REAL FUCHSIAN EQUATIONS AND CONSTANT MEAN CURVATURE SURFACES

J. DORFMEISTER, J.-H. ESCHENBURG

ABSTRACT. In this note we discuss geometric applications of the classical theory (going back to H.A. Schwarz [S]) of second order linear ODEs

$$y'' + py' + qy = 0$$

where p, q are real rational functions with only regular singularities in $\hat{R} = \mathbb{R} \cup \{\infty\}$, so called (real) Fuchsian equations. In particular we investigate in which cases the monodromy group is (up to conjugation) contained in the isometry group of the 2-sphere, the euclidean plane or the hyperbolic plane. As an application we study punctured spheres of constant mean curvature in euclidean 3-space where all punctures lie on a common circle.

1. Introduction

Surfaces of constant mean curvature (cmc) in euclidean 3-space display many similarities to minimal surfaces. They are local area minimizers, however under the constraint that the variations preserve the volume bounded by the surface [BD]. They come in associated families, i.e. isometric one-parameter deformations preserving the principal curvatures while rotating the principal curvature directions. And, most importantly, all cmc surfaces are constructed globally by a generalized Weierstrass representation [DPW] which explicitly involves the deformation parameter λ. The surfaces (immersions) f_λ, defined on a simply connected open domain $S \subset \hat{\mathbb{C}}$, are obtained from a fundamental system of the complex ODE on S with parameter λ,

$$y'' + py' + qy = 0. \tag{1}$$

Here p and q are holomorphic functions of $x \in S$ and $\lambda \in \mathbb{C}^*$; they can be viewed as generalized Weierstrass data for f_λ.

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If S is no longer simply connected, the surfaces f_λ are defined on the universal cover $\tilde{S}$ rather than on S itself. A critical quantity is the *monodromy group* $M_\lambda \subset GL_2(\mathbb{C})$ of the ODE (1), see below. If M_λ is unitarizable for some λ, i.e. if there is a matrix $A \in GL_2(\mathbb{C})$ such that $AM_\lambda A^{-1} \subset U_2$, then the deck transformations of the covering $\tilde{S} \to S$ act as euclidean motions on f_λ, cf. [DW1]. We need an additional closing condition in order to assure that f_λ is defined on S.

In the present paper, S will be the multiply punctured sphere,

$$S = \hat{\mathbb{C}} \setminus \{s_1, \dots, s_k\}$$

with $k \geq 3$. For $k = 2$, examples are the unduloids, i.e. embedded surfaces of revolution of Delaunay, see [E2],[E1]. If we fix the mean curvature, the Delaunay surfaces are determined by a single quantity, the *neck size*. It has been shown in [KKS] that embedded ends look asymptocially like the ends of unduloids. This imposes restrictions on the functions p, q on $\hat{\mathbb{C}} \setminus \{s_1, \dots, s_k\}$. In particular, we assume that p and q are rational functions of x with poles at s_j. The main question is which of the admissible data p, q lead to cmc surfaces with embedded ends, so called *k-noids*. In this case, unitarizability of the monodromy for all λ is already sufficient to ensure that $f = f_1$ is defined on S, see [DW1].

If our k-noid is planar and is symmetric under reflection in the k-noid plane, then all punctures lie on the equator, the real axis $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\} \subset \hat{\mathbb{C}}$. This translates on S to an anti-holomorphic involution fixing the real axis. Thus we are led to assume that p and q are real along the real axis.

The paper is mainly concerned with the differential equation (1), a special type of a Fuchsian equation. The question of unitarizability of the monodromy can be transformed into a geometric question in its own right which has been discussed and partially answered hundred years ago by Hilb [H] and his student Gerstenmeier [G]. We will present and apply some of these ideas. The application to cmc surfaces will be presented in Section 11 at the end of the paper.

2. Regular singular equations

Let us consider the general linear second order ODE

$$y'' + py' + qy = 0 \tag{2}$$

where p, q are holomorphic functions on an open domain $S \subset \hat{\mathbb{C}}$. An isolated singularity $s \in \hat{\mathbb{C}}$ of p or q is called *regular* for (2) if it is a simple pole for p and/or a pole of order at most two for q. We will assume that p, q are real functions (i.e. $p(x), q(x) \in \mathbb{R}$ for all $x \in \mathbb{R}$ where p, q are defined) and that all singularities are real, too.

E.g. $s = 0$ is a regular singularity iff $p = \hat{p}/x$ and $q = \hat{q}/x^2$ with regular $\hat{p}, \hat{q}$. This means that (2) allows solutions of the form $y = x^\alpha \hat{y}$ with regular $\hat{y}$ where α solves the quadratic equation

$$\alpha(\alpha - 1 + p_0) + q_0 = 0 \tag{3}$$

with $p_0 = \hat{p}(0)$ and $q_0 = \hat{q}(0)$. We assume (cf. [DW1]) that

$$(p_0 - 1)^2 > 4q_0, \tag{4}$$

i.e. there exist two real solutions α' and α'' of (2). From (2) we obtain a similar differential equation for $\hat{y} = y/x^{\alpha'}$ where the solutions of the corresponding quadratic equation (3) are shifted to 0 and $\alpha := \alpha'' - \alpha'$. In other words, for the new equation we have $q_0 = 0$ which means that the pole of q is also first order, and

$$p_0 = 1 - \alpha. \tag{5}$$

Then we have a fundamental basis of (2)

$$y_1 = x^\alpha \hat{y}_1 \text{ and } y_2 \tag{6}$$

where $\hat{y}_1$ and y_2 are regular with nonzero value at 0. Of course we may replace 0 with any other isolated singularity $s \in \mathbb{C}$ by changing the independent variable from x to $\tilde{x} = x - s$.

Now we assume $S = \mathbb{C} \setminus \{s_1, \ldots, s_{k-1}\}$ and all singularities s_j with $1 \leq j \leq k-1$ as well as $s_k := \infty$ are regular; then (2) is called a *Fuchsian equation*. If we apply the above transformations at all finite singular points $s_1, \ldots, s_{k-1}$, our coefficient functions p, q are almost determined. Using (5) at any s_j we obtain:

$$p = \sum_{j=1}^{k-1} \frac{1 - \alpha_j}{x - s_j}, \quad q = \frac{f(x)}{\prod_{j=1}^{k-1}(x - s_j)} \tag{7}$$

where

$$f(x) = Ax^{k-3} + Bx^{k-4} + \ldots \tag{8}$$

is a polynomial of degree $k - 3$ in order to ensure that q has a pole of degree two at ∞ (see below). We will assume throughout this paper that $\alpha_j \in (0, 1)$, for all $j = 1, ..., k$. Thus we also assume this for the quantity α_k at ∞. In our application to cmc k-noids, the quantitiy α_j is determined by the neck size of the Delauney end at s_j, cf. [DW1].

