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Symmetric spaces as Grassmannians

1. Introduction

A main problem in Riemannian geometry is understanding the exceptional symmetric spaces. In the present paper we restrict our attention mainly to the type-I case corresponding to symmetric pairs (G, K) where G is a compact simple Lie group, [3]. There are 12 exceptional spaces of this kind. The most prominent examples are the "Rosenfeld planes" $(\mathbb{O} \otimes \mathbb{L})\mathbb{P}^2$ (shortly $\mathbb{O}\mathbb{L}\mathbb{P}^2$) where $\mathbb{L}$ is one of the division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. These are defined in terms of the (half) spin representations of $Spin_{8+l}$ on $\mathbb{R}^n$ with $n = 2^{4+m}$ where $l = \dim_{\mathbb{R}} \mathbb{L} = 2^m$ and $m = 0, 1, 2, 3$. It seems to be impossible to define these spaces really as projective planes over the algebra $\mathbb{A} = \mathbb{O} \otimes_{\mathbb{R}} \mathbb{L}$. However, they behave in certain ways like projective planes. In particular, there is Vinberg's formula ([1, p. 192]) for the Lie algebra of $G = \mathrm{Aut}\,(\mathbb{A}^n, \langle\ ,\ \rangle)$ for $\mathbb{A} = \mathbb{K} \otimes \mathbb{L}$,

$$\mathfrak{g} = \mathfrak{aut}(\mathbb{K}) \oplus A_o(\mathbb{A}^n) \oplus \mathfrak{aut}(\mathbb{L}) \tag{1.1}$$

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where A_o denotes the antihermitian trace zero matrices.[1] This formula makes still sense when $\mathbb{K} = \mathbb{O}$ and $\mathbb{L} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ provided that $n \leq 3$, and it describes the Lie algebras of the groups $G = F_4, E_6, E_7, E_8$ corresponding to the Rosenfeld planes. All other exceptional symmetric spaces (with the only exception $G_2/SO_4 = \{\mathbb{H} \subset \mathbb{O}\}$, the space of quaternion subalgebras of the octonians) are obtained as spaces of certain self-reflective[2] subspaces of the Rosenfeld planes, e.g. $E_6/F_4 = \{\mathbb{O}\mathbb{P}^2 \subset \mathbb{O}\mathbb{C}\mathbb{P}^2\}$, the set of all subspaces congruent to $\mathbb{O}\mathbb{P}^2$ in $\mathbb{O}\mathbb{C}\mathbb{P}^2$. These facts motivated Y. Huang and N.C. Leung to investigate structures related to $\mathbb{A} = \mathbb{K} \otimes \mathbb{L}$ on symmetric spaces [4,5]. In some sense they tried to adapt the classical case to the exceptional one.

The classical type-I symmetric spaces seem to be easy to describe. There are three families:

(1) Grassmannians $\{\mathbb{K}^p \subset \mathbb{K}^n\}$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$,
(2) $\mathbb{R}$-structures on $\mathbb{C}^n$, $\{\mathbb{R}^n \subset \mathbb{C}^n\}$,
$\mathbb{C}$-structures on $\mathbb{H}^n$, $\{\mathbb{C}^n \subset \mathbb{H}^n\}$,
(3) $\mathbb{C}$-structures on $\mathbb{R}^{2n}$, $\{\mathbb{R}^{2n} \cong \mathbb{C}^n\}$,
$\mathbb{H}$-structures on $\mathbb{C}^{2n}$, $\{\mathbb{C}^{2n} \cong \mathbb{H}^n\}$.

The first two families are Grassmannians in a natural way, sets of subspaces W which are invariant (1) or anti-invariant (2) under some complex structure $j \in \mathbb{K}$. Anti-invariance means $jW = W^\perp$; this property is also called *Lagrangian*.[3] Only case (3) is slightly different. How can we assign a subspace to a complex or quaternionic structure J on a real or complex vector space V? We just take the eigenspace W corresponding to the eigenvalue $i = \sqrt{-1}$. But W is contained not in V itself but in $V \otimes \mathbb{C}$. So we arrive at subspaces of $(\mathbb{C} \otimes \mathbb{C})^n$ and $(\mathbb{H} \otimes \mathbb{C})^n$.

