Self similar symmetric planar tilings

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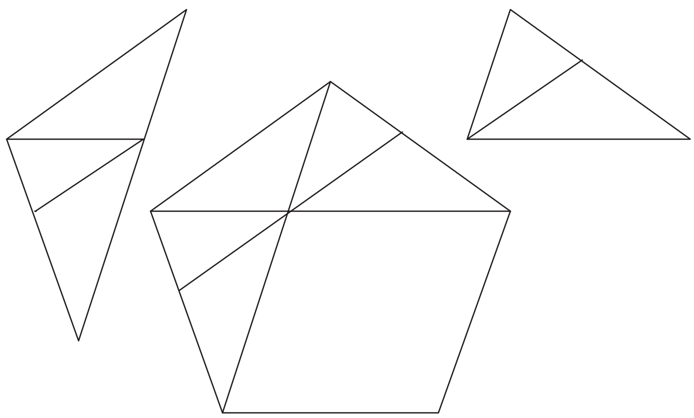
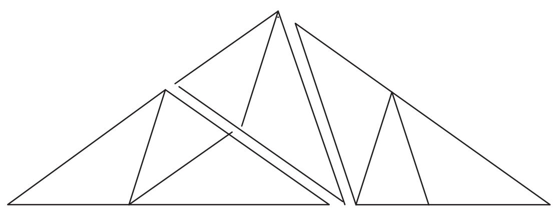


1. Introduction

It is an elementary geometric fact that periodic tilings in dimension 2 and 3 (e.g. crystals) do not allow symmetries of any order different from 2, 3, 4, or 6. Yet there are diffraction patterns of solids showing 5-fold and even 10-fold symmetry[1]. They belong to solids with aperiodic quasi-crystalline structure, discovered in the early 80's [9].

[1] cf. http://www.solid.phys.ethz.ch/ott/staff/beeli/Structural_investigation.html

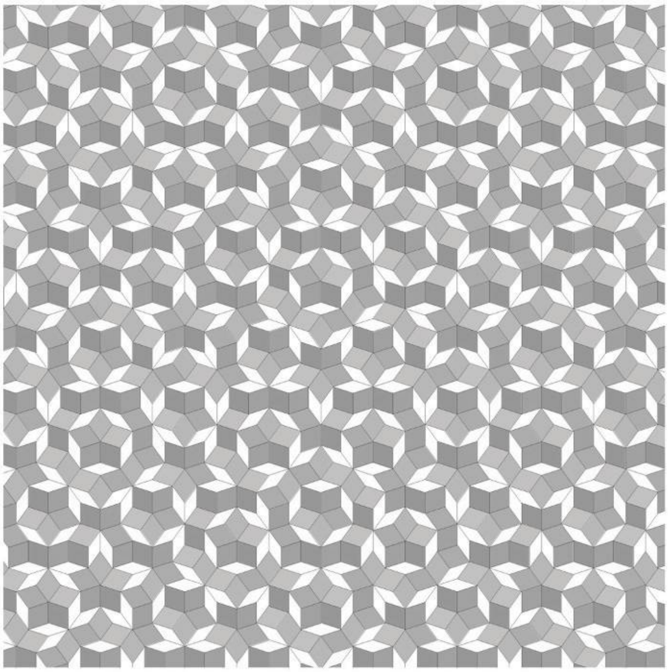
Some years before, Roger Penrose [8] discovered possible models for such quasi-crystals, at least in dimension 2: a class of aperiodic tilings of the Euclidean plane with astonishing properties. As in the case of periodic tilings they have subtilings of any scale with tiles homothetic to the original one. There are two types of tiles, two rhombs (a thick and a thin one) which occur in the regular pentagon; its halves are the two isosceles triangles displayed above. As shown in the figure, these triangles come with a subdivision into smaller triangles of the same shape; the scaling factor is the golden ratio $\varphi = \frac{1}{2}(\sqrt{5} - 1)$. The same subdivisions can also be applied to the small triangles, and when matching across the border lines is required, there is only one way to do it:



(To see uniqueness note that the displayed subdivision of the central triangle is the only one compatible with a subdivision of the one on the right hand side.) By repetition we obtain a subdivision into arbitrary small triangles or rhombs. The *Penrose tilings* have the inverse property: the triangles (half tiles) can be pieced together to form larger and larger triangles or rhombs where the small ones subdivide the large ones as in the above figure, and all large rhombs together form another Penrose tiling.
In the example below[2] we can easily find one of these subtilings: its vertices are the centers of the stars formed by five thick rhombs. Some of the stars have a fully symmetric neighborhood, either pentagonal or star shaped and are surrounded by a chain of thick rhombs; the centers of these stars form the vertices of another tiling with even larger tiles. The displayed example is one of the two Penrose tilings with full pentagonal symmetry (centered at the bottom rose); the other one is given by the subtiling.

It is well known (cf. [1]) that Penrose tilings arise from the integer grid in $\mathbb{R}^5$ by projection onto a certain affine plane $E_a \subset \mathbb{R}^5$. More precisely, there are two-dimensional linear subspaces $E_1, E_2 \subset \mathbb{R}^5$ which are invariant under cyclic coordinate permutations, and we put $E_a = E_1 + a$ for some nonzero $a \in E_2$. The tiles arise from the squares with integer vertex coordinates which lie entirely in the strip $E + I^5$ in 5-space where $I = (0, 1)$ is the open unit interval and I^5 the unit cube in $\mathbb{R}^5$; these squares are projected orthogonally onto E_a. In the present paper we want to investigate a similar construction replacing 5 by other numbers, in particular 7, as was proposed in [10]. Since this means replacing the *pen*tagon by the *hep*tagon, we propose to call these patterns *Heprose tilings* (with apologies

[2] Generated from http://www.geom.umn.edu/apps/quasitiler/start.html



to Professor Penrose). As it will turn out, subtilings of the same sort still exist in all scales. However, they cut the original tiles into pieces which are far more complicated than the triangles in Penrose' case, and moreover the large tiles are subdivided by the small ones in many different ways.

We like to thank L. Danzer and J. Ritter for valuable hints and discussion.

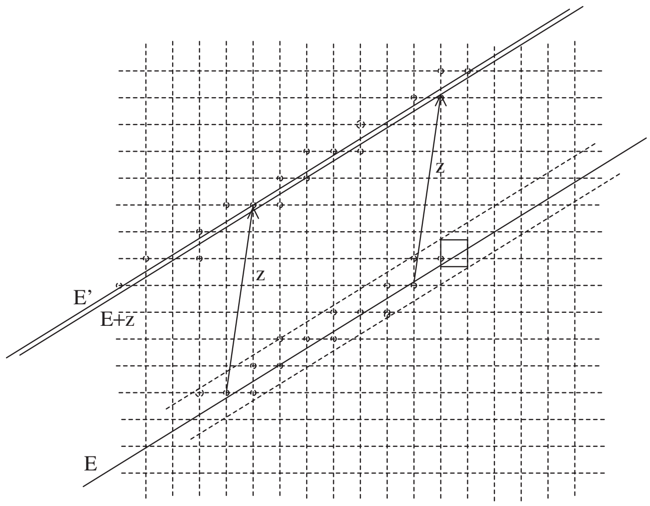
2. Aperiodic tilings via projection

In Euclidean n-space $\mathbb{R}^n$ we consider an r-dimensional subspace E and a parallel affine subspace $E_a = E + a$, $a \in \mathbb{R}^n$ with the following properties:

- E is *irrational*, i.e. $E \cap \mathbb{Z}^n = \{0\}$,
- E_a is in general position, i.e. there are no points on E_a with more than r integer coordinates.

We define a tiling TE_a on E_a (and hence on E) as follows. An integer point $z \in \mathbb{Z}^n$ is *admissible* if z is contained in the *strip* $\Sigma_a = E_a + I^n$ where $I^n = (0, 1)^n$ is the open unit cube. An r-dimensional face of a lattice cube $z + I^n$, $z \in \mathbb{Z}^n$, is *admissible* if it is contained in Σ_a, i.e. all its 2^r vertices are admissible. The orthogonal projections of the admissible

r-faces onto E_a are the tiles; it is a nontrivial fact that these always define a tiling [11]. Since there are only finitely many r-faces up to translation, we get finitely many types of tiles, not more than $\binom{n}{m}$, and each tile is an r-dimensional parallelogram. These tilings TE_a corresponding to affine subspaces $E_a = E + a$ parallel to E will be called *E-tilings*.