3. The singularity at ∞

In order to investigate the singular point at ∞ we are changing the independent variable from x to $\tilde{x} = 1/x$, i.e. we put $y(x) = \tilde{y}(\tilde{x})$ where $\tilde{x} = 1/x$. Then $y' = -\tilde{x}^2\tilde{y}'$ and $y'' = 2\tilde{x}^3\tilde{y}' + \tilde{x}^4\tilde{y}''$, and $\tilde{y}$ satisfies the equation

$$\tilde{y}'' + \tilde{p}\tilde{y}' + \tilde{q}\tilde{y} = 0$$

with

$$\tilde{p} = \frac{2}{\tilde{x}} - \frac{p}{\tilde{x}^2}, \quad \tilde{q} = \frac{q}{\tilde{x}^4}.$$

From (7) we see that the lowest order coefficients are

$$\tilde{p}_o = \sum_{j=1}^{k-1} \alpha_j - (k-3), \quad \tilde{q}_o = A.$$

Thus from (3) we derive that the exponents α at $\tilde{x} = 0$ satisfy

$$\alpha \left(\alpha + \sum_{j=1}^{k-1} \alpha_j - (k-2) \right) + A = 0$$

and hence the solutions α', α'' satisfy

$$\alpha' + \alpha'' = k - 2 - \sum_{j=1}^{k-1} \alpha_j, \quad \alpha'\alpha'' = A.$$

As before we are interested primarily in the difference

$$\alpha_k := \alpha' - \alpha''.$$

Then

$$\begin{aligned} \alpha_k^2 &= (\alpha' + \alpha'')^2 - 4\alpha'\alpha'' \\ &= \left(k - 2 - \sum_{j=1}^{k-1} \alpha_j \right)^2 - 4A. \end{aligned} \tag{9}$$

Thus the condition $\alpha_k \in (0,1)$ amounts to

$$0 < \left(k - 2 - \sum_{j=1}^{k-1} \alpha_j \right)^2 - 4A < 1. \tag{10}$$

Remark. For later use we put

$$\delta = k - 2 - \sum_{j=1}^{k} \alpha_j = \alpha' + \alpha'' - \alpha_k = 2\alpha''. \tag{11}$$

This quantity has a geometric meaning: The numbers $\alpha_j\pi$ will be the angles of a certain planar k-gon with circular arcs (see Theorem 7.1), and $\pi\delta$ is the deviation of $\pi \sum_{j=1}^{k} \alpha_j$ from the angle sum of a planar k-gon with straight edges (which is $(k-2)\pi$). From (9) we obtain

$$\alpha_k^2 - (\alpha_k + \delta)^2 = -4A, \tag{12}$$

Consequently, if $A < 0$, we have $\delta < 0$, using $\alpha_k > 0$,.

4. Simplifying the equation between singular points

We may remove the function p from (2) by replacing x by another parameter t with $x = x(t)$. Denoting $\dot{y} = \frac{dy}{dt}$ while $y' = \frac{dy}{dx}$, we have $y' = \dot{y}t'$ and $y'' = \ddot{y}(t')^2 + \dot{y}t''$, and (2) will be transformed into

$$(t')^2\ddot{y} + (t'' + pt')\dot{y} + qy = 0. \tag{13}$$

In order to remove the $\dot{y}$-Term we choose $t(x)$ such that $t'' + pt' = 0$, e.g. $t' = e^{-\int p}$. For p as in (7) we have

$$\int p = \ln \prod_{j=1}^{k-1} |x - s_j|^{1-\alpha_j}$$

and

$$t' = \prod_{j=1}^{k-1} \frac{1}{|x - s_j|^{1-\alpha_j}}. \tag{14}$$

Now we obtain from (13) and (7) the transformed equation

$$\ddot{y} + ry = 0 \tag{15}$$

with

$$r = q/(t')^2 = \prod_{j=1}^{k-1} \frac{|x - s_j|^{2-2\alpha_j}}{(x - s_j)} f(x).$$

Let us consider a fundamental system y_1, y_2 as in (6), Section 2, replacing $x = 0$ by any of the singular points, say s. We have $\dot{y}_2 = y_2'/t' \to 0$ for $t \searrow t(s)$. Moreover, $y_1 = \tilde{y}(x - s)^\alpha$ for some regular function $\tilde{y}$ with $\tilde{y}(s) \neq 0$, hence $y_1' = \tilde{y}'(x - s)^\alpha + \tilde{y}\alpha(x - s)^{\alpha-1}$, and therefore $\dot{y}_1 = y_1'/t'$ has a finite nonzero limit as $t \searrow t(s)$. Thus we may assume that the initial values at $x = s$ or $t = 0$ are

$$y_1 = 0, \quad \dot{y}_1 = 1, \qquad y_2 = 1, \quad \dot{y}_2 = 0. \tag{16}$$

5. Logarithmic derivatives

From (15) we obtain a nonlinear first order equation for the logarithmic derivative $u = \dot y/y$ of a solution y of (2): Since $\dot u = \frac{\ddot y}{y} - \frac{\dot y^2}{y^2} = -r - u^2$,

$$\dot u + u^2 + r = 0. \tag{17}$$

This Riccati type equation has a nice comparison theory: If we consider two coefficient functions $r, \tilde r$ with $r > \tilde r$ on some open interval I and solutions $u, \tilde u$ of the corresponding equation (17) which are defined on I, the difference $w = \tilde u - u$ satisfies

$$\dot w + (\tilde u + u)w + \tilde r - r = 0.$$

Hence $\dot w > 0$ whenever $w = 0$. Thus w cannot pass the value zero from above. This means: If $w(x_o) \geq 0$ for some $x_o \in I$, then $w(x) > 0$ for all $x \in I$ with $x > x_o$, and if $w(x_0) \leq 0$, then $w(x) < 0$ for all $x \in I$ with $x < x_o$. Thus we have seen:

Lemma 5.1. *Let $u, \tilde u : I \to \mathbb{R}$ be solutions of (17) corresponding to coefficient functions $r, \tilde r$ with $r > \tilde r$. Let $x_o \in I$ and put $I_+ = \{x \in I;\ x > x_o\}$ and $I_- = \{x \in I;\ x < x_o\}$. Then:*
(a) If $u(x_o) \leq \tilde u(x_o)$, then $u < \tilde u$ on I_+.
(b) If $u(x_o) \geq \tilde u(x_o)$, then $u > \tilde u$ on I_-. □

In particular we consider $u_1 = \dot y_1/y_1$ and $u_2 = \dot y_2/y_2$ for $x > s$, starting with $+\infty$ and 0 at the initial point s. Similar, for $x < s$ we consider the solution u_1^- with final value $-\infty$ at s.[1] Since $(x-s)^{\alpha-1}$ is integrable on I_+ (due to $\alpha < 1$), we can choose $t(x)$ with $t(s) = 0$, cf. (14).

