Pluriharmonic maps of maximal rank

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1 Introduction

Let M be a Kähler manifold and J its complex structure. The complexified tangent bundle $T^c M$ splits into the two eigenbundles $T'M$ and $T''M$ of J corresponding to the eigenvalues $\pm i$, and any multi-linear map on $T^c M$ splits accordingly. Further, let $P = G/K$ a Riemannian symmetric space. A smooth map $f : M \to P$ is called *pluriharmonic* if the (1,1)-part of its hessian Ddf (the so-called *Levi form*) vanishes.[1] In

[1] In fact, this does not depend on the choice of the Kähler metric: A map f is pluriharmonic iff $f|_C$ is harmonic for any complex curve $C \subset M$.

To Renato Tribuzy on occasion of his 60th birthday.

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"real" terms this means

$$\mathrm{Ddf}(X,X) + \mathrm{Ddf}(JX,JX) = 0 \tag{1}$$

for any vector field X. If f is an immersion which is also *pluriconformal*, i.e. J is an isometry for the inner product on M induced by f, then f is called *pluriminimal* or *(1,1)-geodesic* (cf. [2]). For pluriharmonic maps, $df_x(T'_xM)$ is known to be a flat subspace of $T^c_{f(x)}P$ for any $x \in M$, i.e. the (complexified) curvature tensor of P vanishes on $df_x(T'_xM)$ (cf. [7] or [3]). Hence the rank of df_x is bounded by the maximal dimension of a flat subspace of $\mathfrak{p}^c = \mathfrak{p} \otimes \mathbb{C}$ where $\mathfrak{p}$ is the Lie triple corresponding to P, i.e. $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the corresponding Cartan decomposition. This dimension can be quite large: If P happens to be hermitian symmetric with complex structure j, then the eigenspaces $\mathfrak{p}'$ and $\mathfrak{p}''$ of j (corresponding to $T'P$ and $T''P$) are flat subspaces. In fact, $df(T'M) \subset f^*T'P$ would mean that f is holomorphic which is stronger than pluriharmonic. But the rank of non-holomorphic pluriharmonic maps must satisfy a more restrictive upper bound: We have to look for maximal flat subspaces $\mathfrak{a} \subset \mathfrak{p}^c$ which are *not* of this type ($\mathfrak{a} \neq \mathfrak{q}', \mathfrak{q}''$ for any hermitian symmetric subtriple $\mathfrak{q} \subset \mathfrak{p}$). In many cases, the maximal dimension r of such abelian subspaces is known (cf. [8,9]). In particular, for complex Grassmannians $P = G_p(\mathbb{C}^{p+q})$ this number is

$$r = (p-1)(q-1) + 1, \tag{2}$$

([9], p. 585; for an elementary proof see [5]). In the present paper, extending the results of [6] we will construct non-holomorphic pluriharmonic maps (immersions) of maximal rank r with values in complex Grassmannians. All constructed maps enjoy the additional property of being *isotropic*. Recall that pluriharmonic maps always come in so called *associated families* depending on an S^1-parameter $\lambda = e^{i\theta}$ (e.g. f. [3]; the best known example is the isometric deformation of the catenoid into the helicoid. If the associated family is trivial, the pluriharmonic map will be called *isotropic*. Such maps are also pluriconformal (cf [2]) and hence (in the immersion case) pluriminimal. We give a classification of all isotropic pluriharmonic maps of maximal rank into complex Grassmannians. However we do not know if there are also non-isotropic pluriharmonic maps which have maximal rank.

2 Isotropic pluriharmonic maps

An isotropic pluriharmonic map with values in a compact symmetric space $P = G/K$ (cf. [3]) is the projection of an holomorphic superhorizontal map into some adjoint orbit $Z = \mathrm{Ad}(G)\xi \subset \mathfrak{g}$ which forms a fibration over P, called *twistor fibration*. More precisely, $\xi \in \mathfrak{g}$ is a so called *canonical element* (cf. [1]) which means that $\sqrt{-1} \cdot \mathrm{ad}(\xi)$ has integer eigenvalues k with corresponding eigenspaces $\mathfrak{g}_k \subset \mathfrak{g}^c$, and $\mathfrak{g}_1 + \mathfrak{g}_{-1}$ generates $\mathfrak{g}^c$ as a Lie algebra. Moreover, $\sum_{k \text{ even}} \mathfrak{g}_k = \mathfrak{k}^c$ and $\sum_{k \text{ odd}} \mathfrak{g}_k = \mathfrak{p}^c$ where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is the Cartan decomposition corresponding to P. The subspace $\mathfrak{p}$ defines a left invariant distribution on the Lie group G which descends to a distribution on Z, the *horizontal distribution* of the canonical projection (*twistor projection*) $\pi : Z \to P$. Similarly, the even smaller subspace $(\mathfrak{g}_1 + \mathfrak{g}_{-1}) \cap \mathfrak{g} \subset \mathfrak{p}$ defines the so called *superhorizontal distribution*, and a map into Z is called superhorizontal if its differential takes values in this subbundle of TZ.