Replacing a by $a + z$ for some $z \in \mathbb{Z}^n$ we obtain an *equivalent* tiling, i.e. the tilings TE_a and TE_{a+z} just differ by a translation. Since no integer translation leaves E_a invariant, the tiling TE_a has no periods. However in any ϵ-neighborhood of E_a there are infinitely many integer points. Let z_1, z_2 be two such points and put $z = z_1 - z_2$, then E_a is 2ϵ-close to E_{a+z} and hence the strips Σ_a and Σ_{a+z} will be almost the same. In other words, for most integer points, the admissibility conditions for E_a and E_{a+z} will agree and hence the two tilings are almost the same. Thus TE_a has no periods, but it has almost periods.

Now let $F \subset \mathbb{R}^n$ be another irrational subspace perpendicular to E. Then the projection of the grid, $\pi_F(\mathbb{Z}^n)$ is dense in F.[3] Thus any two E-tilings TE_a, TE_b with $b - a \in F$ are *almost equivalent*. I.e. there is a translation of TE_a that equals TE_b up to a set of errors which are spread over the tiling with mutual distance greater than R where R can be made arbitrary large[4]. In fact, there is some $z \in \mathbb{Z}^n$ such that $\pi_F(z)$ is arbitrarily close to $b - a$

[3]Claim: $\pi_F(\mathbb{Z}^n)$ is dense in F for any irrational subspace $F \subset \mathbb{R}^n$.In fact, $\pi_F(\mathbb{Z}^n)$ is a subgroup of the additive (translation) group of F. Suppose it is not dense in F. Then its closure is of the form $F' \times \Delta$ where F' is a proper linear subspace of F and Δ a discrete group of translations. Thus $\mathbb{Z}^n \subset (F^\perp \oplus F') \times \Delta$. The subspace $F^\perp \oplus F'$ is generated by integer vectors $v_1, \dots, v_p$. There is a nonzero integer solution x of the integer linear system $\langle v_i, x \rangle = 0, i = 1, \dots, p$ (obtained by applying elementary transformations). In particular $x \perp F^\perp$ and hence x is an integer vector in F.

[4]Note that being "almost equivalent" is stronger than being "locally isometric". While the former feature is a global one, the latter is only local.

and hence E_{a+z} is arbitrarily close to E_b. As before we conclude that the two tilings on E_b and E_{a+z} are almost the same.

3. Symmetry of the tiling

Let G be a subgroup of the permutation group S_n of $\{1, \ldots, n\}$ which acts on $\mathbb{R}^n$ by permuting the coordinates: $(x_1, \ldots, x_n) \mapsto (x_{\sigma 1}, \ldots, x_{\sigma n})$ for $\sigma \in G$. The corresponding permutation matrices are orthogonal and integer. Thus each $g \in G$ preserves the integer lattice and the unit cube I^n. Let $W \subset \mathbb{R}^n$ be a nonzero subspace which is G-invariant and *rational* (i.e. spanned by integer vectors) and which is minimal with respect to these two properties. Such space will be called *rationally irreducible*. Then any nonzero G-invariant subspace $E \subset W$ will be irrational unless $E = W$. In fact, if there is a nonzero integer vector $z \in E$, the orbit Gz consists of integer vectors and spans a rational G-invariant subspace $E' \subset E \subset W$. By rational irreducibility, $E' = W$, hence $E = W$.

While W is rationally irreducible, it may be reducible as a representation of G over $\mathbb{R}$. We let $E \subset W$ be an irreducible G-invariant subspace and F its orthogonal complement in W. Then both E and F are G-invariant and irrational, and any two tilings of E_a and E_b with $a - b \in F$ are almost equivalent in the sense of Section 2.

The diagonal vector

$$d = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right) = \frac{1}{n}\, e$$

where $e = (1, \ldots, 1)$, is fixed by any permutation, hence by G. The affine subspace $E_{kd} = E + kd$ is in general position for any $k \in \{1, \ldots, n-1\}$ and preserved by G. The corresponding tiling TE_{kd} is symmetric under G and all tilings TE_{kd+a} with $a \in F$ are almost equivalent to TE_{kd}, in particular almost G-symmetric.

Important examples are dihedral groups ($r = 2$) and the icosahedral group ($r = 3$). The dihedral group D_n acts by permutations on the n vertices of a regular n-gon, hence $D_n \subset S_n$. The icosahedral group A_5 permutes the 12 vertices, 20 faces and 30 edges of the icosahedron, hence $A_5 \subset S_{12}, S_{20}, S_{30}$. In the present paper we consider the dihedral group D_n for prime numbers n; other examples will be treated in [4].

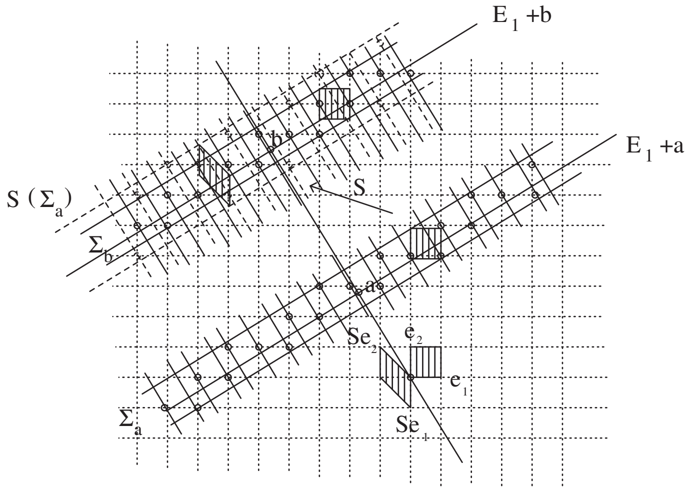
4. Inflation

Now suppose further that E is an eigenspace of a symmetric matrix S which is integer invertible, i.e. it maps the integral lattice $\mathbb{Z}^n$ bijectively onto itself.[5] Then $S(E_a) = E_b$ for $b = Sa$. The set of vertices of TE_a is the E_a-projection of the set $Z_a = \Sigma_a \cap \mathbb{Z}^n$. This is

[5] However in many cases S is somewhat less than integer invertible, see below.

mapped by S bijectively onto $SZ_a = S(\Sigma_a) \cap \mathbb{Z}^n$. Since $S|_E$ is multiplication by a scalar, the projection of SZ_a on E_b is congruent up to scaling to the projection of Z_a onto E_a, hence the tiling of E_a and its image under S are congruent up to scaling. This scaling factor is called the *inflation factor*.

Further we assume that S is an proper expansion on $E^\perp$ (and a contraction on E). Then $S(\Sigma_a) \supset \Sigma_b$ and hence $SZ_a \supset Z_b = \Sigma_b \cap \mathbb{Z}^n$. Thus TE_b with vertex set $\pi_{E_b}(Z_b)$ is refined by a diminution of TE_a with vertex set $\pi_{E_b}(SZ_a)$. Conversely, $S^{-1}(TE_b)$ is a proper subtiling of TE_a in the sense that the vertices of $S^{-1}(TE_b)$ form a subset of the vertex set of TE_a. This phenomenon is called *inflation*, such a linear map S will be called an *inflation map* and $S^{-1}(TE_b)$ an *inflation tiling* of TE_a. In the subsequent figure the situation is drawn for $n = 2$ and $S = \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$. The eigenvalues of S are $-\varphi$ and Φ where $\varphi = \frac{1}{2}(\sqrt{5} - 1)$ is the golden section and $\Phi = \frac{1}{\phi} = \varphi + 1$ its inverse, and E is the $(-\varphi)$-eigenspace of S.



Now let $E \subset \mathbb{R}^n$ be an irreducible representation space of a group $G \subset S_n$ as described in Section 3, and assume that there is no other G-submodule in $\mathbb{R}^n$ which is equivalent to E. Then we want the inflation map S to be G-invariant, i.e. $gS = Sg$ for any $g \in G$. Hence $SE = E$ and in fact E is contained in an eigenspace of S since $\ker(S - \lambda I) \cap E$ is G-invariant. If a tiling TE_a is G-invariant, the same holds for $S(TE_a)$.

