$ where $\omega_\gamma=\tau(\sigma,\gamma)$. Then t is ω_γ-periodic, and therefore $\tau(\sigma,\gamma)=\tau(\bar\sigma,\gamma)=\tau(\tilde\sigma,\vartheta_1^{-1}(\gamma))$, and similar, $\delta(\sigma,\gamma)=\delta(\tilde\sigma,\vartheta_1^{-1}(\gamma))$. So it follows that $\beta'=\beta$ and $\vartheta_1^{-1}(\gamma)=\gamma$.

If Γ_1' contains only translations, we are done. So suppose that there is a reflection γ_1 in Γ_1. Then $\vartheta_1^{-1}(\gamma_1)$ is also a reflection. We must show that $\vartheta_1^{-1}(\gamma_1)=\gamma\gamma_1\gamma^{-1}$ for some translation γ: then ϑ_1^{-1} is the conjugation with γ since Γ_1' is generated by γ_1 and the translation subgroup. If $\beta=0$, this is clear since $I(Y_0)=E(1)$ and any two reflections in $E(1)$ are conjugate under a translation in $E(1)$.

Now let us suppose $\beta>0$. After possibly reflecting the parameterization of $\tilde\sigma=A\circ\sigma_{\beta,R}$, we may assume that A is a proper motion. Since $\sigma=\sigma_{\beta,R'}$ and $\tilde\sigma$ (which is $\bar\sigma$, up to reparametrization) are invariant under the same group of planar translations, A must be a translation, too.

Let $u=k\pi$ be the fixed point of the reflection γ_1 of the parameter of σ. The unit tangent vector of σ at u is $-(-1)^k e_1$, and the sign distinguishes between the two conjugacy classes of reflections of σ. Let n and $\bar n$ be the unit normal fields of the curves σ and $\bar\sigma$ which point to the convex side so that the corresponding curvatures are nonnegative. The ray $r_u(t)=\sigma(u)-tn(u)$ leaves σ towards the concave side and meets $\tilde\sigma$ from the convex side at $\bar\sigma(u)=\tilde\sigma(s(u))$. Moreover, r_u is fixed by the planar reflection $A_1=\varphi(\gamma_1)$, where $\varphi: \Gamma_1'\to E(2)$ is the homomorphism corresponding to the developing map D. So $\vartheta_1^{-1}(\gamma_1)$ is the reflection at $s(u)$ and the intersection of r_u and $\tilde\sigma$ is orthogonal. Hence $n(u)=\bar n(s(u))$ and therefore also the corresponding unit tangent vectors agree. This shows that the reflections γ_1 and $\vartheta_1^{-1}(\gamma_1)$ are conjugate under a translation in $I(Y_\beta)$ which finishes the proof of the lemma and of the theorem.

Finitely, let us classify the 2-dimensional flat ends Y/Γ. In this case, Γ is an infinite cyclic group $\Gamma(\omega)$, generated by the translation $\gamma_\omega: s\to s+\omega$ for some $\omega>0$. The case (a) of the theorem does not occur, hence $Y=Y_\beta$ for some $\beta\in[0,\infty]$. If $\beta\neq 0,\infty$, we have $\omega=2\pi k$ for some integer $k>0$. Therefore we get:

Corollary. *2-dimensional flat ends have the following different isometry types:*

(a) $Y_0/\Gamma(\tau)$ *for* $\tau\in(0,\infty)$ *("truncated cones"),*

(b) $Y_\beta/\Gamma(2\pi k)$ *for* $k\in\mathbb{N}$, $\beta\in(0,\infty)$ *("generalized cylinders"),*

(c) $\mathbb{R}_+\times(\mathbb{R}/\omega\cdot\mathbb{Z})$ *for* $\omega>0$ *("cylinders").*

Remark. It is now easy to describe the ends of complete metrics on $\mathbb{R}^2$ with $K \geqq 0$ or $K \leqq 0$ which are flat outside of a compact set. The Gauß-Bonnet theorem implies that a convex curve surrounding the non-flat region has total curvature $\tau \leqq 2\pi$ (resp. $\tau \geqq 2\pi$) where equality implies global flatness. If $Y_\beta/\Gamma(\omega)$ is the standard end of this metric as described in the corollary above, we therefore must have $\tau(\sigma_\beta, \gamma_\omega) \leqq 2\pi$ (resp. $\geqq 2\pi$). It follows in the case $K \geqq 0$ that only the truncated cones with $\tau \leqq 2\pi$ and the cylinders can occur. For $K \leqq 0$, the only possibilities are the truncated cones with $\tau \geqq 2\pi$ and the generalized cylinders with $k \geqq 2$. Note that in the case $k=1$ we have total curvature 2π, but this generalized cylinder is not isometric to the end of the euclidean plane. Using polar coordinates, one can explicitly construct the possible types.