If P is the *complex Grassmannian* $G_p = G_p(\mathbb{C}^n)$ consisting of all p-dimensional linear subspaces in $\mathbb{C}^n$, the holomorphic superhorizontal maps for all twistor fibrations have been described by F. Burstall (cf. [4], p. 185). The twistor spaces are *classical flag manifolds over* $\mathbb{C}^n$. Recall that a *flag* over $\mathbb{C}^n$ can be viewed in two different ways: as a chain of subspaces $0 = W_0 \subset W_1 \subset \cdots \subset W_s = \mathbb{C}^n$ or else as an orthogonal decomposition $\mathbb{C}^n = E_1 \oplus \cdots \oplus E_s$ where $E_i = W_i \ominus W_{i-1}$. A flag manifold Z is the set of all flags of a fixed type where the *type* of a flag $(W_i) = (E_i)$ is given by the dimensions of the spaces W_j or E_j. A map from a manifold M into Z is a "moving flag" (W_j) or (E_j) where W_j and E_j are "moving spaces", i.e. vector bundles over M or maps from M into the corresponding Grassmannian. The twistor projection $\pi : Z \to P$ is the map $(E_i) \mapsto \sum_{j \text{ odd}} E_j \in P = G_p(\mathbb{C}^n)$ with $p = \sum_{j \text{ odd}} \dim E_j$. If M is a complex manifold, the moving flag (W_j) is holomorphic (i.e. locally spanned by holomorphic $\mathbb{C}^n$-valued functions on M) if for all $j = 1, \ldots, r$

$$\bar{\partial} W_j \subset W_j, \tag{3}$$

and (W_j) is also superhorizontal if additionally

$$\partial W_j \subset W_{j+1}, \tag{4}$$

i.e. for any local section f of W_j and any holomorphic chart $(z^1, \ldots, z^m)$ on M we have $\partial f / \partial \bar{z}^i \in W_j$ and $\partial f / \partial z^i \in W_{j+1}$ for $i = 1, \ldots, m$.

3 Short maximal superhorizontal flags

We start by considering flag manifolds Z containing flags $W_1 \subset W_2 \subset \mathbb{C}^n$ of length $s = 3$. Let $W = (W_1, W_2) : M \to Z$ be a holomorphic superhorizontal map, i.e. $W_1 \subset W_2$ are holomorphic bundles and

$$\partial W_1 \subset W_2. \tag{5}$$

We define $\partial W_1(z)$ for any $z \in M$ as the linear span of the values at z of all f_j and their first partial derivatives $\partial_i f_j$ for any local basis $f_1, \ldots, f_{p_1}$ of W_1. Locally we may assume that ∂W_1 has constant dimension, hence it forms a subbundle of W_2. Now we have the following "moving decomposition"

$$\begin{aligned}
\mathbb{C}^n &= E_1 \oplus E_2' \oplus E_2'' \oplus E_3 \\
E_1 &= W_1, \\
E_2' &= \partial W_1 \ominus W_1 \\
E_2'' &= W_2 \ominus \partial W_1 \\
E_3 &= \mathbb{C}^n \ominus W_2
\end{aligned} \tag{6}$$

and the dimension n decomposes according to (6) as

$$n = p_1 + q_1 + q_2 + p_2. \tag{7}$$

Further we may assume that W is an immersion. Moreover, the map $W_1 : M \to G_{p_1}$ has (locally) constant rank which means that it factorizes over some submersion $\pi : M \to M_1$. This means that in fact the mapping W_1 depends only on part of the variables z, namely on $z_1 = \pi(z) \in M_1$. Hence for fixed $z_1 \in M_1$ the spaces $W_1(z_1)$ and $\partial W_1(z_1)$ do not depend on the actual point z in the fibre $F_{z_1} := \pi^{-1}(z_1)$. Thus