5. Planar tilings with dihedral symmetry

We consider the cyclic permutation $\alpha \in S_n$ with $\alpha j = j + 1$ for all $j = 1, \ldots, n$ mod n. The corresponding linear map A on $\mathbb{R}^n$ permuting the coordinates sends

$(x_1, \dots, x_n) \mapsto (x_2, \dots, x_n, x_1)$. The matrix A is diagonalizable over $\mathbb{C}$ with eigenvectors $v_\omega = (\omega^0, \dots, \omega^{n-1}) \in \mathbb{C}^n$ where ω is any n-th root of unity, $\omega^n = 1$. Apparently $Av_\omega = \omega v_\omega$, hence ω is the corresponding eigenvalue. The only real eigenvalues are 1 with eigenvector $e = (1, \dots, 1)$ and in case of even n also -1 corresponding to the eigenvector $f = (1, -1, \dots, 1, -1)$. All other eigenvalues and eigenvectors come in conjugate pairs, and thus the real and imaginary parts of v_ω span an A-invariant 2-plane $E_k \subset \mathbb{R}^n$ on which A acts by rotation with rotation angle $2\pi k/n$. We identify E_k with the complex plane $\mathbb{C}$ by assigning

$$\text{Re } v_\omega \mapsto 1, \;\; \text{Im } v_\omega \mapsto i = \sqrt{-1}. \tag{1}$$

With this identification, $A|_{E_k}$ becomes the multiplication by $\omega = \omega_k = e^{2\pi i k/n}$, hence $\pi_k e_j$ is identified with $\omega^j = \omega_{kj}$, where π_k is the orthogonal projection onto E_k. Here, e_j is the basis vector $(0, \dots, 0, 1, 0, \dots, 0)$ with zeros on all places but the j'th place.

We have the orthogonal decomposition $\mathbb{R}^n = D + E_1 + \ldots + E_m$ where $m = [\frac{n-1}{2}]$ and D is spanned by e and (if present) f.

The permutation α generates the cyclic group C_n of order n. The dihedral group D_n is the extension of C_n by the permutation $\beta : j \mapsto n - j \bmod n$. Let B be the corresponding linear map on $\mathbb{R}^n$ and $\mathbb{C}^n$. Then apparently $Bv_\omega = \overline{v_\omega}$, and thus the A-invariant planes E_k are also B-invariant for $k = 1, \dots, m$. Hence the E_k are irreducible and inequivalent D_n-modules (even after complexification).

The generalized Penrose tilings ("*n-rose tilings*") arise as follows. Let $E = E_1$ and $F = E_2 + \cdots + E_m$. For any $a \in F$ we put $E_a = E + a$ where $E := E_1$. As described above, the tiles are the orthogonal E_a-projections of all unit squares with integer vertices in the strip $\Sigma_a = E_a + I^n$ where $I = (0, 1)$.

We restrict our attention to the case where n is a prime (cf. [4] for the the composite case). The group D_n has the fixed vector e, but this is not in $W := e^\perp = E + F$. However, if we put $d = \frac{k}{n} e$ for some $k \in \{1, \dots, n-1\}$, then $a = d - (e_1 + \cdots + e_k) \in W$, and the tilings TE_d and TE_a are equivalent. Thus we find $n - 1$ tilings with D_n-symmetry. Since n is prime, the space W is rationally irreducible for D_n and by the results of Section 2, any two such tilings are almost equivalent.[6]

[6] The rational irreducibility of W is seen as follows. Let W^c and E_k^c be the complexifications of W and E_k. The 2-dimensional subspaces E_k^c are inequivalent D_n-modules. Thus any D_n-module $W_1 \subset W$ is a sum of some of the E_k. On the other hand, we claim that a nonzero rational vector $v = \sum \lambda_\omega v_\omega \in W^c \cap \mathbb{Q}^n$ has only nonzero coefficients, $\lambda_\omega \neq 0$ for all ω. In fact, since v is rational and $v_\omega \in \mathbb{K}^n$ where $\mathbb{K} = \mathbb{Q}(\omega)$ (i.e. the smallest field containing $\mathbb{Q}$ and the n-th unit roots), we may assume that all $\lambda_\omega \in \mathbb{K}$. Now $v \in \mathbb{Q}^n$ iff $v^\sigma = v$ for all $\sigma \in G_{\mathbb{K}}$ where $G_{\mathbb{K}}$ denotes the Galois group of $\mathbb{K}$ over $\mathbb{Q}$. Each $\sigma \in G_{\mathbb{K}}$ is of the type $\omega \mapsto \omega^k$ for $k \in \{1, \dots, n-1\}$. Hence $v^\sigma = \sum_\omega \lambda_\omega^\sigma v_{\omega^k}$, and $v^\sigma = v$ iff $(\lambda_{\omega^{-k}})^\sigma = \lambda_\omega$ for all ω. Hence, if $\lambda_\omega \neq 0$ for some ω, then also $\lambda_\beta \neq 0$ for $\beta = \omega^k$. Thus $\lambda_\omega \neq 0$ for all ω, and hence W_1^c cannot miss any E_k^c if it contains a nonzero rational vector v.

6. Dihedral integer matrix invariants

As before, let $n = 2m + 1$ be a prime and consider the dihedral group $D_n \subset S_n$ generated by the permutations $\alpha, \beta \in S_n$ with $\alpha j = j+1$ and $\beta j = n - j$ for all $j = 1, \dots, n \bmod n$. It acts on $\mathbb{R}^n$ by permuting the coordinates; the corresponding integer orthogonal matrices are denoted A, B. For any $l \in \{1, \dots, m\}$ we consider the linear map S_l on $\mathbb{R}^n$ defined by

$$S_l(e_j) = e_{j-l} + e_{j+l}. \tag{2}$$

Apparently, S_l commutes with A and B, and it is a symmetric matrix since

$$S_l = A^l + A^{-l} = A^l + (A^l)^T. \tag{3}$$

Thus S_l commutes with the projections π_k onto the invariant planes E_k defined above, and all these planes are eigenspaces of S_l. Identifying E_k with $\mathbb{C}$ as in (1) above, $A^l|_{E_k}$ is the multiplication with $\omega = e^{2\pi i l k/n}$ and hence $S_l = A^l + A^{-l}$ is the multiplication with the real factor

$$\lambda = \omega + \bar{\omega} = 2\cos(2\pi l k/n). \tag{4}$$

These integer linear maps S_l generate a commutative ring of integer matrix invariants of the group D_n. They are invertible elements (units) in this ring: in fact, $S_1 S_2 \dots S_m = \pm I$ on W since the product of all eigenvalues $\lambda_j = 2\cos(2\pi j/n)$ for $j = 1, \dots, m$ equals ± 1.[7] Hence any $S = \pm S_1^{k_1} S_2^{k_2} \dots S_m^{k_m}$ with nonnegative integers $k_1, \dots, k_m$ is integer invertible on W. Using the identity

$$S_k S_l = S_{k-l} + S_{k+l} \tag{5}$$

(with $S_0 := 2I$) which follows straight forward from the definition, we may easily compute S as an integer linear combination of $S_0, \dots, S_m$. If we manage to arrange the powers

[7] This can be seen for all odd $n = 2m + 1$ as follows. For any n-th unit root $\omega \neq 1$ we put $w_k = \omega^k + \bar{\omega}^k$. We have $\sum_{j=0}^{n-1} \omega^j = 0$ and hence

$$(*) \qquad w_1 + \dots + w_m + 1 = 0.$$

The terms w_k can be expressed by powers of $\lambda = \omega + \bar{\omega}$ as follows: We have $\lambda^1 = w_1$ and $\lambda^2 = w_2 + 2$ and $\lambda^k = w_k + \binom{k}{1} w_{k-2} + \binom{k}{2} w_{k-4} + \dots (+\binom{k}{l})$ where the last term is present only if $k = 2l$. Solving for w_k we obtain