8. Flat Ends in Manifolds of Nonpositive Curvature

Let V be a complete Riemannian manifold of nonpositive sectional curvature with a flat end E. We represent V as H/Γ, where H is a complete simply connected manifold and $\Gamma \approx \pi_1(V)$ the group of deck transformations (compare [EO, BGS]). H is diffeomorphic to $\mathbb{R}^n$ and convex in the sense that any two points in H can be joined by a unique geodesic.

It turns out that either E has a standard description or we are in a special situation, where we control the global structure of the manifold. We first describe the special cases:

Type A: H splits isometrically as $H' \times \mathbb{R}^{n-2}$ with a twodimensional factor H' which is flat outside of a compact set. Γ is a Bieberbach group of rank $n-2$, every $\gamma \in \Gamma$ splits as (γ_1, γ_2) with $\gamma_1 \in I(H')$, $\gamma_2 \in E(n-2)$. All elements γ_1 are elliptic with a common fixed point in H' and the group formed by all γ_2 operates freely with compact quotient on $\mathbb{R}^{n-2}$.

Type B: H is isometric to $H' \times \mathbb{R}^{n-2}$ and H' is flat outside of a horoball $B \subset H'$. Γ is a Bieberbach group of rank $n-1$ and every $\gamma \in \Gamma$ splits as (γ_1, γ_2). The elements γ_1 leave B invariant and Γ operates with compact quotient on $\partial B \times \mathbb{R}^{n-2}$.

Type C: There exists a totally geodesic hyperplane S in H isometric to $\mathbb{R}^{n-1}$ invariant under Γ. The group Γ operates with compact quotient on S, in particular Γ is a Bieberbach group of rank $n-1$. There exists an element $\gamma \in \Gamma$ such that γ interchanges the two components of $H \setminus S$. Furthermore H is flat outside of the a distance tube of radius a around S for some $a \geqq 0$.

For a more detailed description of manifolds of type C see Sect. 9.

Theorem 2. *Let $V = H/\Gamma$ be a complete manifold of nonpositive curvature with a flat end E. Assume that V is neither flat nor of type A, B, C. Then we have*

(i) *There exists a totally geodesic compact flat hypersurface T dividing M into the pieces $E(T)$ and $M \setminus E(T)$.*

(ii) *The closure of $E(T)$ is diffeomorphic under the normal exponential map to $T \times \mathbb{R}_+$.*

(iii) *The closure of $E(T)$ is isometric to $Q \times \mathbb{R}^{n-2}/\Delta$, where Q is a closed convex 2-dimensional subset of H with totally geodesic boundary ∂Q isometric to $\mathbb{R}$. There exists a number $a > 0$ such that Q is flat outside of the a-distance* *tube of ∂Q. Δ is a Bieberbach group of rank $n-1$ which operates with compact quotient on $\partial Q \times \mathbb{R}^{n-2}$.*

Proof. By Proposition 1, there exists a compact set Ω', such that $E(\Omega')$ is isometric to the *interior* of a concave flat manifold M' with compact boundary. Thus $E(\Omega')$ contains the concave flat manifold $M=M'_t$ where $t>0$. Note that $M?$ is the closure of $E(\Omega)$ for a suitable compact set $\Omega \supset \Omega'$. Let $\pi: H \to V$ be the universal covering of V, and let W be a connected component of $\pi^{-1}(M) \subset H$. Then W is precisely invariant under the operation of Γ, i.e. either $\gamma W = W$ or $\gamma W \cap W = \emptyset$ for all $\gamma \in \Gamma$. We identify M with W/Γ_w where $\Gamma_w = \{\gamma \in \Gamma \mid \gamma W = W\}$. Then $\pi|_w: W \to M$ is a covering map.