Huang and Leung have studied systematically the various types of Grassmannians over $V = \mathbb{A}^n$ for $\mathbb{A} = \mathbb{K} \otimes \mathbb{L}$ with $\mathbb{K}, \mathbb{L} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. They distinguish four kinds of such Grassmannians $\{W \subset V\}$:

(1) pure, $\{\mathbb{A}^p \subset \mathbb{A}^n\}$,
(2) Lagrangian, $\{\mathbb{A}_1^n \subset \mathbb{A}^n\}$ where $\mathbb{A}_1 = \mathbb{K}_1 \otimes \mathbb{L}$ and $\mathbb{K}_1 \subset \mathbb{K}$ is the half-dimensional subalgebra,
(3) double Lagrangian, $\{\mathbb{A}_2^n \oplus j\hat{j}\mathbb{A}_2^n \subset \mathbb{A}^n\}$ where $\mathbb{A}_2 = \mathbb{K}_1 \otimes \mathbb{L}_1$ and $\mathbb{K} = \mathbb{K}_1 + j\mathbb{K}_1$, $\mathbb{L} = \mathbb{L}_1 + \hat{j}\mathbb{L}_1$,

[1] Equation (1.1) gives only the vector space decomposition; the Lie bracket is more complicated: For any $A = (a_{ij})$, $B = (b_{ij}) \in A_o(\mathbb{A}^n)$ we put $[A, B] = (AB - BA)_o + \frac{1}{3}\sum_{ij} D_{a_{ij},b_{ij}}$ where $()_o$ denotes the traceless part and $D_{a\otimes a',b\otimes b'} = \langle a', b'\rangle D_{a,b} + \langle a, b\rangle D_{a',b'}$ and where $D_{a,b}c = [[a, b], c] - 3((ab)c - a(bc))$ for any $a, b, c \in \mathbb{K}$, see [1].

[2] A reflective submanifold Q of a symmetric space P is a fixed set component of some involution r on P. Reflective submanifolds come in pairs: For any $q \in Q$ there is another reflective submanifold Q' intersecting Q perpendicularly at q, a fixed set component of the involution rs_q of P, where s_q denotes the symmetry at q. If Q and Q' are congruent, the submanifold is called self-reflective.

[3] The 2-form $\omega(x, y) = \langle jx, y\rangle$ is a symplectic form, and W is a Lagrangian subspace for $\omega \iff jW = W^\perp$.

(4) isotropic, $\{W = (sW)^{\perp} \subset \mathbb{A}^n\}$ where s is a paracomplex structure[4] commuting with the scalars in $\mathbb{A}$.

All these sets of linear subspaces have in common that they are preserved under the reflection at any of their elements, thus they define a symmetric subspace of the corresponding pure Grassmannian. Moreover, the defining involution descends to an involution of the projective space $\mathbb{P}(\mathbb{A}^n) = \{\mathbb{A}^1 \subset \mathbb{A}^n\}$. Thus the linear subspaces of $\mathbb{A}^n$ become self-reflective subspaces of $\mathbb{P}(\mathbb{A}^n)$. It seems that the latter property survives for $\mathbb{K} = \mathbb{O}$ when $n \leq 3$ while the linear algebra description breaks down: By lack of associativity, $\mathbb{A}^n$ is not an $\mathbb{A}$-module for $\mathbb{A} = \mathbb{O} \otimes \mathbb{L}$.

We are using a different approach which unifies the Grassmannians of type 2,3,4. We first investigate involutions (order-2 automorphims) σ of the algebra $\mathbb{A} = \mathbb{K} \otimes \mathbb{L}$ which have eigenspaces of equal dimensions (balanced involutions). There are two types of such involutions which lead to the Lagrangian and double Lagrangian subspaces when applied component-wise to $\mathbb{A}^n$. We get a few more examples by extending our investigation also to the fixed algebra $\mathbb{B} \subset \mathbb{A}$ of every balanced involution and to the tensor products $\mathcal{C}\mathbb{A}$ of $\mathbb{A}$ with the paracomplex numbers $\mathcal{C}$. Then we determine the orthogonal automorphism groups of $\mathbb{A}$ and $\mathbb{B}$ which consist of all $\mathbb{R}$-linear maps commuting with the scalar multiplication by the generators of $\mathbb{A}$ or $\mathbb{B}$. Thus we obtain the usual coset representation of the corresponding symmetric spaces $\{\mathbb{A}^p \subset \mathbb{A}^n\}$ and $\{\mathbb{B}^n \subset \mathbb{A}^n\}$. We end with a glance onto the exceptional spaces by comparing the tables of [4,5] with the lists of self-reflective submanifolds in [2] and [6].