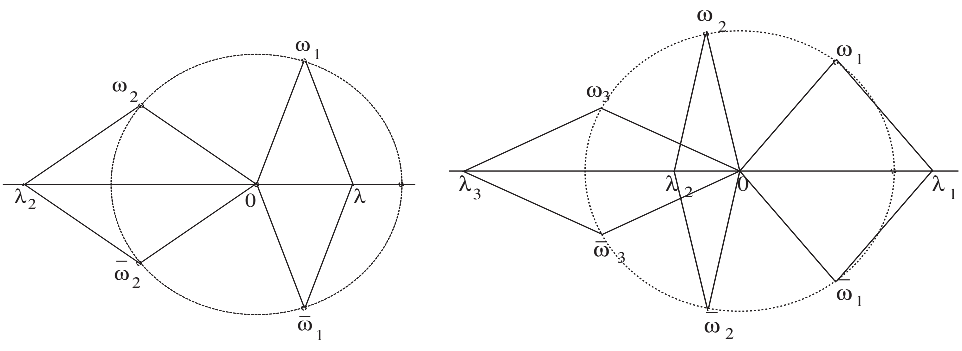
$$(**) \qquad w_k = \lambda^k - \binom{k}{1} w_{k-2} - \binom{k}{2} w_{k-4} - \dots (-\binom{k}{l})$$

(the last term is present only if $k = 2l$). When we apply the same formula to w_{k-2}, w_{k-4}, etc., w_k becomes a polynomial of degree k in λ. Inserting into $(*)$ we obtain a polynomial equation $p_m(\lambda) = 0$ of degree m, and the polynomial p_m is independent of the choice of ω. Hence the solutions are $\lambda_1, \dots, \lambda_m$, and their product is the λ^0-coefficient a_0 of $p_m(\lambda)$. E.g. we have $p_2(\lambda) = \lambda^2 + \lambda - 1$ (golden section) and $p_3(\lambda) = \lambda^3 + \lambda^2 - 2\lambda - 1$. Let v_k denote the λ^0-term of w_k. Apparently, $v_k = 0$ if k is odd. For even $k = 2j$, we claim $v_k = (-1)^j \cdot 2$. In fact, this is true for $j = 1$. By induction hypothesis we have $v_{2i} = (-1)^i \cdot 2$ for $i < j$. The induction step is obtained from $(**)$: $v_{2j} = -\binom{2j}{1} v_{2j-2} - \binom{2j}{2} v_{2j-4} - \dots - \binom{2j}{j} = -(-1)^j \sum_{i=1}^{2j-1} \binom{2j}{i} (-1)^i = (-1)^j \cdot 2$. This proves the claim. Now from $(*)$ we see $a_0 = 1 - v_2 - v_4 - \dots = (-1)^k$ for $k = [\frac{m}{2}]$.

$k_1, \ldots, k_m$ so that the eigenvalues μ_j of S satisfy

$$|\mu_j| > 1 \quad \forall j = 2, \ldots, m, \tag{6}$$

then S can serve as inflation map (cf. Section 4).



However, S is not integer invertible on all of $\mathbb{R}^n$: in the diagonal direction we have $S_l d = 2d$ for all l and hence $Sd = sd$ with

$$s = \pm 2^k, \quad k = \sum k_j. \tag{7}$$

Thus S^{-1} has an integer eigenvector $e = nd = \sum_j e_j$, with a non-integer eigenvalue $\frac{1}{s}$. We will address this problem in the next section.

7. The index of admissible points

For any integer point $z \in \mathbb{Z}^n$ the number $\langle z, e\rangle = \sum_j z_j = p \in \mathbb{Z}$ (where $e = \sum_j e_j$) will be called the *index* of z. The whole grid $\mathbb{Z}^n$ is contained in the union of the hyperplanes

$$H_p = \{x \in \mathbb{R}^n;\ \langle x, e\rangle = p\} = W + pd = W + e^p$$

for all $p \in \mathbb{Z}$ where $W = H_0 = e^\perp$ and $e^p \in \mathbb{Z}^n$ is any integer vector with index p; e.g. we may choose $e^p = e_1 + \cdots + e_p$. The integer points in H_p form the index p lattice

$$\Gamma_p = H_p \cap \mathbb{Z}^n = (W + e^p) \cap \mathbb{Z}^n = \Gamma_0 + e^p$$

where $\Gamma_0 = W \cap \mathbb{Z}^n$. When projected orthogonally onto W, any two of these lattices Γ_p, Γ_q differ by a translation, and they are equal iff $q \equiv p \bmod n$; then Γ_p and Γ_q differ by an integer multiple of $e = e^n \perp W$. Since our projection plane $E_a = E_1 + a$ is contained in W, it suffices to consider only the projections onto W, hence the index need to be considered only modulo n. If $z \in \mathbb{Z}^n$ is *admissible* for E_a, then $z - x = u \in I^n$ for some $x \in E_a$ and hence $\langle e, z\rangle = \langle e, u\rangle \in \{1, \ldots, n-1\}$. Thus admissible points have index $p \in \{1, \ldots, n-1\}$.

Our possible inflation map $S = \pm S_1^{k_1} \ldots S_m^{k_m}$ changes the index from p to sp since $Sd = sd$, cf. (7). However because S is invertible on Γ_0, it maps $\Gamma_p = \Gamma_0 + e^p$ bijectively onto $\Gamma_0 + Se^p = \Gamma_{sp}$.

S is an *inflation map* (cf. Section 4) iff

$$\pi_b(S(\Sigma_a \cap \mathbb{Z}^n)) \supset \pi_b(\Sigma_b \cap \mathbb{Z}^n) \tag{8}$$

where $b = Sa$ and $\Sigma_a = E_a + I^n$, and where π_b denotes the orthogonal projection onto E_b. Remembering the index we can rewrite this as

$$\pi_b(S(\Sigma_a \cap \Gamma_p)) \supset \pi_b(\Sigma_b \cap \Gamma_q) \tag{9}$$

where $q = sp$; in particular we need $sp \not\equiv 0 \bmod n$ if $p \not\equiv 0 \bmod n$. Since we already know that S maps Γ_p bijectively onto Γ_q, this is equivalent to

$$\pi_b(S(\Sigma_a \cap H_p)) \supset \pi_b(\Sigma_b \cap H_q). \tag{10}$$

We have $\Sigma_a = E_a + I^n = E_a + \pi_\perp(I^n)$ where $\pi_\perp$ is the projection onto $E_1^\perp \subset \mathbb{R}^n$, and hence

$$\Sigma_a \cap H_p = E_a + W_p, \quad W_p := \pi_\perp(I^n \cap H_p) \tag{11}$$

The set $W_p \subset E_1^\perp \cap H_p$ is sometimes called *window*. Hence $S(\Sigma_a \cap H_p) = S(E_a) + S(W_p) = E_b + S(W_p)$, and (10) becomes

$$\pi_b(SW_p) \supset \pi_b(W_q) \tag{12}$$

for all $p \in \{1, \ldots, n-1\}$, where $q \equiv sp \bmod n$. This shows that the condition (6) is not sufficient for S to be an inflation map, but it will suffice for a suitably large power S^k.

How does W_p look like? It is a projection of $I_n \cap H_p$ which in turn is the convex hull of the set V_p formed by the vertices of I^n with precisely p nonzero coordinates. The elements of V_p are parameterized by the subsets $J \subset \{1, \ldots, n\}$ with $|J| = p$; in fact, if we put $e_J = \sum_{j \in J} e_j$ for each such J, we have

$$V_p = \{e_J;\ J \subset \{1, \ldots, n\},\ |J| = p\}$$

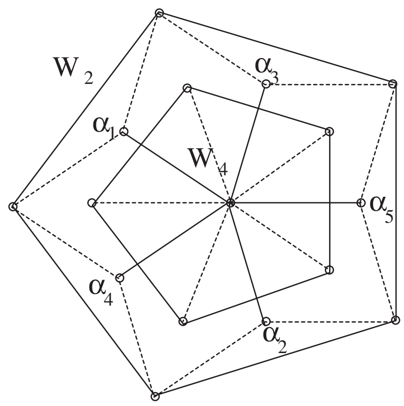
and W_p is the convex hull of $\pi_\perp(v_p)$. If $\bar{J}$ denotes the complement of J, then $e_{\bar{J}} = e - e_J$ and hence (after projecting to $W = e^\perp$) we have

$$W_{n-p} = -W_p\,. \tag{13}$$

In particular we see from (13) that a sign change does not matter: The condition (12) remains the same when passing from S to $-S$ since $W_{-q} = W_{n-q} = -W_q$.