If $r(t) > 0$ for all $t > 0$, then (u_1, u_2) roughly behave like $(\cot, -\tan)$: We can compare with the solutions $1/(t - t_o)$ and 0 for (17) with $\tilde r = 0$, and if r is sufficiently positive in a fixed subinterval I', say $r > 1/\omega^2$ on I, then u_1 and u_2 attain a given number of poles on I' as we see by comparing with the solution $\tilde u = \omega \cot \omega(t - t_o)$ of (17) for $\tilde r = 1/\omega^2$. If however $r < 0$, then we may compare with $\tilde r = 0$ or $\tilde r = -1/\omega^2$ (reversing the above rôles of r and $\tilde r$), and (u_1, u_2) behave roughly like $(\coth, \tanh)$: On I_+ they are above $1/t)$ and 0, respectively, and in particular they have both the same sign. On I_- in turn, u_1^- and u_2^- are below $1/t$ and 0, respectively.

[1] We have $y_1(x) \to 0$ and $\dot y_1(x) \to 1$ for $x \searrow s$, thus $u_1(x) \to +\infty$. Similar, for $x \nearrow s$ we have $y_1^-(x) \to 0$ and $y_1'(x) \to -1$ and thus $u_1^-(x) \to -\infty$.

6. Monodromy

As before, we consider the multiply connected punctured sphere $S = \hat{\mathbb{C}} \setminus \{s_1, ..., s_k\}$ which, in view of $s_k = \infty$, we can interpret as $S = \mathbb{C} \setminus \{s_1, ..., s_{k-1}\}$. In the preceeding sections we have discussed the solutions to the ODE (2),(7) near the points $s_j, j = 1, ..., k$. Since S is not simply connected, there is *monodromy*: If we extend a solution along a closed path which is defined in the domain where p, q are defined and which is not contractible, then after returning to the starting point we will end up in general with a solution which is different from the original one. Starting from a fundamental system y_1, y_2, we obtain at the end of the path another fundamental system $\tilde{y}_1 = ay_1 + by_2$, $\tilde{y}_2 = cy_1 + dy_2$. The matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{C})$ is called *monodromy matrix*. More generally, the monodromy is a group homomorphism $\mu : \pi_1(S) \to GL_2(\mathbb{C})$ where $\pi_1(S)$ denotes the fundamental group of S. In our geometric applications the monodromy matrices will depend on the additional "loop parameter" $\lambda \in S^1$ and will be contained in $SL_2(\mathbb{C})$.

Example. Let us consider (2) with a regular singularity at 0 as in Section 2. Then $S = \mathbb{C}^*$, and the solutions of (2) are $y_1 = x^{\alpha'}\hat{y}_1$ and $y_2 = x^{\alpha''}\hat{y}_2$ where $\hat{y}_1, \hat{y}_2$ are holomorphic at 0. Thus along any simply closed curve around 0, the solution y_1 changes to $\tilde{y}_1 = e^{\alpha'(\ln x + 2\pi i)}\hat{y}_1 = e^{2\pi i\alpha'}x^{\alpha'}\hat{y}_1 = e^{2\pi i\alpha'}y_1$, and similarly y_2 changes to $e^{2\pi i\alpha''}y_2$, hence

$$\tilde{y}_1 = e^{2\pi i\alpha'}y_1, \quad \tilde{y}_2 = e^{2\pi i\alpha''}y_2, \tag{18}$$

and the monodromy matrix is

$$M = \begin{pmatrix} e^{2\pi i\alpha'} & 0 \\ 0 & e^{2\pi i\alpha''} \end{pmatrix}. \tag{19}$$

The same result is obtained if one replaces $\mathbb{C}^*$ by any small disk with its center removed.

It is easy to see that the class of Fuchsian equations is invariant under fractional linear transformations $\tilde{x} = \frac{ax+b}{cx+d}$ of the independent variable, and such transformations do not change the monodromy. In particular, (18) and (19) remain true for any of the singularities s_j which we see by passing to $\tilde{x} = x - s_j$ or to $\tilde{x} = 1/x$.

However, the monodromy does depend on the fundamental system: Let $\mathsf{y} = \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right)$ and $\tilde{\mathsf{y}} = \left(\begin{smallmatrix} \tilde{y}_1 \\ \tilde{y}_2 \end{smallmatrix}\right)$ be two fundamental systems with $\tilde{\mathsf{y}} = W\mathsf{y}$ and M and $\tilde{M}$ the monodromy matrices for y and $\tilde{\mathsf{y}}$ with respect to some fixed closed curve. Then $\tilde{M}\tilde{\mathsf{y}} = WM\mathsf{y}$ and hence

$$\tilde{M} = WMW^{-1}. \tag{20}$$

7. Schwarz's function

Now let (y_1, y_2) be a fundamental system for (2),(7). The discussion of the quotient

$$\eta = y_1/y_2 \tag{21}$$

was suggested by H.A. Schwarz [S]. We have $\eta' = w/y_2^2$ with $w = y_1'y_2 - y_1y_2'$. From $w' = y_2''y_1 - y_2y_1'' = -pw$ we see that

$$w = C \cdot e^{-\int p} \tag{22}$$

for some real constant C whenever the antiderivative $\int p$ of p exists. In particular, $\int p$ exists on the upper half plane

$$\mathbb{H} = \{x = u + iv;\ u, v \in \mathbb{R},\ v > 0\}$$

and η is a local diffeomorphism on $\mathbb{H}$. Further, η is strongly monotonic, $\eta' > 0$ or $\eta' < 0$, on any nonsingular intervall

$$I_j = (s_j, s_{j+1}) \subset \mathbb{R} = \partial\mathbb{H}$$

if y_1, y_2 are real on I_j. But note that the values of $\eta|_{I_j}$ may pass through ∞, since y_2 may vanish at points of I_j.