Aside from the exceptional spaces, the main subject of the present paper is a representation of classical symmetric spaces as Grassmannians which seems to be of some value on its own. There are other representations of a symmetric space $P = G/K$ as a Grassmannian. The easiest way is probably to assign to every point $p \in P$ the (-1)-eigenspace of the involution $\sigma = Ad(s_p)$ on $\mathfrak{g}$ where s_p is the symmetry at p. However, this requires already the knowledge of the Lie algebra $\mathfrak{g}$ of our group G. Our representation is different: The algebras $\mathbb{A}$ and $\mathbb{B}$ are not directly related to P. Thus they define a nontrivial new structure on P. Moreover it is clear from this representation that there is a larger noncompact group acting on P, namely the group of all $\mathbb{A}$-linear automorphism of $\mathbb{A}^n$ (not just the orthogonal ones). This shows immediately that all these spaces are symmetric R-spaces (which does not hold for most of the exceptional spaces). However, not all symmetric R-spaces can be represented this way though we have covered all infinitesimal irreducible types.

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[4] A paracomplex structure is an involution with eigenspaces of equal dimensions.

2. Tensor products of division algebras and involutions

Let $\mathbb{A} = \mathbb{K} \otimes_{\mathbb{R}} \mathbb{L} =: \mathbb{K}\mathbb{L}$ where $\mathbb{K}, \mathbb{L}$ are two associative division algebras, $\mathbb{K}, \mathbb{L} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. This becomes an associative algebra with the multiplication

$$(u \otimes x)(v \otimes y) = uv \otimes xy \tag{2.1}$$

for $u, v \in \mathbb{K}$ and $x, y \in \mathbb{L}$. Let $\sigma \in \mathrm{Aut}\,(\mathbb{A})$ be an involutive automorphism ($\sigma^2 = I$). Assume that there is some invertible element $a_o \in A$ with $\sigma(a_o) = -a_o$. Then the (left or right) multiplication with a_o anticommutes with σ since $\sigma(a_o b) = -a_o\sigma(b)$. Thus it interchanges the fixed and the anti-fixed spaces of σ and consequently, the fixed algebra

$$\mathbb{B} = \mathbb{A}^\sigma = \{a \in \mathbb{A} : \sigma(a) = a\} \tag{2.2}$$

has half dimension. We will call such involutions *balanced*. We have

$$\mathbb{A} = \mathbb{B} \oplus a_o\mathbb{B} = \mathbb{B} \oplus \mathbb{B}a_o. \tag{2.3}$$

We see two kinds of balanced involutions on $\mathbb{A}$: those of type $\sigma \otimes I$ or $I \otimes \tau$ and those of type $\sigma \otimes \tau$, where σ and τ are nontrivial involutions of $\mathbb{K}$ and $\mathbb{L}$, respectively. We will apply the involutions to "vectors" in $\mathbb{A}^n$ rather than to "scalars" in $\mathbb{A}$. The fixed spaces of involutions of the first (second) kind are called *Lagrangian* (*double Lagrangian*) subspaces in [4,5]. In Table 1, everything with $\hat{}$ refers to the second tensor factor. By $\mathcal{C}$ we denote the paracomplex numbers, $\mathcal{C} = \mathbb{R} + s\mathbb{R}$ with $s^2 = 1$. We have $s = j_1\hat{j}_2$ for two complex structures $j_1 \in \mathbb{K}$ and $j_2 \in \mathbb{L}$.

In the second part of the table we are starting with the three new algebras $\mathcal{C}$, $\tilde{\mathcal{C}}\mathbb{C} = \mathcal{C}\tilde{\otimes}\mathbb{C}$, $\tilde{\mathcal{C}}\mathbb{C}\mathbb{C} = \mathcal{C}\tilde{\otimes}\mathbb{C}\mathbb{C}$ which we have obtained as subalgebras. Note that the multiplication on $\tilde{\mathcal{C}}\mathbb{A} = \mathcal{C}\tilde{\otimes}\mathbb{A}$ is different: the generator s of $\mathcal{C}$ anticommutes with the complex structures (i and $i, \hat{i}$, respectively) generating $\mathbb{A}$, see Table 1, No. 7

Table 1.