$E_2''(z)$ is a q_2-dimenional subspace of $\partial W_1(z_1)^\perp$, i.e. it belongs to the Grassmannian $G_{q_2}(\partial W_1(z_1)^\perp)$. Since W is an immersion on F_{z_1} while W_1 is constant along F_{z_1}, the mapping $z \mapsto E_2''(z) : F_{z_1} \to G_{q_2}(\partial W_1(z_1)^\perp)$ must be an immersion. Obviously, the dimension of M can be maximal only if this immersion is a diffeomorphism, or equivalently if $F_{z_1} = G_{q_2}(\partial W_1(z_1)^\perp)$. Hence in order to maximize dimension we must assume that M is a bundle over M_1 with fibres $G_{q_2}(\partial W_1(z_1)^\perp)$.

Example 0 (cf. [2]) Let $M \subset \mathbb{CP}^{n-1}$ be a complex submanifold. Each tangent space $T_{[x]}\mathbb{CP}^{n-1}$ of $\mathbb{CP}^{n-1}$ (with $x \in \mathbb{C}^n \setminus \{0\}$) can be viewed as the complex subspace $(\mathbb{C}x)^\perp \subset \mathbb{C}^n$, and consequently the tangent and normal subspaces $T_zM, N_zM \subset T_z\mathbb{CP}^{n-1}$ also become complex subspaces of $\mathbb{C}^n$ for any $z \in M$. If M is a hypersurface, i.e. $\dim M = n-2$, we consider the *Gauss map* $f : M \to G_2\mathbb{C}^n$ which maps $z \in M$ to the 2-dimensional subspace $z + N_zM \subset \mathbb{C}^n$. In this case we have $M_1 = M$, and the moving flag is $W_1 \subset W_2 = \partial W_1$ with $W_1(z) = z$ and $\partial W_1(z) = z + T_zM$. The rank is maximal since $p = 2$, $q = n-2$ and $r = (p-1)(q-1)+1 = n-2 = \dim M$.

Example 1 We can extend this example to the case of a complex submanifold $M_1 \subset \mathbb{CP}^{n-1}$ of arbitrary dimension $m_1 \le n-2$. Like in the real case, we obtain a hypersurface from of a submanifold M_1 of higher codimenison by passing to the “tube” around M_1; in the holomorphic framework this is the projectivized normal bundle,

$$M = \mathbb{P}NM_1 \tag{8}$$

with $\dim M = m_1 + (n-1-m_1-1) = n-2$. In fact, since there are plenty of normal lines at each point of a submanifold of higher codimension, we pass to the set M of *all* normal lines. This fibres naturally as $\pi : M \to M_1$. Now the corresponding maximal isotropic pluriharmonic immersion is

$$f : M \to G_2(\mathbb{C}^n), \quad f(z) = z + \pi(z). \tag{9}$$

Here $W_1(z) = \pi(z)$, $\partial W_1(z) = \pi(z) + T_{\pi(z)}M_1$ and $W_2(z) = \partial W_1(z) + z$ for all $z \in M$. As above f is maximal since $\dim M = n-2 = r$.

Example 2 Let $M_1 \subset \mathbb{CP}^{n-1}$ be a one-dimensional submanifold (a complex projective curve) and NM_1 its normal bundle. Fix $p \le n-1$ and let $q = n-p$. Let M be the Grassmann bundle $G_{q-1}(NM_1)$ of all $(q-1)$-planes normal to M_1. This fibres over M_1 with projection $\pi : M \to M_1$. Let W_1 be the inclusion map of M_1 into $\mathbb{CP}^{n-1}$, considered as a line bundle over M_1. Then $\partial W_1(z_1) = z_1 + T_{z_1}M_1$. Put $W_2(z) = \partial W_1(\pi(z)) + z$ and note that $\dim W_2(z) = 2 + q - 1 = q + 1$. We consider the map

$$f : M \to G_p(\mathbb{C}^n), \quad f(z) = \pi(z) + W_2(z)^\perp \tag{10}$$

Since $\dim M = 1 + (q-1)(n-2-(q-1)) = r$, it is maximal.