If we pass to another fundamental basis

$$\tilde{y}_1 = ay_1 + by_2, \quad \tilde{y}_2 = cy_1 + dy_2,$$

the corresponding Schwarz functions η and $\tilde{\eta}$ differ only by a fractional linear transformation: $\tilde{\eta} = f \circ \eta$ with $f(x) = \frac{ax+b}{cx+d}$. In particular this applies to the fractional linear transformations representing some monodromy matrix. If the monodromy is as in (18), then

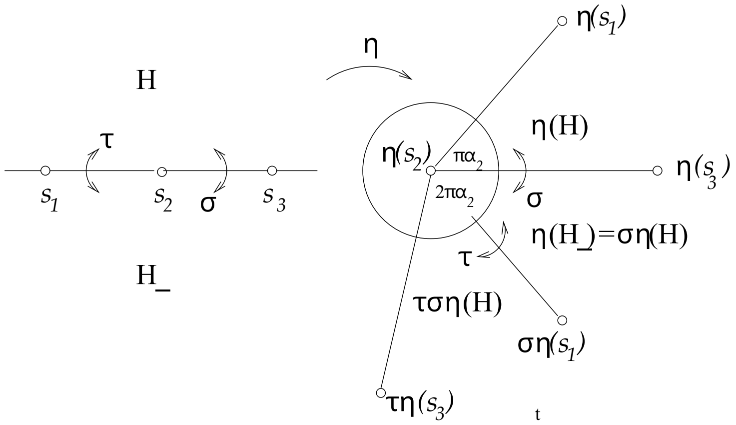
$$\tilde{\eta} = \frac{\tilde{y}_1}{\tilde{y}_2} = e^{2\pi i\alpha}\eta \tag{23}$$

where $\alpha = \alpha' - \alpha''$ as defined earlier.

Theorem 7.1. *The Schwarz function $\eta = y_1/y_2$ maps the upper half plane $\mathbb{H}$ onto a k-gon $\eta(\mathbb{H}) \subset \hat{\mathbb{C}}$, bounded by k circular arcs $\eta(I_j)$, with interior angles $\pi\alpha_j$, $j = 1, \dots, k$ ("Schwarz polygon").*

Proof. Our assertion is invariant under fractional linear transformations. Therefore in our proof we can choose any fundamental system. Let y_1, y_2 be a fundamental system which is real on $I_1 = (s_1, s_2)$, e.g. the one chosen in (6). Then η maps I_1 monotonically onto a segment of the augmented real line $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ which we consider as the equator of $\mathbb{S}^2 = \hat{\mathbb{C}}$. We may extend the solutions y_1, y_2 to $\mathbb{H}$. Likewise, for any $j \in \{2, , \dots, k\}$ there is a fundamental system $\tilde{y}_1, \tilde{y}_2$ which is real on $I_j = (s_j, s_{j+1})$ (where j is taken mod k) and which also extends to

$\mathbb{H}$. The corresponding Schwarz function $\tilde{\eta} = \tilde{y}_1/\tilde{y}_2$ maps this intervall I_j monotonically onto a segment $J \subset \hat{\mathbb{R}}$. Since $\eta = m \circ \tilde{\eta}$ for some fractional linear transformation m, we see that η maps I_j onto $m(J)$ which is part of a circle. Thus $\eta(\mathbb{H})$ is bounded by circular arcs (which might wrap around the circle more than once).



To determine the angle of the Schwarz polygon $\eta(\mathbb{H})$ at, say, $\eta(s_2)$, we extend η further to the lower half plane $\mathbb{H}_-$, using the Schwarz reflection principle over the interval $(s_2, s_3) \subset \mathbb{R}$. Then we reflect back to the upper half plane H over the neighboring interval (s_1, s_2). So we have analytically extended η along a circle around s_2 (wrongly oriented). Thus the composition $\tau\sigma$ of the two reflections σ, τ is the inverse monodromy transformation around s_2 which changes η to $e^{-2\pi i \alpha_2}\eta$, cf. (23). This shows that the angle of $\eta(\mathbb{H})$ at $\eta(s_2)$ must be half of the rotation angle $2\pi\alpha_2$ of the monodromy at s_2. The same argument can be applied to the other singular points, replacing η by $\tilde{\eta}$. □

In our geometric applications it will be important to understand when the monodromy group is unitary. Using stereographic projection, we may identify $\hat{\mathbb{C}} = \mathbb{CP}^1$ with the standard sphere $\mathbb{S}^2$. The fractional linear transformations act on $\mathbb{CP}^1$ as the projective group $PGL_2(\mathbb{C}) = SL_2(\mathbb{C})/\{\pm I\}$, and the subgroup $PU_2 \cong SO_3$ acts as the standard rotation group on $\mathbb{S}^2$. By (19) and (20), the monodromy matrices have $|\det M| = 1$, and thus $M \in U_2$ iff its action on $\mathbb{CP}^1$ is in $PU_2 = SO_3$. From this observation we obtain:

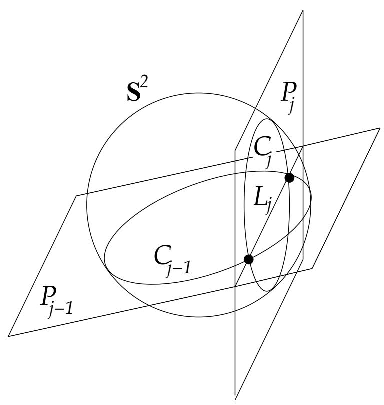
Corollary 7.2. *The monodromy group of* (2),(7) *for the fundamental system* y_1, y_2 *is unitary if and only if all edges of the Schwarz polygon* $\eta(\mathbb{H}) \subset \hat{\mathbb{C}} = \mathbb{S}^2$ *are great circle segments.*