No.	$\mathbb{A}$	Generators	σ	a_o	$\mathbb{B}$	Generators
1	$\mathbb{C}$	i	κ	i	$\mathbb{R}$	–
2	$\mathbb{H}$	i, j	$Ad(i)$	j	$\mathbb{C}$	i
3	$\mathbb{C}\mathbb{C}$	$i, \hat{i}$	κ	i	$\mathbb{C}$	i
4	$\mathbb{C}\mathbb{C}$	$i, \hat{i}$	$\kappa\hat{\kappa}$	$i, \hat{i}$	$\mathcal{C}$	$i\hat{i}$
5	$\mathbb{H}\mathbb{C}$	$i, j, \hat{i}$	$\hat{\kappa}$	$\hat{i}$	$\mathbb{H}$	i, j
6	$\mathbb{H}\mathbb{C}$	$i, j, \hat{i}$	$Ad(i)$	j	$\mathbb{C}\mathbb{C}$	$i, \hat{i}$
7	$\mathbb{H}\mathbb{C}$	$i, j, \hat{i}$	$Ad(i)\hat{\kappa}$	$j, \hat{i}$	$\tilde{\mathcal{C}}\mathbb{C}$	$i, j\hat{i}$
8	$\mathbb{H}\mathbb{H}$	$i, j, \hat{i}, \hat{j}$	$Ad(i)$	j	$\mathbb{C}\mathbb{H}$	$i, \hat{i}, \hat{j}$
9	$\mathbb{H}\mathbb{H}$	$i, j, \hat{i}, \hat{j}$	$Ad(i)Ad(\hat{i})$	$j, \hat{j}$	$\tilde{\mathcal{C}}\mathbb{C}\mathbb{C}$	$i, \hat{i}, j\hat{j}$
10	$\mathcal{C}$	s	$\tilde{\kappa}$	s	$\mathbb{R}$	–
11	$\tilde{\mathcal{C}}\mathbb{C}$	$s, \hat{i}$	$\tilde{\kappa}$	s	$\mathbb{C}$	i
12	$\tilde{\mathcal{C}}\mathbb{C}$	s, i	κ	i	$\mathcal{C}$	s
13	$\tilde{\mathcal{C}}\mathbb{C}\mathbb{C}$	$s, i, \hat{i}$	$\tilde{\kappa}$	s	$\mathbb{C}\mathbb{C}$	$i, \hat{i}$
14	$\tilde{\mathcal{C}}\mathbb{C}\mathbb{C}$	$s, i, \hat{i}$	κ	i	$\tilde{\mathcal{C}}\mathbb{C}$	$s, \hat{i}$

and 9. These new algebras also allow balanced involutions σ. By $\tilde{\kappa}$ we denote the conjugation $x + sy \mapsto x - sy$ in $\mathcal{C}$ (extended to tensor products with $\mathcal{C}$). Table 1 can be further extended, see Sect. 6.

3. The automorphism group of the spaces $\mathbb{A}^n$

We are considering the free $(\mathbb{K} \otimes \mathbb{L})$-module $V = \mathbb{A}^n \cong \mathbb{R}^{dn}$ with its canonical inner product, where $d = \dim_{\mathbb{R}} \mathbb{A} = \dim \mathbb{K} \dim \mathbb{L}$. The elements of $\mathbb{A}^n$ are rows $x = (x_1, \dots, x_n)$ with $x_1, \dots, x_n \in \mathbb{A}$, and scalars $a \in \mathbb{A}$ act from the left, $ax = (ax_1, \dots, ax_n)$. An $\mathbb{R}$-linear map A on $\mathbb{A}^n$ is $\mathbb{A}$-linear if it commutes with this scalar multiplication. If we want to express A by a matrix with entries in $\mathbb{A}$, we have to let A act from the right. However, we will consider A as a map on $\mathbb{A}^n$ rather than as a matrix, thus we keep writing $A(x)$. First we determine the *automorphism group* of $(\mathbb{A}^n, \langle\ ,\ \rangle)$, consisting of all orthogonal $\mathbb{A}$-linear maps on $\mathbb{A}^n$. We just have to find those orthogonal maps on $\mathbb{A}^n$ which commute with the generators of $\mathbb{A}$ (see Table 1).

3.1. $\mathbb{A} = \mathbb{C} \otimes \mathbb{C}$

On the real vector space $V = (\mathbb{C} \otimes \mathbb{C})^n$ we have two complex structures $i = i \otimes 1$ and $\hat{i} = 1 \otimes i$ which commute with each other. Thus, the composition $S = i\hat{i}$ is self adjoint with $S^2 = I$. Hence V is decomposed into the two eigenspaces $V_+ = \{S = I\}$ and $V_- = \{S = -I\}$. On the latter space V_-, the two complex structures agree, for $i\hat{i} = -1 \iff \hat{i} = i$, while they differ by a sign on V_+. Each orthogonal and $(\mathbb{C} \otimes \mathbb{C})$-linear map A on V preserves i, $\hat{i}$, S, and the subspaces V_+, V_- together with their complex structures are kept invariant, so A defines a pair of unitary linear maps (A_+, A_-) on (V_+, V_-). Vice versa, each such pair defines an orthogonal $(\mathbb{C} \otimes \mathbb{C})$-linear map on V. Let $\hat{\kappa}$ denote the conjugation in the second tensor factor, $\hat{\kappa}(z \otimes w) = z \otimes \bar{w}$. This commutes with i and anticommutes with $\hat{i}$, thus it anticommutes with $S = i\hat{i}$. Therefore $\hat{\kappa}$ interchanges V_+ and V_- which shows that V_+ and V_- have the same dimension, i.e. both are isomorphic to $\mathbb{C}^n$. Thus