General case For a local construction of *all* maximal isotropic pluriharmonic immersions, we start with some m_1-dimensional complex manifold M_1 and a holomorphic immersion $W_1 : M_1 \to G_{p_1}$ which can be considered as a q_1-dimensional vector bundle over M_1. Adding the first partial derivatives of sections $w_1, \dots, w_{q_1}$ forming a local basis of W_1, on a dense open subset we obtain a larger bundle ∂W_1 of dimension $p_1 + q_1$ which contains W_1. Locally, we may choose a subspace $\mathbb{C}^{p_2+q_2} \subset \mathbb{C}^{p+q}$ which is a complement to $\partial W_1(z_1)$ for all z_1 in some open subset of M_1. We put $M = M_1 \times M_2$ with $M_2 = G_{q_2}(\mathbb{C}^{p_2+q_2})$. Then we let $W_2(z_1, z_2) = \partial W_1(z_1) + z_2$ and put

$$f : M \to G_p(\mathbb{C}^n), \quad f(z_2, z_2) = W_1(z_1) \oplus W_2(z_1, z_2)^\perp. \tag{11}$$

Now $m = \dim M$ satisfies

$$\begin{aligned} m &= \dim M_1 + \dim G_{q_2}(\mathbb{C}^{p_2+q_2}) \\ &= m_1 + q_2 p_2 \\ &= m_1 + (q - q_1)(p - p_1). \end{aligned} \tag{12}$$

This has to be compared to the upper bound

$$r = 1 + (q-1)(p-1). \tag{13}$$

Lemma 1

$$m_1 \le p_1 q_1 \tag{14}$$

$$q_1 \le m_1 p_1 \tag{15}$$

Proof By assumption $W_1 : M_1 \to G_{p_1}$ is an immersion, hence the differential $(\mathrm{d}W_1)_{z_1}$ is injective at each point $z_1 \in M_1$. Let $t \mapsto z_1(t)$ be a smooth curve in M_1 with $z_1(0) = z_1$ and $z_1'(0) = v \in T_{z_1}M_1$. Then $\mathrm{d}W_1.v = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} W_1(z_1(t))$ is a homomorphism of $W_1(z_1)$ into $W_1(z_1)^\perp$, in fact into $\partial W_1(z_1) \ominus W_1(z_1)$.[2] Hence $m_1 \le \dim \mathrm{Hom}(W_1, \partial W_1 \ominus W_1) = p_1 q_1$ which proves (14).

If $w_1, \dots, w_{p_1}$ denotes a local basis of W_1, then ∂W_1 is spanned by the w_j and their partial derivatives $\partial_i w_j$ where $i = 1, \dots, m_1$ and $j = 1, \dots, p_1$. Hence $q_1 = \dim(\partial W_1 \ominus W_1) \le m_1 p_1$ which proves (15). □

Lemma 2 *Up to equivalence, we have $m = r$ if and only if $p = 2$ or $m_1 = 1$.*

Proof Let us first consider the special case $p_1 = 1$. Then by (14) and (15) we get $m_1 = q_1$, and from (13)–(12) we see

$$\begin{aligned} r - m &= (p-1)(m_1 - 1) - (m_1 - 1) \\ &= (p-2)(m_1 - 1) \\ &= 0 \iff p = 2 \quad \text{or } m_1 = 1. \end{aligned} \tag{16}$$

In the general case we get

$$\begin{aligned} r - m &= p(q_1 - 1) + q(p_1 - 1) - p_1 q_1 - m_1 - 2 \\ &\overset{(14)}{\ge} p(q_1 - 1) + q(p_1 - 1) - 2p_1 q_1 + 2 \\ &\overset{(*)}{\ge} (p_1 + 1)(q_1 - 1) + (q_1 + 2)(p_1 - 1) - 2p_1 q_1 + 2 \\ &= 0. \end{aligned} \tag{17}$$

where the inequality at $(*)$ comes from $p_2, q_2 \ge 1$. Hence $r - m = 0$ implies equality at $(*)$, in particular $p_2 = 1$. But the roles of p_1 and p_2 are interchangeable since we may pass to the orthogonal flag $W_2^\perp \subset W_1^\perp$ which is holomorphic with respect to the negative complex structure on M. Thus we are back to the special case above. □

[2] For any local section w of W_1 we have $(\mathrm{d}W_1 \cdot v) \cdot w(z_1) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} w(z_1(t)) \right)^{W_1^\perp}$; this lies in $\partial W_1 \ominus W_1$ since $\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} w(z_1(t)) \in \partial W_1$.