8. The monodromy axes of η

As we have seen, the angles of the Schwarz polygon $\eta(\mathbb{H})$ can easily be read off the coefficients of (2),(7). But these do not yet fully determine the monodromy matrices. In order to find the missing piece we use another model of the conformal geometry of $\mathbb{S}^2 = \hat{\mathbb{C}}$ by embedding it into real projective 3-space. More precisely, $\mathbb{S}^2$ is the projectivized light cone in Minkowski space:

$$\mathbb{S}^2 = \{[v];\ v \in \mathbb{R}^4 \setminus \{0\},\ \langle v, v\rangle_- = 0\} \subset \mathbb{RP}^3 \tag{24}$$

where $\langle v, w\rangle_- = v_1w_1 + v_2w_2 + v_3w_3 - v_4w_4$ is the Lorentz scalar product. The conformal group $PGL_2(\mathbb{C}) = SL_2(\mathbb{C})/\{\pm I\}$ acting on $\hat{\mathbb{C}} = \mathbb{CP}^1$ by fractional linear transformtions now becomes the Lorentz group $PSO_{3,1}$. Since the monodromy transformations $M \in PSL_2(\mathbb{C}) = PSO_{3,1}$ are conjugate to rotations (which are elements of $PU_2 = SO_3$), they have a one-dimensional axis, a fixed line in $\mathbb{RP}^3$. How can we determine this axis? The edges of the Schwarz polygon are subsets of circles C_j, intersections of $\mathbb{S}^2$ with planes Π_j, and two neighboring planes Π_{j-1}, Π_j have a common line of intersection L_j passing through $\eta(s_j)$. This is the axis of the monodromy matrix corresponding to the point s_j.



In fact, as we have seen, the monodromy matrix is the composition of the reflections at the two circles C_{j-1}, C_j which extend to (Lorentz) reflections at the corresponding planes Π_{j-1}, Π_j, thus the common intersection of these planes is fixed.

Remark. We observe, in particular, that two neighboring axes L_j, L_{j+1} lie in a common plane, the plane Π_j of the connecting edge $\eta(I_j)$. Consequently, L_j and L_{j+1} intersect. This is a special feature of the "real" case at hand where the coefficients and the singularities of (7) are real.

9. Where neighboring monodromy axes meet

How can we compute the axes from (7)? At any fixed singularity s_j let $\mathsf{y} = (y_1, y_2)$ be the fundamental system with standard initial conditions (16) at s_j; this fundamental system is real valued on $I_j = (s_j, s_{j+1})$. From the example in Section 6 we see that the monodromy is the multiplication by $\zeta = e^{2\pi i \alpha_j}$ with its two fixed points 0 and ∞ on $\hat{\mathbb{C}}$. Thus the axis is the vertical line passing through the poles $[1, 0]$ and $[0, 1]$ of $\mathbb{S}^2 = \mathbb{CP}^1$.

Now let $\tilde{\mathsf{y}} = (\tilde{y}_1, \tilde{y}_2)$ be the fundamental system which is still real on I_j but with "final" conditions

$$\tilde{y}_1 = 0, \quad \dot{\tilde{y}}_2 = 0 \tag{25}$$

at the final t-value corresponding to s_{j+1}. Likewise, the monodromy axis of $\tilde{\eta} = \tilde{y}_1/\tilde{y}_2$ at s_{j+1} is the vertical axis through the poles $0 = [1, 0]$ and $\infty = [0, 1]$ in $\mathbb{CP}^1$. What is the monodromy axis L of η at s_{j+1}? We have $\tilde{\mathsf{y}} = A\mathsf{y}$ with $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ or

$$\begin{aligned} \tilde{y}_1 &= ay_1 + by_2 \\ \tilde{y}_2 &= cy_1 + dy_2 \end{aligned} \tag{26}$$

and we may assume that $ad - bc = \det A = 1$. Then $A^{-1} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$.

Lemma 9.1. *The axis L meets $\mathbb{S}^2 = \hat{\mathbb{C}}$ in the points $-b/a$ and $-d/c$.*

Proof. We have $\tilde{y} = A\mathsf{y}$ with $\mathsf{y} = \left(\begin{smallmatrix} y_1 \\ y_0 \end{smallmatrix}\right)$ and $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Thus A transforms $\eta = \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right] = [\mathsf{y}]$ into $\tilde{\eta} = [\tilde{\mathsf{y}}]$ and the monodromy axis of η at s_{j+1} is mapped onto that of $\tilde{\eta}$. But the monodromy axis of $\tilde{\eta}$ at s_{j+1} is the vertical axis passing through the poles $0 = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ and $\infty = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. Thus the monodromy axis of y at s_{j+1} passes through

$$A^{-1}\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} d \\ -c \end{smallmatrix}\right] = -d/c \quad \text{and} \quad A^{-1}\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} -b \\ a \end{smallmatrix}\right] = -b/a. \qquad \square$$

Lemma 9.2. *Putting $\tilde{u}_j = \dot{\tilde{y}}_j/\tilde{y}_j$ we have*

$$-b/a = -1/\tilde{u}_1(s_j), \quad -d/c = -1/\tilde{u}_2(s_j). \tag{27}$$

Proof. We have

$$\tilde{u}_1 = \frac{a\dot{y}_1 + b\dot{y}_2}{ay_1 + by_2}, \quad \tilde{u}_2 = \frac{c\dot{y}_1 + d\dot{y}_2}{cy_1 + dy_2},$$

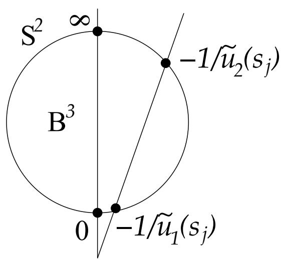
and the values at the point s_j are

$$\dot{y}_1 = 1, \quad \dot{y}_2 = 0, \quad y_1 = 0, \quad y_2 = 1. \tag{9}$$

Thus we obtain $\tilde{u}_1(s_j) = a/b$ and $\tilde{u}_2(s_j) = c/d$ as claimed. $\square$

Corollary 9.3. *If $r < 0$ throughout (s_j, s_{j+1}) for some $j \leq k-2$, the monodromy axes cannot meet inside $\mathbb{B}^3$.*

Proof. We have seen that the monodromy axis of η at s_{j+1} is given by the value of the logarithmic derivatives $\tilde{u}_1$ and $\tilde{u}_2$ at s_j. From Section 5 we know how these functions behave in the case $r < 0$: The function $\tilde{u}_1 = \dot{\tilde{y}}_1/\tilde{y}_1$ approaches $-\infty$ for $x \nearrow s_{j+1}$, and it will never be positive for $x < s_j$. Moreover, $\tilde{u}_2$ is zero at s_{j+1} and stays below zero for $x < s_{j+1}$. Thus these values have the same sign and therefore the monodromy axis at s_{j+1} which meets $\hat{\mathbb{R}} \subset \hat{\mathbb{C}} = \mathbb{S}^2$ at $-1/u_1(s_j)$ and $-1/u_2(s_j)$ does not cross the vertical axis (the monodromy axis at s_j) inside $\mathbb{B}^3$.