8. The Penrose case

For $n = 5$ the situation has often been described, (cf. [1], [6], [7]). We have only S_1, S_2 with $S_1 + S_2 = -I$ and $S_1S_2 = -I$ on W. Choosing $S = -S_1$ with eigenvalues $\lambda_1 = -\frac{1}{\Phi}$ and $\lambda_2 = \Phi$ with $\Phi = \frac{1}{2}(1+\sqrt{5})$ (golden section), we have $q = -2p$ and hence $p = 1, 2, 3, 4$ corresponds to $q = 3, 1, 4, 2$ modulo 5. The sets $W_p \subset E_2$ are as follows. Let $\alpha_j = \pi_2(e_j)$. Clearly W_1 is the convex hull of $\alpha_1, \dots, \alpha_5$, a pentagon. W_2 in turn is the convex hull of all $\alpha_j + \alpha_k$ with $j \neq k$. The extremal points (generating the convex hull) are sums of direct neighbors. These are the points $\alpha_{i-1} + \alpha_{i+1} = S_1(\alpha_i)$ (recall that $\alpha_i \in E_2$ corresponds to the unit root ω_{2i} under the identification (1)). Hence W_2 is the pentagon with vertices $S_1(\alpha_j)$ and $W_3 = -W_2$ the one with vertices $-S_1(\alpha_j) = S(\alpha_j)$. Consequently $S(W_1) = W_3$, and the condition (12) is satisfied for $p = 1, q = 3$, (cf. the figure below). The same argument holds for $p = 4, q = 2$, and the condition is trivially satisfied for the two remaining pairs (p, q) since $W_p \supset W_q$. Hence S is an inflation map; in fact it defines the subdivision described in the introduction. The inflation tiling is obtained by applying $S^{-1} = S_2$.



9. The "Heprose" case

In the case $n = 7$, the map S_1 has three *eigenvalues* $\lambda_k = 2\cos(2\pi k/7)$ for $k = 1, 2, 3$ with $\lambda_1 \approx 1.247$, $\lambda_2 \approx -0.445$, $\lambda_3 \approx -1.802$ (cf. figure in Section 6), corresponding to the eigenspaces E_1, E_2, E_3. According to (4), S_2 and S_3 have the same eigenvalues in different order as displayed below. In particular the table shows that S_2 is an expansion on $F = E_2 + E_3$ and hence some power of S_2 is an inflation map.

The *index* is changed by every S_j from p to $q \equiv 2p \bmod 7$, i.e. $p \mapsto q$ is the order 3 permutation

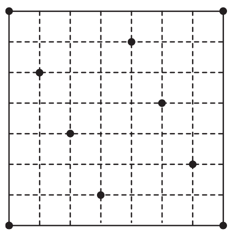
$$1 \mapsto 2 \mapsto 4 \mapsto 1, \quad 3 \mapsto 6 \mapsto 5 \mapsto 3. \tag{14}$$

	E_1	E_2	E_3
S_1	λ_1	λ_2	λ_3
S_2	λ_2	λ_3	λ_1
S_3	λ_3	λ_1	λ_2

The *windows* look more complicated than in the case $n = 5$: As explained in Section 7, W_1 is the convex hull of $\pi_F(e_j) = \pi_2(e_j) + \pi_3(e_j)$ for $j = 1, \dots, 7$. Using (1) we identify F with $\mathbb{C}^2$ and we get $\pi_F(e_j) = (\omega^{2j}, \omega^{3j})$ where $\omega := e^{2\pi i/7}$. Hence W_1 is the convex hull of the set

$$P_1 = \{(\omega^{2j}, \omega^{3j});\ j = 1, \dots, 7\} \subset \mathbb{C}^2$$

which lies on a Clifford torus $S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C}$ as drawn in the subsequent figure (identify parallel edges of the square).



W_2 in turn is the convex hull of $\pi_F(e_{j+k} + e_{j-k}) = \pi_F(S_k e_j) = (S_k \pi_2 e_j, S_k \pi_3 e_j)$ for $k = 1, 2, 3$. The corresponding point sets are contained in three different Clifford tori:

$$\begin{aligned} P_{21} &= \{(\lambda_2\omega^{2j}, \lambda_3\omega^{3j});\ j = 1, \dots, 7\}, \\ P_{22} &= \{(\lambda_3\omega^{2j}, \lambda_1\omega^{3j});\ j = 1, \dots, 7\}, \\ P_{23} &= \{(\lambda_1\omega^{2j}, \lambda_2\omega^{3j});\ j = 1, \dots, 7\}. \end{aligned}$$

Now we see that S_2 is not an inflation map: It maps W_1 perfectly onto the convex hull of P_{22}, but this is a proper subset of W_2 as we shall see below.

W_3 at last is the convex hull of the points $\pi_F(e_j + e_{j+k} + e_{j-l})$ with $j = 1, \dots, 7$ and $(k, l) = (1, 1), (2, 2), (3, 3), (1, 2), (2, 1)$. In the first three cases we have $\pi_F(e_j + e_{j-k} + e_{j+k}) = \pi_F((I + S_k)e_j)$ and we obtain the three point sets

$$\begin{aligned} P_{31} &= \{((1 + \lambda_2)\omega^{2j}, (1 + \lambda_3)\omega^{3j});\ j = 1, \dots, 7\}, \\ P_{32} &= \{((1 + \lambda_3)\omega^{2j}, (1 + \lambda_1)\omega^{3j});\ j = 1, \dots, 7\}, \\ P_{33} &= \{((1 + \lambda_1)\omega^{2j}, (1 + \lambda_2)\omega^{3j});\ j = 1, \dots, 7\}. \end{aligned}$$

In the two remaining cases we get for $j = 7$:

$$\pi_F(e_7 + e_1 + e_5) = (\omega^7 + \omega^2 + \omega^3,\ \omega^7 + \omega^3 + \omega^1) =: (\eta_1, \eta_2),$$

$$\pi_F(e_7 + e_2 + e_6) \;=\; (\omega^7 + \omega^4 + \omega^5,\ \omega^7 + \omega^6 + \omega^4) = (\bar{\eta}_1, \bar{\eta}_2),$$

and hence the corresponding point sets are

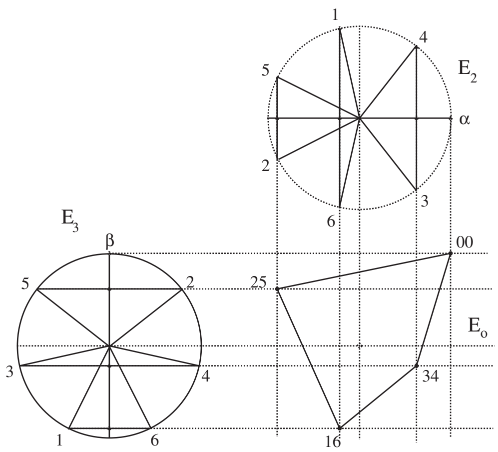
$$\begin{aligned} P_{34} &= \{(\omega^{2j}\eta_1, \omega^{3j}\eta_2);\ j = 1, \ldots, 7\}, \\ P_{35} &= \{(\omega^{2j}\bar{\eta}_1, \omega^{3j}\bar{\eta}_1);\ j = 1, \ldots, 7\}. \end{aligned}$$

(It might be interesting to note that $|\eta_i| = \sqrt{2}$ for $i = 1, 2.$[8]) Since $W_{7-p} = -W_p$, we have finished the description of the windows.

Now we have seen that all W_p are convex hulls of a union of point sets $P = P_{ij}$ which have a common form:

$$P = \{(\omega^{2j}\alpha, \omega^{3j}\beta);\ j = 1, \ldots, 7\} \tag{15}$$

for some $\alpha, \beta \in \mathbb{C}^*$. How can we understand the convex hull $\mathrm{CH}(P)$ of the set P? This is a subset of $\mathbb{C} \times \mathbb{C} = \mathbb{R}^4$ which may be hard to imagine, but we can draw its intersection with the plane $E_o := \mathbb{R}\alpha \times \mathbb{R}\beta$ which is at the same time its projection onto this plane (cf. subsequent figure).