Theorem 3 *Let $f : M \to G_p(\mathbb{C}^{p+q})$ be an isotropic pluriharmonic immersion which is neither holomorphic nor antiholomorphic, where M is any complex manifold. Then M has maximal dimension $m = (p-1)(q-1)+1$ if and only if f is locally of type (11) with either $p = 2$ (Example 1) or $m_1 = 1$ (Example 2).*

Proof Since f is isotropic pluriharmonic, it is the projection of some holomorphic superhorizontal map $W : M \to \mathsf{F}$ where F is some flag manifold over $G_p(\mathbb{C}^{p+q})$. If the flags in F are short ($s = 3$), the assertion follows from the preceding discussion and Lemma 1. In the next section we will show that F cannot consist of flags with length $s \geq 4$ which will finish the proof. □

4 Long superhorizontal flags are not maximal

In this section we consider a length-s flag manifold, i.e. F consists of decompositions $\mathbb{C}^n = E_1 \oplus \cdots \oplus E_s$. It fibres over the Grassmannian $G_p(\mathbb{C}^n)$ with

$$p = \sum_{j \text{ odd}} n_j, \quad q = n - p = \sum_{k \text{ even}} n_k \tag{18}$$

with $n_i := \dim E_i$. Throughout this section we will assume $s \geq 4$ ("long" flags). The superhorizontal space at some flag $\mathsf{E} = (E_1, \ldots, E_s) \in \mathsf{F}$ is

$$\mathsf{H}_\mathsf{E} = \bigoplus_{i=1}^{s-1} \mathrm{Hom}(E_i, E_{i+1}). \tag{19}$$

Hence any $a \in \mathsf{H}_\mathsf{E}$ has a decomposition

$$a = (a_1, \ldots, a_{s-1}) \tag{20}$$

with $a_i \in \mathrm{Hom}(E_i, E_{i+1})$.

Now let $\mathsf{A} \subset \mathsf{H}_\mathsf{E}$ be some maximal abelian subalgebra, i.e.

$$\dim \mathsf{A} = r = (p-1)(q-1)+1. \tag{21}$$

Let $\pi_i : \mathsf{A} \to \mathrm{Hom}(E_i, E_{i+1})$ be the projection related to the decomposition (19). We may assume $\pi_i(\mathsf{A}) \neq 0$ for all $i \in \{1, \ldots, s-1\}$ since otherwise we could reduce the dimension or split.[3]

From the assumption $s \geq 4$ we see that $n_j \leq p-1$ for all odd j and $n_k \leq q-1$ for all even k, hence

$$\dim \mathrm{Hom}(E_i, E_{i+1}) = n_i n_{i+1} \; < \; (p-1)(q-1)+1 = r \tag{22}$$

for all $i \in \{1, \ldots, s-1\}$. Consequently, each of the projections $\pi_i : \mathsf{A} \to \mathrm{Hom}(E_i, E_{i+1})$ must have a nonzero kernel.

Now we may assume that there is some $a \in \mathsf{A}$ and $i \in \{2, \ldots, s-1\}$ with

$$a_{i-1} = 0, \quad a_i \neq 0. \tag{23}$$

In fact, any nonzero $a = (a_1, \ldots, a_{s-1}) \in \mathsf{A}$ has some nonzero component a_k. If $a_j = 0$ for some $j < k$, it is easy to find such an index i satisfying (23). Otherwise, we pass to

[3] For this argument we must exclude the holomorphic and antiholomorphic cases where every second component of A vanishes.

the conjugate abelian subalgebra $\mathsf{A}^* = \{a^*;\ a \in \mathsf{A}\}$ (with $a^* = \bar{a}^T$) where the order in the string (20) is reversed.

Let $\mathsf{A}_i = \ker \pi_{i-1}$. For any $a \in \mathsf{A}_i$ and $b \in \mathsf{A}$ we have

$$0 = [a,b]|_{E_{i-1}} = a_i b_{i-1} - b_i a_{i-1} = a_i b_{i-1} \tag{24}$$

since $a_{i-1} = 0$. Thus $a_i(\text{im } b_{i-1}) = 0$ (where "im" stands for "image"). In other words, for all $b \in \mathsf{A}$ we have

$$\text{im } b_{i-1} \subset E_i' := \bigcap_{a \in \mathsf{A}_i} \ker a_i. \tag{25}$$