□

Theorem 9.4. *If $A \geq 0$, the monodromy of (2),(7) is not unitarizable.*

Proof. We need to avoid $r < 0$ in any of the bounded regular intervalls $I_j = (s_j, s_{j+1})$ for $j = 1, \dots, k-2$. The sign of $r_o = r/f$ (cf. (15)) on each of the intervals I_j is alternately positive and negative: positive on $I_{k-1} = (s_{k-1}, \infty)$, negative on $I_{k-2} = (s_{k-2}, s_{k-1})$, again positive on I_{k-3} etc. We call a bounded regular intervall I_j "positive" if $r_o > 0$ on I_j and "negative" if $r_o < 0$ on I_j. Then $r|_{I_j} < 0$ for some $j \in \{1, \dots, k-2\}$ unless $\{f < 0\}$ intersects each negative intervall and $\{f > 0\}$ each positive intervall. Hence f must change sign at least $k-3$ times which means that its degree is $\geq k-3$. By (8) this implies $A \neq 0$.

If $A > 0$, then $f(x)$ is positive for all $x \in (x_1, \infty)$ where x_1 is the largest real root of f. In order to avoid $r < 0$ on I_{k-2} we need to have $x_1 \geq s_{k-2}$. Let $x_2 < x_1$ be the next largest root. Then $f < 0$ on (x_2, x_1) and hence we need $x_2 \geq s_{k-3}$ in order to avoid $r < 0$ on I_{k-3}. By induction we need $x_j \geq s_{k-j-1}$ for all real roots x_j. But $j \leq k-3 =$ degree(f), hence $x_j \geq s_3$ for all j, and therefore r has the wrong sign on either I_1 or I_2.

□

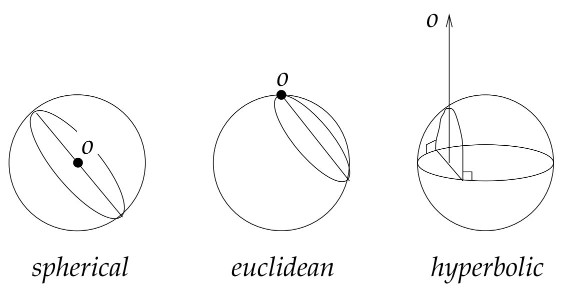
10. Unitary monodromy

Lemma 10.1. *In the real case, considered exclusively in this paper, all monodromy axes* $L_1, \dots, L_k$ *meet at a common point* $o \in \mathbb{RP}^3$ *iff the axes* $L_1, \dots, L_{k-1}$ *meet in a common point.*

Proof. Suppose that the axes $L_1, \dots, L_{k-1}$ meet at a point o. Then L_{k-1} and L_1 are coplanar. Likewise, the neighboring pairs of axes L_{k-1}, L_k and L_k, L_1 are coplanar. Thus L_k, L_{k-1}, L_1 meet in a common point (which must be o) unless they lie in a common plane. But in the latter case the angle $\alpha_k \pi$ at $\eta(s_k)$ equals 0 or π which is impossible since $\alpha_k \in (0,1)$. Thus all axes $L_1, \dots, L_k$ meet at o. $\square$

Lemma 10.2. *Suppose that all axes* $L_1, \dots, L_k$ *meet in a common point* $o \in \mathbb{RP}^3$*. Let* $\mathbb{B}^3 \subset \mathbb{RP}^3$ *denote the open 3-ball bounded by* $\mathbb{S}^2$*. Then* $\eta(\mathbb{H})$ *is conformally equivalent to a geodesic polygon in either the spherical or euclidean or hyperbolic metric, depending on the position of* o*:*

$$\begin{array}{lcccc} \textit{spherical} & \Longleftrightarrow & o & \in & \mathbb{B}^3 \\ \textit{euclidean} & \Longleftrightarrow & o & \in & \mathbb{S}^2 \\ \textit{hyperbolic} & \Longleftrightarrow & o & \in & \mathbb{RP}^3 \setminus \overline{\mathbb{B}^3} \end{array}$$



Proof. Case 1: Assume $o \in \mathbb{B}^3$. Recall that $PSO_{3,1}$ acts transitively on $\mathbb{B}^3$ as the hyperbolic isometry group. Thus, after applying an appropriate transformation in $PSO_{3,1}$ (acting conformally on $\mathbb{S}^2 = \partial\mathbb{B}^3$), we may assume that o is the center of euclidean $\mathbb{B}^3$. Thus all planes Π_j defining the edges of $\eta(\mathbb{H})$ pass through the center o which implies that the edges are subsets of great circles.

Case 2: Assume $o \in \mathbb{S}^2$. In this case we apply the stereographic projection with center o (= "north pole"). Then any plane through o (with the exception of the tangent plane to $\mathbb{S}^2$ at o) will be projected onto a straight line in the euclidean plane, hence the Schwarz polygon is mapped conformally onto a polygon with straight edges in euclidean plane.

Case 3: Assume $o \in \mathbb{RP}^3 \setminus \overline{\mathbb{B}^3}$. In this case we use a transformation in $PSO_{3,1}$ moving o into the plane at infinity in $\mathbb{RP}^3$. Thus in the affine picture, o becomes a certain direction in 3-space, say the vertical direction, and all planes through o are parallel to this direction, hence vertical. Thus they intersect $\mathbb{S}^2$ in a circle which meets the horizontal equator perpendicularly, and such circles are hyperbolic geodesics in the Poincaré model on the upper half sphere. □

Theorem 10.3. *The monodromy group is unitarizable, i.e. conjugate to a subgroup of $PU_2 \subset PGL_2(\mathbb{C})$, if and only if the monodromy axes meet at a common point inside $\mathbb{B}^3$.*

Proof. If the monodromy group is unitarizable for some fundamental system $\tilde{\mathsf{y}}$, we find $W \in GL_2(\mathbb{C})$ such that $\mathsf{y} = W\tilde{\mathsf{y}}$ has unitary monodromy. Then the corresponding transformations on $\mathbb{CP}^1 = \mathbb{S}^2$ lie in $PU_2 = SO_3$ and thus their axes pass through the origin $0 \in \mathbb{B}^3$.