Let $\psi: X \to M$ be the universal covering of M. Then we may lift ψ to a covering $\varphi: X \to W$. Let $X = X' \times \mathbb{R}^k$ the splitting of the euclidean de Rham factor. For $p \in X'$, $\varphi: \{p\} \times \mathbb{R}^k \to H$ is an isometric immersion, and thus $\varphi(\{p\} \times \mathbb{R}^k)$ is a k-flat in H, i.e. a totally geodesic submanifold isometric to $\mathbb{R}^k$. Note that $\varphi(\{p_1\} \times \mathbb{R}^k)$ and $\varphi(\{p_2\} \times \mathbb{R}^k)$ have bounded distance from each other and thus are parallel by Eberlein's "Sandwich Lemma" [BGS, 2.3].

It follows that $\varphi(X) \subset P_F$ where P_F is the set of all parallels to F [BGS, 2.4]. P_F is a closed convex subset of H which splits as $P'_F \times \mathbb{R}^k$ and is invariant under Γ_w. Thus the map $\varphi: X \to P_F$ which is a covering onto its image induces an isometric map $\varphi': X' \to P'_F$ which is also a covering onto its image $W' = \varphi'(X') \subset P'_F$.

By Proposition 3, X' is either the complement of a relatively compact convex set in $\mathbb{R}^m$ or of dimension $\leqq 2$. In the first case we can extend $\varphi': X' \to P'_F$ to an isometry $\Phi: \mathbb{R}^m \to P'_F$ by [BGS, Corollary 2, p. 67]. Thus P'_F is flat and hence V is also flat. Hence we can assume that $\dim X' = \dim P'_F \leqq 2$.

We first consider the case $\dim X' = 2$. Note that W' is a concave 2-dimensional flat manifold with boundary $\partial W'$. We consider the case that $\partial W'$ is diffeomorphic to S^1. Then one proves easily that W' is the complement of a convex subset of P'_F and P'_F is complete without boundary. It follows that H itself splits isometrically as $H' \times \mathbb{R}^{n-2}$ and H' is flat outside of a compact set. We may assume that H' is not flat. Then there exists a smallest ball in H' containing the points with negative curvature [BGS, p. 10]. The center o of this ball is fixed by all isometries of H'. Thus every $\gamma \in \Gamma$ splits as (γ_1, γ_2) with an elliptic isometry γ_1. The group Γ leaves $\{o\} \times \mathbb{R}^{n-2}$ invariant and operates with compact quotient on this $\mathbb{R}^{n-2}$ since H/Γ has a flat end. Thus V is of type A.

We now assume that $\partial W'$ is diffeomorphic to $\mathbb{R}$. In this case, $\varphi: X \to P_F$ and $\varphi': X' \to P'_F$ are injective and we can identify X with W and X' with W'. Γ_w operates with compact quotient on $\partial W = \partial W' \times \mathbb{R}^{n-2}$. Hence Γ_w is a Bieberbach group of rank $n-1$. Every $\gamma \in \Gamma_w$ respects the splitting of ∂W and leaves $P_F = P'_F \times \mathbb{R}^{n-2}$ invariant. It follows that there exists a nontrivial isometry α of P'_F which leaves W' invariant and translates the boundary $\partial W'$.

Let Q be the convex hull of W' in P'_F, i.e. Q is the smallest closed convex subset of P'_F containing W'. We identify $\partial W'$ with $\mathbb{R}$. Then let g_i be the geodesic in P'_F from i to $-i$. Then we have the following alternatives:

(i) g_i has an accumulation geodesic g. Then it is not difficult to prove that Q has boundary $\partial Q = g$. Furthermore g is an axis of the isometry α and the distance to W' is bounded on ∂Q.

(ii) The sequence g_i diverges. Then larger and larger distance ball of a point $o \in \partial W'$ are contained in Q. Then Q is complete without boundary. The isometry α is not hyperbolic, since any axis of α would bound a convex halfspace containing W'. It follows that α is a parabolic isometry and one verifies that V is of type B.