Consider the orthogonal splitting $E_i = E_i' \oplus E_i''$ which is nontrivial, due to (23). Denote the corresponding dimensions by $n_i = n_i' + n_i''$. Now we remove the E_i''-components of all elements of A: For any $b = (b_1, \dots, b_{s-1}) \in \mathsf{A}$ we let

$$b' = \pi'(b) = (b_1, \dots, b_i', \dots, b_{s-1}), \quad b_i' = b_i|_{E_i'} \tag{26}$$

and we put

$$\mathsf{A}' = \pi'(\mathsf{A}) = \{b';\ b \in \mathsf{A}\}. \tag{27}$$

This is still abelian since for all $b, c \in \mathsf{A}$,

$$\begin{aligned} [b', c']|_{E_{i-1}} &= b_i' c_{i-1} - c_i' b_{i-1} = b_i c_{i-1} - c_i b_{i-1} = [b,c]|_{E_{i-1}} = 0 \\ [b', c']|_{E_i'} &= b_{i+1} c_i' - c_{i+1} b_i' = [b,c]|_{E_i} = 0. \end{aligned}$$

However, the vector space $\mathbb{C}^n$ is replaced by $\mathbb{C}^n \ominus E_i'' = \mathbb{C}^{n-n_i''}$. Hence we may view A' as an abelian subspace for another Grassmannian $G_{p'}(\mathbb{C}^{p'+q'})$ where either p' or q' is diminished by n_i''. Interchanging the roles of p and q if necessary, we may assume $p' = p - n_i''$, $q' = q$. From the dimension restriction (2) with p, q replaced by p', q' we obtain the upper bound

$$\dim \mathsf{A}' \le r' = (p - n_i'' - 1)(q-1) + 1 = r - n_i''(q-1). \tag{28}$$

On the other hand, $\dim \mathsf{A}' = \dim \mathsf{A} - \dim \ker \pi'$ where

$$\ker \pi' = \{b \in \mathsf{A};\ b_i|_{E_i'} = 0,\ b_j = 0\ \forall j \neq i\}. \tag{29}$$

For any $b \in \ker \pi'$, the only remaining nonzero component is the one in $\text{Hom}(E_i'', E_{i+1})$, hence $\dim \ker \pi' \le n_i'' n_{i+1}$, and we get the lower bound

$$\dim \mathsf{A}' \ge r - n_i'' n_{i+1}. \tag{30}$$

Comparing (28) and (30) we end up with

$$q - 1 \le n_{i+1}. \tag{31}$$

This has strong consequences. Since $n_{i-1} \neq 0$ and $q \ge n_{i-1} + n_{i+1}$, it implies $n_{i-1} = 1$ and equality in (31) which in turn implies equality in (28) and (30). In particular,

$$\text{Hom}(E_i'', E_{i+1}) = \ker \pi' \subset \mathsf{A}. \tag{32}$$

But then $b_{i+1} = 0$ for all $b \in \mathsf{A}$ since otherwise we would find some $c = c_i \in \text{Hom}(E_i'', E_{i+1}) \subset \mathsf{A}$ with

$$[b,c]|_{E_i''} = b_{i+1} c_i \neq 0.$$

Hence the string ends at E_{i+1}, i.e. $s = i + 1$. Since $s \geq 4$ and $q = n_{i-1} + n_{i+1}$, we must have $s = 4$ and $i - 2 = 1$ (but we will keep the old notation with i).

Let $b \in \mathsf{A}$ with $b_{i-1} \neq 0$. Since $n_{i-1} = 1$, this means that b_{i-1} is injective. Then we have for all $a \in \mathsf{A}_i$:

$$a_{i-2} = 0 \tag{33}$$

since $0 = [b, a]|_{E_{i-2}} = b_{i-1}a_{i-2}$. Since also $a_{i-1} = 0$ and $a_i|E_i' = 0$, we obtain

$$\ker \pi_{i-1} = \mathsf{A}_i = \mathrm{Hom}(E_i'', E_{i+1}). \tag{34}$$

But then $\pi_{i-1}|_{\mathsf{A}'}$ is injective which implies

$$(p' - 1)(q - 1) + 1 = \dim \mathsf{A}' \leq \dim \mathrm{Hom}(E_{i-1}, E_i') = n_{i-1}n_i' = n_i'. \tag{35}$$

This is a contradiction since $p' - 1 \geq n_i'$ and $q - 1 \geq 1$. Hence we have shown that long flags cannot be maximal and thus we have completed the proof of Theorem 3.

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