Vice versa, if all monodromy axes pass through $o \in \mathbb{B}^3$, we may assume $o = 0$ after applying a transformation in $PSO_{3,1}$. Thus $\eta(\mathbb{H})$ is a geodesic spherical polygon. Therefore the reflections at the circles are isometries of $\mathbb{S}^2$, and the same holds for the monodromy elements which are generated by compositions of two reflections. Thus the monodromy matrices modulo complex scalars are in $SO_3 = PU_2$. But from (19) and (20) we know that $|\det M| = 1$ for any monodromy matrix $M \in GL_2(\mathbb{C})$, thus $M \in U_2$. □

Theorem 10.4. *The monodromy is unitarizable if and only if all (up to possibly one) monodromy axes pass through a common point and $A < 0$ in (8).*

Proof. If $A \geq 0$, the monodromy is not unitarizable by Theorem 9.4 (unless the monodromy is trivial). Thus we may assume $A < 0$. From Lemma 10.1 we know that all k axes pass through a common point if $k-1$ of them do. Then one of the cases of Lemma 10.2 applies. The three cases can be distinguished by the angle defect = "angle sum minus euclidean angle sum" for the geodesic k-gon $\eta(\mathbb{H})$. This is $-\pi\delta$ with δ as in (11). From (12) and $A < 0$ we see $\delta < 0$, thus we are in the spherical case and hence $o \in \mathbb{B}^3$. Now the claim follows from the previous theorem. □

Particular Cases:

For $k = 3$, the first condition of Theorem 10.4 holds trivially. Thus we have unitarizable monodromy if and only if $A < 0$.

For $k = 4$, we assure the first condition of Theorem 10.4 by a symmetry assumption: We assume that our ODE is invariant not only under the transformation $x \mapsto \bar{x}$ of the domain (the reality condition), but also under $x \mapsto -x$, and we assume in addition that two of the four singularities are 0 and ∞, the fixed points of this transformation. Then the other two singularities are $a > 0$ and $-a$, and we may choose $a = 1$. Now by the symmetry assumptions the coefficients of (2),(7) are

$$p = \frac{1-\alpha}{x+1} + \frac{1-\alpha}{x-1} + \frac{1-\beta}{x}, \quad q = \frac{A}{x^2-1} \tag{28}$$

with constants $\alpha, \beta \in (0, 1)$ and $A < 0$.

For k arbitrary, the first condition of Theorem 10.4 can be satisfied again by symmetry if the singularities s_j are equidistant on $\hat{\mathbb{R}}$ (viewed as a great circle in $\mathbb{S}^2$) and all α_j are the same for $j = 1, \dots, k$.

11. Construction of CMC-4-noids

In this section we will outline how one can use the previous sections to construct CMC-4-noids of genus $g = 0$ and embedded ends, using the so-called "loop group method" [DPW].

Roughly speaking, the construction goes as follows: Let $S \subset \hat{\mathbb{C}}$ be a simply connected open domain. Let $\nu(z, \lambda), \tau(z, \lambda)$ be holomorphic functions on $S \times \mathbb{C}^*$ with a simple pole at 0 in the second variable λ. Let $\eta = \left(\begin{smallmatrix} 0 & \nu \\ \tau & 0 \end{smallmatrix}\right)$ and solve the matrix ODE

$$H' = H\eta \tag{29}$$

(where $'$ denotes the complex derivative $\partial/\partial z$) for $H : S \times \mathbb{C}^* \to SL_2(\mathbb{C})$ with initial value $H(z_*, \lambda) = I$ at some fixed $z_* \in S$. Then apply the so called Iwasawa splitting $H = FV_+$ where $F(z, \lambda)$ takes values in SU_2 for all $\lambda \in \mathbb{S}^1$ and $V_+ = \sum_{k \geq 0} V_k(z)\lambda^k$ is a power series in λ. Put $F_\lambda(z) = F(z, \lambda)$. Viewing the Lie algebra $\mathfrak{su}_2$ as euclidean 3-space with basis $e_1 = \left(\begin{smallmatrix} i & \\ & -i \end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$, $e_3 = \left(\begin{smallmatrix} & i \\ i & \end{smallmatrix}\right)$, the map $h_\lambda = \mathrm{Ad}(F_\lambda)\, e_3 : S \to \mathbb{S}^2$ is harmonic and hence the Gauss map of a cmc surface $f_\lambda : S \to \mathfrak{su}_2 = \mathbb{R}^3$ which we obtain by the Sym-Bobenko formula $f_\lambda = g_\lambda + h_\lambda$ where $g_\lambda = \left(\frac{\partial}{\partial\theta} F_\lambda\right) F_\lambda^{-1}$, putting $\lambda = e^{-i\theta}$. (cf. [EQ] for a geometric interpretation of this formula).

The matrix valued first order ODE (29) can be changed into a scalar valued second order equation. In fact it is straightforward to see that the solutions H of (29) are of the type

$$H = \begin{pmatrix} y_1'/\nu & y_1 \\ y_2'/\nu & y_2 \end{pmatrix} \tag{30}$$

where y_1, y_2 form a fundamental system of the scalar ODE

$$y'' - \frac{\nu'}{\nu} y' - \nu\tau\, y = 0, \tag{31}$$

and the assumptions imply that the coefficients $\frac{\nu'}{\nu}$ and $\nu\tau$ have poles of the right order turning (31) into a regular singular equation.

We need to start by choosing a potential η. Following [DPW] we will obtain a CMC-immersion from very many such potentials. To obtain some CMC-4-noid we need to choose η just right. In a first step we choose η as in section 5.1. of [DW1] with singularities at the "ends", $0, 1, a \in \mathbb{R}, \infty$. Then [DPW] produces some CMC-immersion.

Rewriting the matrix ODE (29) we obtain the second order scalar ODE (31) , which has the same regular singular points as the matrix ODE.

Following standard ODE theory as explained in Section 2, we transform our equation equivalently into a standardized equation, namely where the exponents of all finite singular points have one exponent equal to 0.