In the figure we have drawn the planes E_2, E_3 and E_o. E.g. the point $25 \in E_o$ is the intersection of E_o with line segment between the points $2 := (\omega^{2\cdot 2}\alpha, \omega^{3\cdot 2}\beta)$ and $5 := (\omega^{2\cdot 5}\alpha, \omega^{3\cdot 5}\beta)$. Hence $\mathrm{CH}(P) \cap E_o$ is the convex hull or the four points $00 = (\alpha, \beta)$, 25, 16 and 34 as shown in the figure below. The coordinates of these vertices with respect to the basis α, β are easily read off from the figure: $00 = (1, 1)$, $25 = \frac{1}{2}(\lambda_3, \lambda_1)$, $16 = \frac{1}{2}(\lambda_2, \lambda_3)$, $34 = \frac{1}{2}(\lambda_1, \lambda_2)$.

Now we can test which of the maps $S = \pm S_1^{k_1} S_2^{k_2} S_3^{k_3}$ are inflation maps. A necessary condition is that $S(W_1) \supset W_s$ where $s = \pm 2^k$ with $k = k_1 + k_2 + k_3$ is the diagonal

[8] $(1 + \omega + \omega^3)(1 + \bar{\omega} + \bar{\omega}^3) = \sum_{k=1}^{3}(\omega^k + \bar{\omega}^k) + 3 = -1 + 3.$

eigenvalue of S. If $k = 1$ then S_2 is the only candidate which is an expansion on F. But it fails to satisfy $S_2(W_1) \supset W_2$ since $S_2(W_1) \not\supset \mathrm{CH}(P_{21})$: the point $(\lambda_2, \lambda_3) \in E_o \cap \mathrm{CH}(P_{21})$ is not contained in $E_o \cap \mathrm{CH}(S_2 P_1) \subset S_2(W_1)$ which is the convex hull of the points $S_2(00) = (\lambda_3, \lambda_1)$, $S_2(25) = \frac{1}{2}(\lambda_3^2, \lambda_1^2)$, $S_2(16) = \frac{1}{2}(\lambda_2\lambda_3, \lambda_3\lambda_1)$, $S_2(34) = \frac{1}{2}(\lambda_1\lambda_3, \lambda_2\lambda_1)$ all of whose second coordinates are bigger than λ_3.

For similar reasons, $S_2^2 = 2I + S_3$ (cf. (5)) as well fails to be an inflation map: Now we have $k = 2$ and $s = 4$, but $S_2^2(W_1) \not\supset W_4$ since the point $-(1+\lambda_3, 1+\lambda_1) \in -P_{32} \subset -W_3 = W_4$ is not contained in $\mathrm{CH}((2 + S_3)W_1) \cap E_o$: The second coordinates of the generating points are $\frac{1}{2}(2 + \lambda_1)$ multiplied by $2, \lambda_1, \lambda_3, \lambda_2$, and all these numbers are larger than $-(1 + \lambda_1)$. There are no further candidates with $k = 2$ since $S_i S_j = S_l^{-1}$ when $\{i, j, l\} = \{1, 2, 3\}$, and S_l^{-1} has two contracting eigenvalues.

The next case is $k = 3$ and $s = 8 \equiv 1 \bmod 7$, thus S preserves the index. In order to show the inflation property it suffices to show that $S(\mathrm{CH}(P)) \subset \mathrm{CH}(P)$ for each of the sets $P = P_{ij}$.

LEMMA 9.1. *S with $s \equiv 1$ mod 7 is an inflation map if and only if*

$$S(W_1 \cap E_o) \supset W_1 \cap E_o. \tag{16}$$

Proof. Clearly the condition (16) is necessary. But it is also sufficient: First of all, if the condition holds for $W_1 \cap E_o = \mathrm{CH}(P_1) \cap E_o$, it also holds for $\mathrm{CH}(P) \cap E_o$ for each of the sets $P = P_{ij}$ since S acts in the same way in all cases. Further, due to the D_7-invariance of S, the condition still holds when we replace the plane $E_o = \mathbb{R}\alpha + \mathbb{R}\beta$ by one of the corresponding planes $\mathbb{R}\omega^{2j}\alpha + \mathbb{R}\omega^{3j}\beta$. But then in particular $(\omega^{2j}\alpha, \omega^{3j}\beta) \in S(\mathrm{CH}(P))$, and since $\mathrm{CH}(P)$ is the convex hull of these points, we obtain $S(\mathrm{CH}(P)) \supset \mathrm{CH}(P)$ and hence $S(W_j) \supset W_j$. □

THEOREM 9.2. *$S = S_2^3$ is an inflation map for any Heprose tiling.*

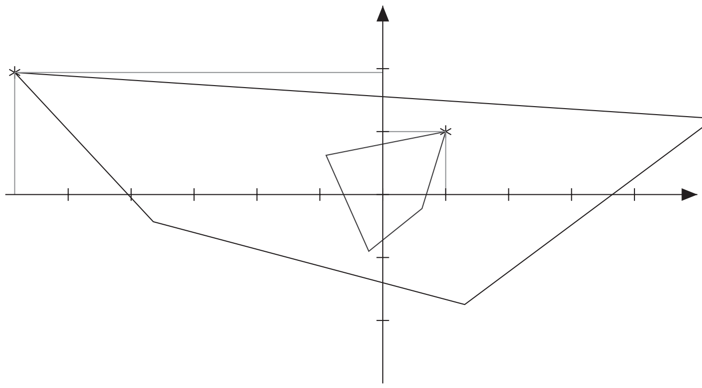
Proof. We just have to show that the plane quadrangle with vertices (λ_3, λ_1), (λ_2, λ_3), (λ_1, λ_2), $(2, 2)$ is contained in its image under the diagonal matrix $diag(\lambda_3^3, \lambda_1^3)$, see the figure below. □

The *inflation subtiling* of a given tiling is obtained by applying $S^{-1} = S_2^{-3}$, (cf. Section 4). We have $S_2^{-1} = S_1 S_3 = -(S_1 + 1)$, hence

$$S_2^{-3} = -(S_1 + 1)^3 = -(S_1^3 + 3S_1^2 + 3S_1 + 1).$$

Using $S_1^2 = 2 + S_2$ and $S_1^3 = (2 + S_2)S_1 = 2S_1 - S_2 - 1$ we obtain

$$S_2^{-3} \;=\; -(2S_1 - S_2 - 1 + 6 + 3S_2 + 3S_1 + 1)$$



$$
\begin{aligned}
&= -(2S_1 - S_2 + 6 - 3S_3 - 3)\\
&= -2S_1 + S_2 + 3S_3 - 3
\end{aligned}
$$

where we have used the identity $S_1 + S_2 + S_3 + 1 = 0$. Thus the inflation factor is $2|\lambda_1| + |\lambda_2| + 3|\lambda_3| + 3$ which we can see in the tilings displayed below.

There are still other candidates S of order $k = 3$: consider the map

$$S := S_1 S_2^2 = S_1(2 + S_3) = 2S_1 - S_1 - 1 = S_1 - 1.$$

Its eigenvalues $1 - \lambda_1 \approx 0.247$, $1 - \lambda_2 \approx 1.445$, $q - \lambda_3 \approx 2.802$ have the right behavior. However, as it turns out, the condition (16) is violated (cf. the following figure). On the other hand, only a small portion of $W_1 \cap E_o$ is outside $S(W_1 \cap E_o)$. Hence very few vertex points of the inflation tiling do not belong to the original tiling. In the example displayed below there are no errors visible. Passing to $S^2 = S_1^2 S_2 S_2^3$, we get a true inflation map; this is independent of the previous one, S_2^3, since $S_1^2 S_2$ is not an inflation map. Note that

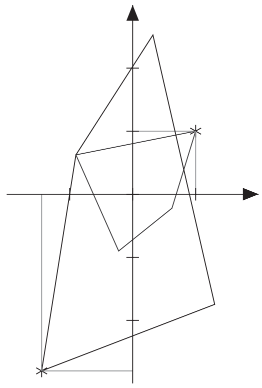
$$S' := S_1 S_3^2 = S_1(2 + S_1) = 2S_1 + S_2 + 2 = S_1 - S_3 + 1$$

is the inverse of S since $SS' = (S_1 S_2 S_3)^2 = I$. This shows the inflation factor of S, and we also see that there are no further examples with $k = 3$.