It remains to consider the case (i). Note that $Q \times \mathbb{R}^{n-2}$ is the convex hull of W. Let $S = \partial(Q \times \mathbb{R}^{n-2})$, then S is a totally geodesic hypersurface isometric to $\mathbb{R}^{n-1}$. There exists a constant $a > 0$ such that $\operatorname{dist}(p, W) \leq a$ for all $p \in S$. Let C be the connected component of $H \backslash S$ which contains W. Thus C is the interior of $Q \times \mathbb{R}^{n-2}$.

We claim that S is precisely invariant under Γ, i.e. either $\gamma S \cap S = \emptyset$ or $\gamma S = S$. Let therefore $\gamma \in \Gamma$ such that $\gamma S \cap S$ is not empty. Then the Hausdorff distance between S and γS is infinite and therefore there exists a point $p \in S$ such that $\gamma p \in C$ with $\operatorname{dist}(\gamma p, S) \geq 2a$. Let $q \in W$ with $\operatorname{dist}(p, q) \leq a$. Then $\operatorname{dist}(p, \gamma q) \leq a$ and hence $\gamma q \in C$ and $\operatorname{dist}(\gamma q, S) \geq a$, thus $\gamma q \in W$. Therefore $\gamma W \cap W$ is not empty and since W is precisely invariant it follows that $\gamma W = W$. Then γ leaves also the convex hull of W and S invariant.

Let $\Gamma_s = \{\gamma \in \Gamma \mid \gamma S = S\}$, then $\Gamma_w \subset \Gamma_s$. Let us assume that there exists an element $\alpha \in \Gamma_s \backslash \Gamma_w$. Then α interchanges the components of $H \backslash S$. In particular, the component αC is isometric to C. Let γ be an arbitrary element of γ. Then γW is either contained in C or in αC. Thus $\gamma W = W$ or $\gamma W = \alpha W$. It follows that $\gamma \in \Gamma_w$ or $\alpha\gamma \in \Gamma_w$. So Γ_w is a subgroup of index 2 in γ and $\Gamma = \Gamma_s$. It follows that V is of type C.

If $\Gamma_w = \Gamma_s$ then let $T = \partial S / \Gamma_w$. Then $Q \times \mathbb{R}^{n-2} / \Gamma_w$ is diffeomorphic to $T \times [0, \infty)$. One easily checks now (i), (ii) and (iii) of the theorem.

At last it remains to consider the case that $\dim(X') = 1$. In this situation, W is isometric to $\mathbb{R}^{n-1} \times \mathbb{R}_+$. The group Γ_w operates with compact quotient on $\partial W = \mathbb{R}^{n-1}$ and $S = \partial W$ is precisely invariant with $\Gamma_w \subset \Gamma_s$. If Γ_w and Γ_s are not equal, the H is flat everywhere. If $\Gamma_w = \Gamma_s$, then $\mathbb{R}^{n-1} \times \mathbb{R}_+ / \Gamma_w$ is indeed isometric to $T \times \mathbb{R}_+$ with $T = S / \Gamma_w$.

The above theorem implies in particular that the isometry type of a flat end in manifolds of nonpositive curvature is restricted:

Corollary. *Let V be a complete manifold of nonpositive curvature with a flat end E. If V is not flat, then there exists a compact set Ω in V such that $E(\Omega)$ is diffeomorphic to $T \times \mathbb{R}_+$ where T is a compact flat space form.*

Remark. We observe that any flat end in a manifold of nonpositive curvature which is neither flat nor of type A, B, C has the topological type of an end of finite volume in a manifold of constant negative curvature.

9. Examples and Remarks

1. To construct nonpositively curved manifolds with flat ends, we start with a complete noncompact hyperbolic manifold $M(K \equiv -1)$ of finite volume. Let E be an end of M. For suitable Ω, $E(\Omega)$ has topologically the structure $T \times [0, \infty)$ where T is a compact spaceform and the metric is a warped product ds^2

$= e^{-2t} d\sigma^2 + dt^2$ where $d\sigma^2$ is a flat metric on T (cf. [E]). We produce a new metric $ds^2 = f^2 d\sigma^2 + dt^2$ on the end, where f is a convex function which coincides with e^{-t} for small t and is a positive constant for t large. By the warped product formula [BO, 7.7] the new metric has nonpositive curvature and by construction the end is finally isometric to $T' \times [0, \infty)$ for a flat spaceform T'. Thus the end is of type (c) of Theorem 1.