$$\mathrm{Aut}\,(\mathbb{C}\mathbb{C})^n \cong U_n \times U_n. \tag{3.1}$$

3.2. $\mathbb{A} = \mathbb{H} \otimes \mathbb{C}$

On the real vector space $V = (\mathbb{H} \otimes \mathbb{C})^n$, we have the two anticommuting complex structures i, j (and $k = ij$) of $\mathbb{H} \otimes 1$ and a third one, $\hat{i}$, of $1 \otimes \mathbb{C}$, which commutes with the two others. As before we define the self adjoint involution $S = i\hat{i}$ with its eigenspaces V_+, V_-. The remaining complex structure j interchanges these subspaces because j anticommutes with S. In particular, the two eigenspaces $V_\pm$ must have the same dimension. Since V has complex dimension $4n$ (with respect to the complex structure $\hat{i}$), $V_- \cong \mathbb{C}^{2n}$. Any $\mathbb{H} \otimes \mathbb{C}$-linear isometry A on V commutes with S. Therefore the space V_- is invariant under A. The symmetry A is already determined by $A' = A|_{V_-}$, because for $x \in V_+$ we have $jAx = Ajx = A'jx$. Conversely, each $\mathbb{C}$-linear isometry A' on V_- defines an $(\mathbb{H} \otimes \mathbb{C})$-linear isometry A on V where A on V_+ is defined by $Ajx = jA'x$, $x \in V_-$. Thus

$$\text{Aut}\,(\mathbb{H}\mathbb{C})^n \cong U_{2n}. \tag{3.2}$$

3.3. $\mathbb{A} = \mathbb{H} \otimes \mathbb{H}$

On the real vector space $V = (\mathbb{H} \otimes \mathbb{H})^n$, we have two pairs of anticommuting complex structures i, j and $\hat{i}$, $\hat{j}$, and complex structures from different pairs commute with each other. We form two self-adjoint involutions $S_i = i\hat{i}$ and $S_j = j\hat{j}$, which also commute with each other: because of the two sign changes, we get

$$S_i S_j = i\hat{i}j\hat{j} = ij\hat{i}\hat{j} = ji\hat{j}\hat{i} = j\hat{j}i\hat{i} = S_j S_i.$$

These have a simultaneous eigenspace decomposition

$$V = V_-^- \oplus V_-^+ \oplus V_+^- \oplus V_+^+ \tag{*}$$

where $V_-^+ = V_- \cap V^+$ etc. Each $(\mathbb{H} \otimes \mathbb{H})$-linear isometry A on V commutes with S_i and S_j and therefore it leaves the four subspaces invariant. In particular it defines an orthogonal map A' on V_-^-. The mapping S_i commutes with $i, \hat{i}$ and anticommutes with j, $\hat{j}$ and for S_j, it is vice versa. Therefore, i interchanges the spaces V_-^-, V_-^+ and similarly V_+^-, V_+^+ (the lower index is preserved), while j interchanges V_-^- with V_+^- and V_-^+ with V_+^+ (the upper index is preserved). In particular, V_-^- is mapped onto V_-^+, V_+^-, V_+^+ by i, j, ij, respectively. Since A commutes with these maps, it is already completely determined by its restriction $A' = A|_{V_-^-}$. Conversely, each orthogonal map A' on V_-^- can be extended uniquely to a $(\mathbb{H} \otimes \mathbb{H})$-linear isometry of A on V by applying i, j, ij. Since all summands of the decomposition $(*)$ have the same dimension $4n$ (a quarter of $\dim(\mathbb{H}\mathbb{H})^n = 16n$), we have

$$\text{Aut}\,(\mathbb{H}\mathbb{H})^n \cong O_{4n}.$$

3.4. $\mathbb{A} = \mathcal{C}$

On $V = \mathcal{C}^n$, the paracomplex structure $s \in \mathcal{C}$ acts by scalar multiplication and decomposes V into two eigenspaces $V_\pm$ which have to be preserved by any automorphism A. Since s anti-commutes with the paracomplex conjugation $\tilde{\kappa}$, the two eigenspaces have equal dimension. Thus A splits into two orthogonal maps $A_\pm = A|_{V_\pm}$, and vice versa, any pair of such maps defines an automorphism of $\mathcal{C}^n$. Hence we obtain

$$\text{Aut}\,\mathcal{C}^n \cong O_n \times O_n.$$

Table 2.