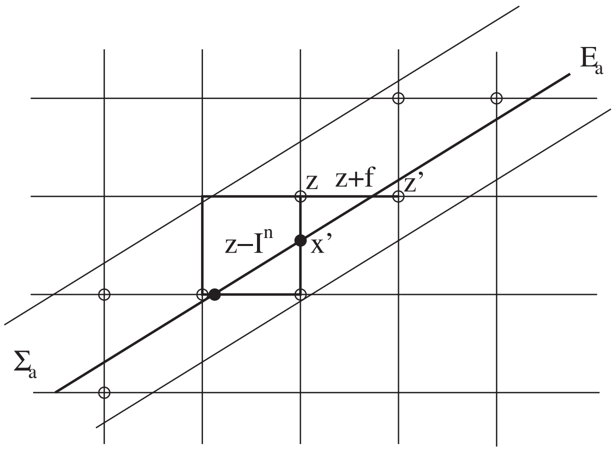
10. Displaying the new patterns

Let E_a be a r-dimensional affine subspace of $\mathbb{R}^n$, $r < n$, whose points have at most r integer coordinates ("general position"). An integer grid point $z \in \mathbb{Z}^n$ is admissible iff there is some $x \in E_a$ such that $z \in x + I^n$, or in other words, $(z - I^n) \cap E_a \neq \emptyset$.

Stated slightly differently, $z \in \mathbb{Z}^n$ is admissible if the subspace E_a intersects the closed n-cube $z - \bar{I}^n$ in a point x' with r integer values, say $x'_{i_1} = z_{i_1} - \epsilon_1, \ldots, x'_{i_r} = z_{i_r} - \epsilon_r$



where $1 \leq i_1 < \cdots < i_r \leq n$ and $\epsilon_1, \ldots, \epsilon_r \in \{0, 1\}$ (cf. next figure). Conversely, for any point $x' \in E_a$ with r integer coordinates the integer vector $z = \lceil x' \rceil$ is admissible where $\lceil x' \rceil$ denotes the integer vector z with coordinates $z_j \in [x'_j, x'_j + 1)$.



Any such admissible point z is the lowest vertex of an admissible r-cube $z + f$ where $f \subset \bar{I}^n$ is an r-face of the unit cube $\bar{I}^n$, and these r-cubes $z + f$ project onto tiles in E_a. The neighbors of z in $z + f$ are precisely the $2^r - 1$ integer points $z' \neq z$ with $z'_j - z_j = 0$ for $j \notin \{i_1, \ldots, i_r\}$ and $z'_j - z_j \in \{0, 1\}$ for $j \in \{i_1, \ldots, i_r\}$.

Checking admissibility of a tile reduces therefore to check for admissibility of its vertex z with lowest coordinate values. This is simply done by finding the corresponding x' from z by solving the following equations and then checking if $z' = \lceil x' \rceil$:

$$\begin{aligned} x'_{i_k} &= \; z'_{i_k} && \text{for } k = 1, \ldots, r \\ \langle x', v_j \rangle &= \; \langle a, v_j \rangle && \text{for } j = 1, \ldots, n - r, \end{aligned}$$

where $a \in E_a$, and $(v_j)_{j=1,\ldots,n-r}$ is a basis of the linear subspace $E_a^\perp$ perpendicular to E_a.

In the "n-rose" case with $E_a = E_1 + a$ we choose for (v_j) the set of those eigenvectors v_ω of the cyclic permutation matrix A which are perpendicular to E_1, i.e. the v_ω with $\omega \neq \omega_1, \bar{\omega}_1$ where $\omega_1 := e^{2\pi i/n}$ (cf. Section 5). Since x' in the above equations is real, it does not matter if v_ω is a complex vector.

Below, we have displayed the three symmetric Heprose tilings for E_{jd} with $j = 1, 2, 3$ together with their inflation tilings by $-S_1^{-3}$ and the quasi-inflation tilings by $(S_1 S_2^2)^{-1}$ (see subsequent figures). The big and tiny numbers[9] denote the indices of the vertex points of the inflation tiling and the original one.

11. "Elevenrose" tilings

We briefly discuss the next prime number $n = 11$.[10] Here we have five planes E_k, $k = 1, \ldots, 5$ and five generating invariant matrices $S_1, \ldots, S_5$ with eigenvalues $\lambda_k = 2\cos(2\pi k/11)$. According to (4), the eigenvalues of S_l on E_k are as in the table below. The

	E_1	E_2	E_3	E_4	E_5
S_1	λ_1	λ_2	λ_3	λ_4	λ_5
S_2	λ_2	λ_4	λ_5	λ_3	λ_1
S_3	λ_3	λ_5	λ_2	λ_1	λ_4
S_4	λ_4	λ_3	λ_1	λ_5	λ_2
S_5	λ_5	λ_1	λ_4	λ_2	λ_3
$S_2 S_3^3$	-0.019	9.255	1.101	1.356	-3.780

last line shows that $S = S_2 S_3^3$ satisfies the condition (6) and hence some power of S is an inflation map. We conjecture that there are inflation maps for any "n-rose" tiling if $n \geq 5$ is prime.

12. Discussion

There are three types of "Heprose" tilings with full D_7-symmetry, and each of them is self similar in the sense that its vertex set is mapped to a proper subset by a homothety, even in several different ways. Further, there is an uncountable number of tilings without D_7-symmetry, and again a subset of its vertex set is a homothetic image of the vertex set of (another) such tiling. Moreover, each of these tilings is arbitrarily close to the symmetric ones after some (maybe very large) translation. All these properties are shared with the Penrose tilings.

[9] The tiny numbers become visible only when the picture is enlarged.
[10] "Elevenrose" or Greek: "Endecrose"