Let us consider the universal covering $X = \mathbb{R}^{n-1} \times [0, \infty)$ of this end with deckgroup Γ. If Γ respects a factor of $\mathbb{R}^{n-1}$ (e.g. this holds if Γ contains only translations), then X splits Γ-invariantly as $\mathbb{R}^{n-2} \times \mathbb{R} \times [0, \infty)$. Now we can change the metric on the factor $\mathbb{R} \times [0, \infty)$ such that the new metric coincides with the old one near the boundary and is flat outside of a distance tube of the boundary. We can do this in a way that all elements $\gamma \in \Gamma$ operate as isometries with respects to the new metric. Thus we can open the end in a two-dimensional subspace and obtain a flat end of type (b).

2. Nonpositively curved manifolds of type C: Consider a metric of nonpositive curvature on $S^1 \times [0, \infty)$ which is flat outside of a compact set and isometric to $S^1_\varepsilon \times [0, 1)$ near the boundary where S^1_ε is the circle of length ε. Let W be the Riemannian product of this manifold with S^1_ε. The boundary of W has a colloring isometric to $[0, 1) \times S^1_\varepsilon \times S^1_\varepsilon$. Let W' be an isometric copy of W. We glue W and W' along the boundaries interchanging the S^1_ε-factors. We obtain a complete graph manifold V' of nonpositive curvature (constructions of this type are due to Gromov ([G], also compare [E], [Sc]). One checks that V' allows an isometry of order 2 without fixed points which leaves the common boundary torus $S^1_\varepsilon \times S^1_\varepsilon$ invariant interchanging its factors which maps W onto W'. Dividing V' by this involution we obtain a manifold of type C.

We can construct type C manifolds more generally in the following way: Let Γ be a Bieberbach group on $\mathbb{R}^{n-1}$ such that there are parallel 1-dimensional distributions D_1 and D_2 with $\gamma D_i = D_i$ or $\gamma D_i = D_j$ for all $\gamma \in \Gamma$. We consider the operation of Γ on $H = \mathbb{R}^{n-1} \times \mathbb{R}$ with $\gamma(x, t) = (\gamma x, \rho(\gamma) \cdot t)$ where $\rho(\gamma) = 1$ if γ preserves the distributions and $\rho(\gamma) = -1$, if γ interchanges the distributions. The upper halfspace H_+ splits as $\mathbb{R}^{n-2} \times \mathbb{R} \times [0, \infty)$ where the $\mathbb{R}$-factor corresponds to the distribution D_1 and the lower halfspace H_- splits as $\mathbb{R}^{n-2} \times \mathbb{R} \times (-\infty, 0]$ where the $\mathbb{R}$-factor corresponds to D_2. In general the splittings do not agree on the hyperplane $\mathbb{R}^{n-1}$. By construction, Γ respects the product structures. One can change the metric on the factors $\mathbb{R} \times [0, \infty)$ of H_+ and $\mathbb{R} \times (-\infty, 0]$ of H_- such that the new metric is nonflat with $K \leqq 0$ and Γ still acts by isometries such that H/Γ is of type C.

3. On the other hand, let us consider a complete manifold M of nonnegative sectional curvature with a flat end E. If M has more than one end, there exists a geodesic line in M and M splits isometrically as $M' \times \mathbb{R}$ by Toponogov's theorem. Since M has a flat end, M' is flat. Thus we may assume that M has only one end and hence M is flat outside of a compact set.

If M is in addition simply connected at infinity, then M is isometric to $\mathbb{R}^n$ by a result of [GW]. In general, the following classification holds [SZ]: Let M be complete of curvature $K \geqq 0$ and flat outside of a compact set. Then M is flat or the universal covering X of M splits isometrically as $X' \times \mathbb{R}^{n-2}$ where X' is diffeomorphic to $\mathbb{R}^2$ and flat outside of a compact set. The deck group Γ respects the splitting, operates as a Bieberbach group on $\mathbb{R}^{n-2}$ and has a fixed point on X'.

Thus these manifolds are very similar to the nonpositively curved manifolds of type A. In particular they also carry a flat metric.

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