No.	1	2	3	4	5	6	7	8	9
$\mathbb{A}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{CC}$	$\mathbb{HC}$	$\mathbb{HH}$	$\mathcal{C}$	$\tilde{\mathcal{C}}\mathbb{C}$	$\tilde{\mathcal{C}}\mathbb{CC}$
Aut $\mathbb{A}^n$	O_n	U_n	Sp_n	U_n^2	U_{2n}	O_{4n}	O_n^2	O_{2n}	O_{2n}^2

3.5. $\mathbb{A} = \mathcal{C}\tilde{\otimes}\mathbb{C}$

Besides the paracomplex structure s we have the complex structure i acting on $V = (\mathcal{C}\tilde{\otimes}\mathbb{C})^n$, and the two structures i, s anticommute. Thus i interchanges the two s-eigenspaces, $V_+ = iV_-$. Any automorphism A on V commutes with both structures, hence A_+ is determined by A_-, more precisely, $A_+(iv_-) = iA_-(v_-)$ for all $v_- \in V_-$. Vice versa, any orthogonal map A_- on $V_- \cong \mathbb{R}^{2n}$ extends to an automorphism A on V, using $A(iv_-) = iA(v_-)$. Hence we obtain

$$\text{Aut}\,(\tilde{\mathcal{C}}\mathbb{C})^n \cong O_{2n}.$$

3.6. $\mathbb{A} = \mathcal{C}\tilde{\otimes}(\mathbb{C} \otimes \mathbb{C})$

Now we have two commuting complex structures $i, \hat{i}$ which both anti-commute with the paracomplex structure s. Then $S = i\hat{i}$ commutes with s, and V has another splitting by the eigenspaces of S, called $V^\pm$, compatible with the previous one. While i and $\hat{i}$ commute with S and anticommute with s, preserving $V^\pm$ and interchanging $V_\pm$, the complex conjugation $\kappa(z \otimes w) = \bar{z} \otimes w$ on $\mathbb{C} \otimes \mathbb{C}$ commutes with s and anticommutes with S. Thus it preserves V_+ and V_- while interchanging V^+ and V^- and consequently, all four intersections V_-^-, V_+^-, V_-^+, V_+^+ have equal dimension $\frac{1}{4}dn = 2n$ for $d = \dim \tilde{\mathcal{C}}\mathbb{CC} = 8$. As before, an automorphism A is determined by A_-, its restriction to V_-, because $V_+ = iV_-$. Moreover, since A commutes with both s and S, it preserves the splitting $V_- = V_-^- \oplus V_-^+$. Thus A_- splits into orthogonal maps A_-^- and A_-^+ on V_-^- and V_-^+, respectively, and vice versa, any such pair defines an automorphism of V. Thus

$$\text{Aut}\,(\tilde{\mathcal{C}}\mathbb{CC})^n \cong O_{2n} \times O_{2n}.$$

We insert the results of this section into Table 2.

4. Grassmannians

A (pure) Grassmannian $G_p(\mathbb{A}^n)$ for $\mathbb{A}$ is the space of free submodules of $\mathbb{A}^n$ with rank p; these are $\mathbb{R}$-linear subspaces which are mapped to the standard space $\mathbb{A}^p \subset \mathbb{A}^n$ for $p \leq n/2$ under some automorphims of $\mathbb{A}^n$. Symbolically we write $G_p(\mathbb{A}^n) = \{\mathbb{A}^p \subset \mathbb{A}^n\}$. We may also consider $G_p(\mathbb{A}^n)$ as the space of decompositions isomorphic to the standard decomposition $\mathbb{A}^n = \mathbb{A}^p \oplus \mathbb{A}^q$ with $n = p + q$. The group Aut $(\mathbb{A}^n)$ acts transitively on this space, and the isotropy group of the standard decomposition is Aut $(\mathbb{A}^p) \times$ Aut $(\mathbb{A}^q)$. Any such Grassmannian is a compact

Table 3.