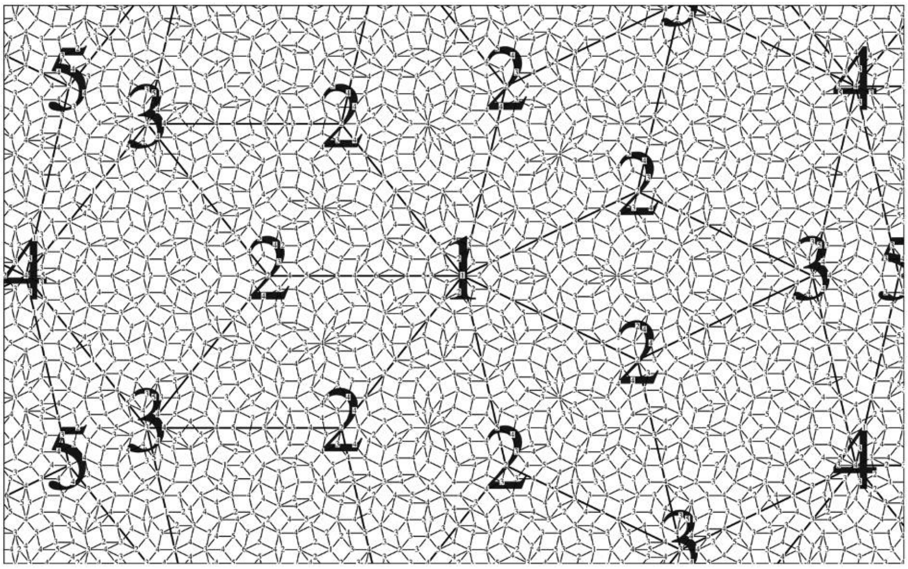


Figure 1 $S = S_2^3$, index=1

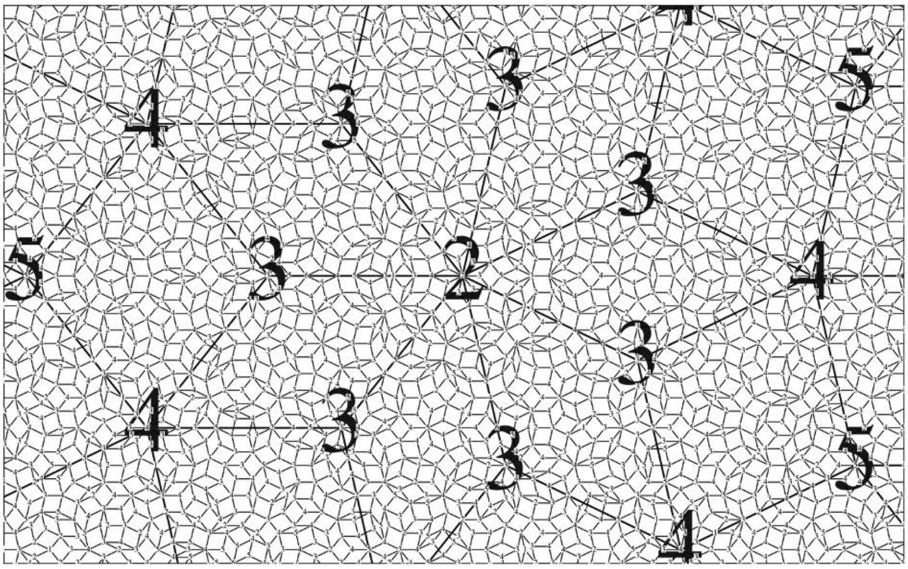


Figure 2 $S = S_2^3$, index=2

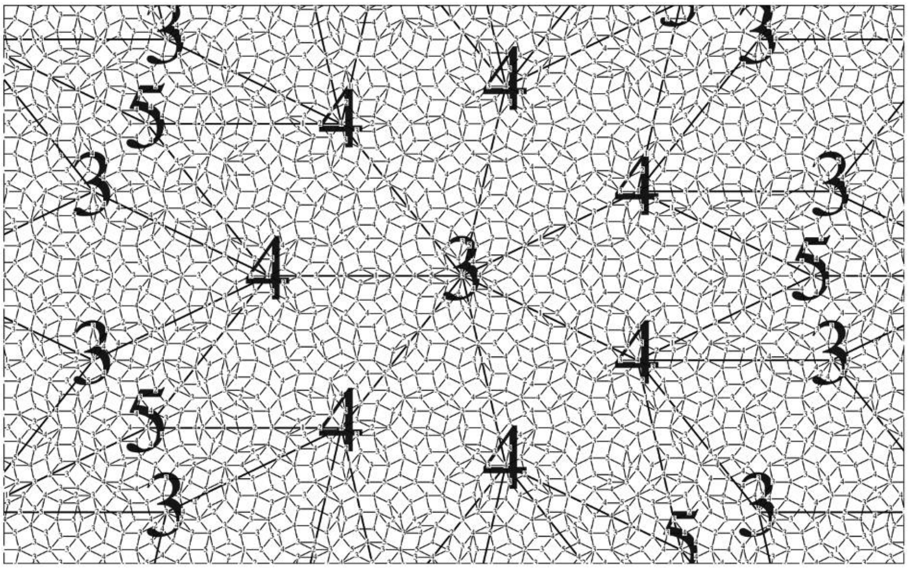


Figure 3 $S = S_2^3$, index=3

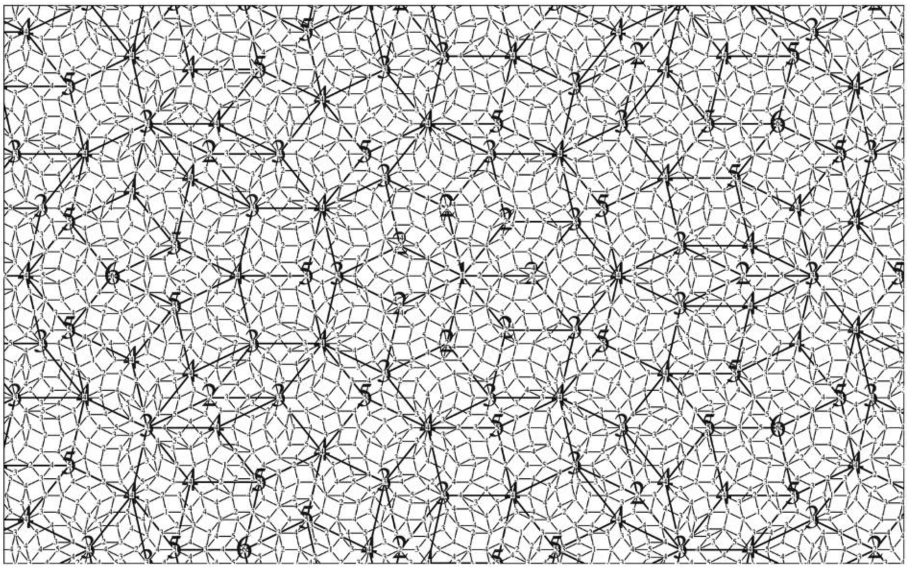


Figure 4 $S = S_1 S_2^2$, index=1

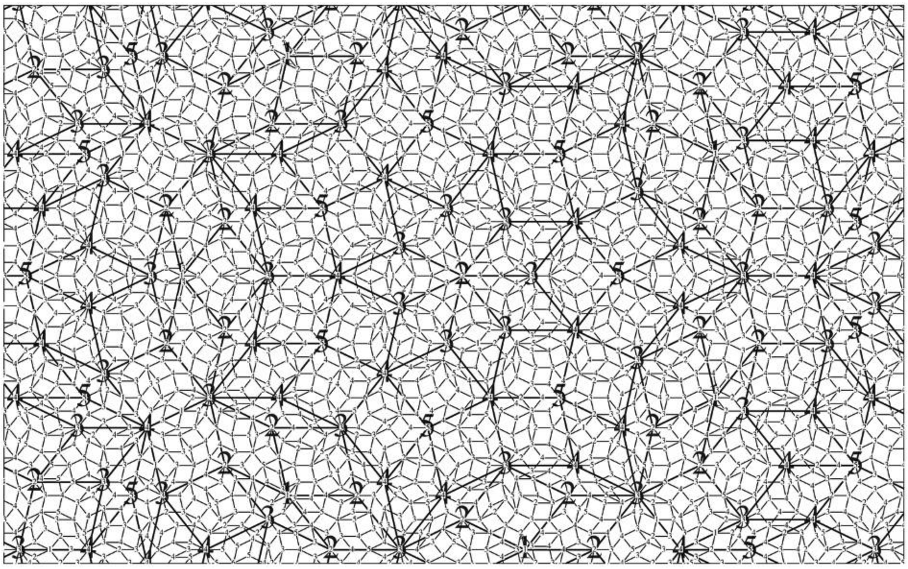


Figure 5 $S = S_1 S_2^2$, index=2

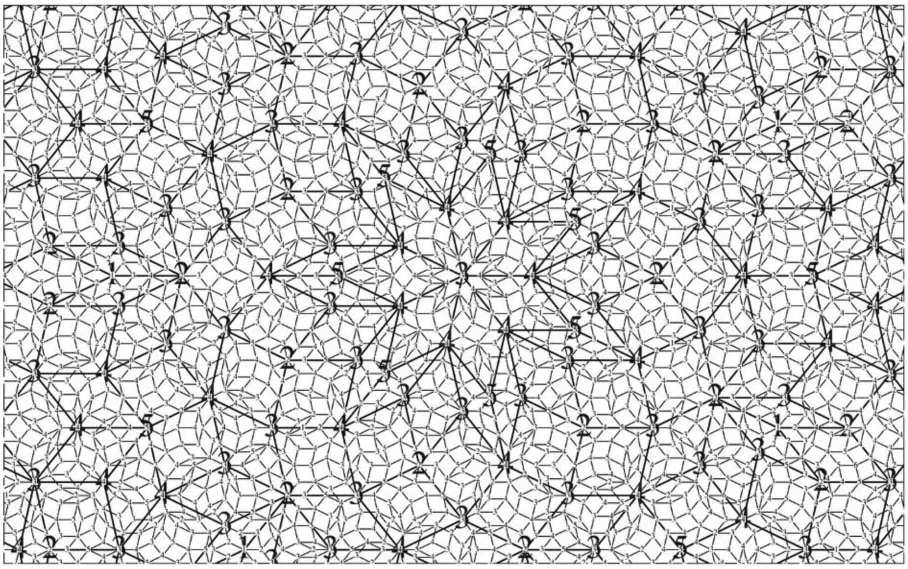


Figure 6 $S = S_1 S_2^2$, index=3

However, the tiles are cut into rather complicated pieces by the edges of the coarser tiling, and each large tile allows a huge (but finite) number of different subtilings. Therefore, unlike in the Penrose case, it would not be possible to obtain such tilings from repeatedly subdividing a tile and enlarging the pieces: since the decomposition is not unique, we do not know the right subdivision to take.

There are other tilings related to the heptagon where the tiles do have a fixed decomposition (cf. [2], [3]), however they were not constructed by the projection method and they do not allow a precise D_7 symmetry. For these reasons, the original Penrose tiling is extremely special since it satisfies all nice properties together.

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