For 4-noids, i.e. equations with four singular points this is the *Heun equation*. The transformation to Heun form is done by writing the solution y to equation (31) in the form

$$y(z) = z^{r_{0-}}(z-1)^{r_{1-}}(z-a)^{r_{2-}} w(z), \tag{32}$$

where the exponents r_{j-} are chosen as in [DW1], equation (3.3.8). Substituting this expression into (31) we obtain the differential equation (see (7), (12) and [DS]):

$$\begin{aligned} w'' &+ \left(\frac{1-2\mu_0}{z} + \frac{1-2\mu_1}{z-1} + \frac{1-2\mu_2}{z-a} \right) w' \\ &+ \frac{Az+B}{z(z-1)(z-a)}\, w = 0. \end{aligned} \tag{33}$$

where

$$A = (\mu_0 + \mu_1 + \mu_2 - \mu_\infty - 1)(\mu_0 + \mu_1 + \mu_2 + \mu_\infty - 1) \tag{34}$$

and

$$B = -\frac{1}{2}(1-2\mu_0)(1-2\mu_1)a - \frac{1}{2}(1-2\mu_0)(1-2\mu_2) + ac_0. \tag{35}$$

The expressions μ_j are given in [DW1], formula (5.11) and are called $\mid D_j \mid$ there. They are functions of the loop parameter λ and encode all properties of the Delaunay surfaces which are asymptotic to the ends of the 4-noid to be constructed. More precisely, for $\lambda = 1$ they encode the neck sizes of the asymptotic ends. From the definition (see loc.cit.) it is clear that the μ_j are real and attain valued between 0 and 1/2.

Thus the expressions $2\mu_j$ are real and attain values between 0 and 1. It is clear that they correspond to the expressions α_j of the previous sections. In particular, A is real.

At this point we need to remember that our chosen potential will produce for every $\lambda \in \mathbb{S}^1$ some CMC-immersion, but not necessarily a 4-noid.

The main result of [DW1], Theorem 5.4.1, states that, starting from a potential as in [DW1], we obtain for $\lambda = 1$ a CMC-4-noid if the monodromy matrices corresponding to the ends are simultaneously r-unitarizable (see below).

Actually, if and only if is true.

Let's note first that the monodromy groups considered in this paper all are generated by the monodromy matrices associated to the four ends and let's explain the terms:

First of all, it is easy to see that the conjugacy classes of monodromy groups defined for the matrix ODE and the associated scalar second order ODE are the same. Next we note that the monodromy matrices occurring naturally in the loop group method are holomorphic in $\lambda \in C^*$. At values of λ, where the chosen (λ-dependent) fundamental system of solutions becomes linearly dependent, the monodromy matrices will become singular. Due to the holomorphic dependence of the monodromy matrices, we can find some number $0 < r_0 < 1$ such that for all r between r_0 and 1 the determinant of each generator of the monodromy group never vanishes on the circle of radius r.

The term "r-unitarizable" now means that one can conjugate simultaneously the four generators of the monodromy group by some matrix which is holomorphic in the disk of radius r (and possibly multivalued outside) such that, after conjugation, the new monodromy matrices are still holomorphic in $\mathbb{C}^*$, but now also are unitary on $\mathbb{S}^1$, cf. [DW2].

The question now is:

How can we make sure that the monodromy matrices associated with the given potential η can be simultaneously r-unitarized?

In view of Theorem 10.4 we need to make sure that $A < 0$ and that at least three of the four monodromy axes pass through one point. Actually, we need to require this for all but finitely many $\lambda \in \mathbb{S}^1$.

Since $\mu_0+\mu_1+\mu_2-\mu_\infty-1<\mu_0+\mu_1+\mu_2+\mu_\infty-1$ and the product of the two sides is A, the condition $A<0$ is equivalent to

$$1<\mu_0+\mu_1+\mu_2+\mu_\infty, \tag{36}$$

$$\mu_0+\mu_1+\mu_2-\mu_\infty \;<\; 1 \tag{37}$$

for all but finitely many $\lambda\in\mathbb{S}^1$. Condition (37) also shows up in a related context. One usually considers the spherical 4-gon which is dual to the Schwarz 4-gon considered in this paper and which has side lengths $2\pi\nu_j$ with $\nu_j=\frac{1}{2}-\mu_j$. Thus a necessary condition for the simultaneous unitarizability of the monodromy matrices are the 4-gon inequalities ([DS], Section 6 and the literature quoted there):

$$\sum_{i\in P}\nu_i-\sum_{j\in P'}\nu_j-\frac{1}{2}(|P|-1)\le 0 \tag{38}$$

where $P\subset\{0,1,a,\infty\}$ with $|P|$ odd and $P'=\{0,1,a,\infty\}\setminus P$. We note that there are two types of inequalities: $|P|=1$ and $|P|=3$. Clearly, one of the inequalities of type $|P|=1$ is (37). The other $|P|=1$ inequalities follow from Theorem 10.4 by interchanging $j\in\{0,1,a\}$ with ∞. One of the inequalities of type $|P|=3$ follows from (37) and the other ones again by interchanging $j\in\{0,1,a\}$ with ∞.

Altogether, these inequalities give severe constraints on the μ_0,μ_1,μ_2, μ_∞ for the monodromy matrices to be simultaneously unitarizable. However, it is known that these inequalities are not sufficient. This is at least plausible since in (33) the coefficient B, see (35), contains a function $c_0=c_0(\lambda)$ for which no condition was given yet. In Theorem 10.4 the function c_0 is fixed implicitly by the condition that at least three monodromy axes meet in one point. Unfortunately, this condition is not explicit and is uncheckable in most cases.

However, in the last example of Section 10, given by (28), it is possible to check everything. For this example we put $a=-1$ and thus have singularities in (32) and (33) at $z=x=-1,0,1,\infty$, like in (28). The symmetry condition for this example mentioned in Section 10 implies that the differential equation is invariant under the transformation $z\mapsto -z$ which fixes 0 and ∞ and interchanges 1 and -1. Applying this to (33), we obtain

$$\mu_1=\mu_2,\quad B=0. \tag{39}$$

Note that all equalities hold for all $\lambda\in\mathbb{S}^1$ and all inequalities for all but finitely many $\lambda\in\mathbb{S}^1$.

From Theorem 10.4 we obtain that for all but finitely many $\lambda\in\mathbb{S}^1$ the monodromy matrices are simultaneously unitarizable. It is known that this implies that the monodromy matrices are simultaneously r-unitarizable.

Thus we have shown:

Theorem 11.1. *Let η be chosen as above with ends at $-1, 0, 1, \infty$ and assume (39). Then the procedure presented in* [DPW] *yields for $\lambda = 1$ a CMC-4-noid of genus $g = 0$ with embedded ends.*

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E-mail address: `dorfm@ma.tum.de`
E-mail address: `eschenburg@math.uni-augsburg.de`