No.	$\mathbb{A}$	$\mathbb{AP}^{n-1}$	$\mathbb{AP}^2$	$\mathbb{AP}^1$
1	$\mathbb{R}$	$\mathbb{RP}^{n-1}$	$\mathbb{RP}^2$	S^1
2	$\mathbb{C}$	$\mathbb{CP}^{n-1}$	$\mathbb{CP}^2$	S^2
3	$\mathbb{H}$	$\mathbb{HP}^{n-1}$	$\mathbb{HP}^2$	S^4
4	$\mathbb{CC}$	$(\mathbb{CP}^{n-1})^2$	$(\mathbb{CP}^2)^2$	$S^2 \times S^2$
5	$\mathbb{HC}$	$G_2(\mathbb{C}^{2n})$	$G_2(\mathbb{C}^6)$	$G_2(\mathbb{C}^4)$
6	$\mathbb{HH}$	$G_4(\mathbb{R}^{4n})$	$G_4(\mathbb{R}^{12})$	$G_4(\mathbb{R}^8)$
7	$\mathcal{C}$	$(\mathbb{RP}^{n-1})^2$	$\mathbb{RP}^2 \times \mathbb{RP}^2$	$S^1 \times S^1$
8	$\tilde{\mathcal{C}}\mathbb{C}$	$G_2(\mathbb{R}^{2n})$	$G_2(\mathbb{R}^6)$	$G_2(\mathbb{R}^4)$
9	$\tilde{\mathcal{C}}\mathbb{CC}$	$G_2(\mathbb{R}^{2n})^2$	$G_2(\mathbb{R}^6)^2$	$G_2(\mathbb{R}^4)^2$

symmetric space where the symmetry at the standard decomposition is given by the reflection $\begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}$ at the subspace $\mathbb{A}^p$.

The case $p = 1$ is of particular importance; the Grassmannian $G_1(\mathbb{A}^n)$ is called the $(n-1)$-dimensional projective space over $\mathbb{A}$, denoted $\mathbb{P}(\mathbb{A}^n) = \mathbb{AP}^{n-1}$. The elements of $\mathbb{P}(\mathbb{A}^n)$ are free $\mathbb{A}$-submodules of rank one; they are of the form $[v] = \mathbb{A}v$ where $v \in \mathbb{A}^n$ has at least one invertible component. We list these spaces in Table 3.

It seems disturbing that the Grassmannians for $A = \tilde{\mathcal{C}}\mathbb{C}$ and $A = \tilde{\mathcal{C}}\mathbb{CC}$ are not complex manifold. Its tangent vectors at $\mathbb{A}^p$ are $\mathbb{A}$-linear maps $f : \mathbb{A}^p \to \mathbb{A}^q$. But observe that the map *if* is no longer $\mathbb{A}$-linear since $if(sx) = isf(x) = -sif(x)$.

5. Grassmannians of subalgebras

The second type of Grassmannians is the set $\{\mathbb{B}^n \subset \mathbb{A}^n\}$ or more precisely the set of linear subspaces of $\mathbb{A}^n$ which are mapped onto $\mathbb{B}^n$ by some automorphism of $\mathbb{A}^n$. Here, $\mathbb{B} \subset \mathbb{A}$ is one of the inclusions of Table 1. Clearly, any automorphism of $\mathbb{A}^n$ which preserves the standard subspace $\mathbb{B}^n$ restricts to an automorphism of $\mathbb{B}^n$, but also the converse is true: Since $\mathbb{A}^n = \mathbb{B}^n + a_o\mathbb{B}^n$, any automorphism B of $\mathbb{B}^n$ has a unique extension to an automorphism A of $\mathbb{A}^n$ by putting $A(a_o w) = a_o B(w)$ for each $w \in \mathbb{B}^n$. Thus A commutes with all $b \in \mathbb{B}$ since $ba_o = a_o b'$ with $b' \in \mathbb{B}$. Moreover, A commutes with a_o and hence with all $a = ba_o \in \mathbb{A}$:

$$A(ba_o w) = A(a_o b' w) = a_o B(b' w) = a_o b' Bw = ba_o Bw = bA(a_o w).$$

This Grassmannian is also a symmetric space where the symmetry at the standard subspace $\mathbb{B}^n$ is given by the automorphism σ, and we have $\{\mathbb{B}^n \subset \mathbb{A}^n\} = \mathrm{Aut}\,(\mathbb{A}^n)/\,\mathrm{Aut}\,(\mathbb{B}^n)$. We insert the results into Table 4.[5]

Last we insert these results into the classification scheme of Cartan and Helgason [3] which shows that all classical symmetric spaces (up to coverings and S^1-factors) can be viewed as Grassmannians over $\mathbb{A}^n$, some of them even in several ways (Table 5).

[5] It is not difficult to see that the inclusions of Aut $(\mathbb{B}^n)$ into Aut $(\mathbb{A}^n)$ as given in Table 2 are conjugate to the standard inclusions of these groups.

Table 4.

No.	$\mathbb{A}$	$\mathbb{B}$	$\{\mathbb{B}^n \subset \mathbb{A}^n\}$	$\{\mathbb{BP}^{n-1} \subset \mathbb{AP}^{n-1}\}$
1	$\mathbb{C}$	$\mathbb{R}$	U_n/O_n	$\{\mathbb{RP}^{n-1} \subset \mathbb{CP}^{n-1}\}$
2	$\mathbb{H}$	$\mathbb{C}$	Sp_n/U_n	$\{\mathbb{CP}^{n-1} \subset \mathbb{HP}^{n-1}\}$
3	$\mathbb{CC}$	$\mathbb{C}$	$U_n^2/U_n = U_n$	$\{\mathbb{CP}^{n-1} \subset (\mathbb{CP}^{n-1})^2\}$
4	$\mathbb{CC}$	$\mathcal{C}$	$(U_n/O_n)^2$	$\{(\mathbb{RP}^{n-1})^2 \subset (\mathbb{CP}^{n-1})^2\}$
5	$\mathbb{HC}$	$\mathbb{H}$	U_{2n}/Sp_n	$\{\mathbb{HP}^{n-1} \subset G_2(\mathbb{C}^{2n})\}$
6	$\mathbb{HC}$	$\mathbb{CC}$	U_{2n}/U_n^2	$\{(\mathbb{CP}^{n-1})^2 \subset G_2(\mathbb{C}^{2n})\}$
7	$\mathbb{HC}$	$\tilde{\mathcal{C}}\mathbb{C}$	U_{2n}/O_{2n}	$\{G_2(\mathbb{R}^{2n}) \subset G_2(\mathbb{C}^{2n})\}$
8	$\mathbb{HH}$	$\mathbb{CH}$	O_{4n}/U_{2n}	$\{G_2(\mathbb{C}^{2n}) \subset G_4(\mathbb{R}^{4n})\}$
9	$\mathbb{HH}$	$\tilde{\mathcal{C}}\mathbb{CC}$	O_{4n}/O_{2n}^2	$\{G_2(\mathbb{R}^{2n})^2 \subset G_4(\mathbb{R}^{4n})\}$
10	$\mathcal{C}$	$\mathbb{R}$	$O_n^2/O_n = O_n$	$\{\mathbb{RP}^{n-1} \subset (\mathbb{RP}^{n-1})^2\}$
11	$\tilde{\mathcal{C}}\mathbb{C}$	$\mathbb{C}$	O_{2n}/U_n	$\{\mathbb{CP}^{n-1} \subset G_2(\mathbb{R}^{2n})\}$
12	$\tilde{\mathcal{C}}\mathbb{C}$	$\mathcal{C}$	O_{2n}/O_n^2	$\{(\mathbb{RP}^{n-1})^2 \subset G_2(\mathbb{R}^{2n})\}$
13	$\tilde{\mathcal{C}}\mathbb{CC}$	$\mathbb{CC}$	$(O_{2n}/U_{2n})^2$	$\{(\mathbb{CP}^{n-1})^2 \subset G_2(\mathbb{R}^{2n})^2\}$
14	$\tilde{\mathcal{C}}\mathbb{CC}$	$\tilde{\mathcal{C}}\mathbb{C}$	$O_{2n}^2/O_{2n} = O_{2n}$	$\{G_2(\mathbb{R}^{2n}) \subset G_2(\mathbb{R}^{2n})^2\}$

Table 5.

Type	Space	Dim.	Rank	Linear algebra	Symmetric spaces
AI	U_n/O_n	$\frac{n(n+1)}{2}$	n	$\{\mathbb{R}^n \subset \mathbb{C}^n\}$	$\{\mathbb{RP}^{n-1} \subset \mathbb{CP}^{n-1}\}$
	U_{2n}/O_{2n}			$\{\tilde{\mathcal{C}}\mathbb{C}^n \subset \mathbb{HC}^n\}$	$\{G_2(\mathbb{R}^{2n}) \subset G_2(\mathbb{C}^{2n})\}$
AII	U_{2n}/Sp_n	$2n(n-1)$	n	$\{\mathbb{H}^n \subset \mathbb{HC}^n\}$	$\{\mathbb{HP}^{n-1} \subset G_2(\mathbb{C}^{2n})\}$
$AIII$	$U_{p+q}/(U_pU_q)$	$2pq$	p	$\{\mathbb{C}^p \subset \mathbb{C}^{p+q}\}$	$\{\mathbb{CP}^{p-1} \subset \mathbb{CP}^{p+q-1}\}$
	$U_{2n}/(U_{2p}U_{2n-2p})$			$\{\mathbb{HC}^p \subset \mathbb{HC}^n\}$	$\{G_2(\mathbb{C}^{2p}) \subset G_2(\mathbb{C}^{2n})\}$
	$U_{2n}/(U_nU_n)$	$2n^2$	n	$\{\mathbb{CC}^n \subset \mathbb{HC}^n\}$	$\{(\mathbb{CP}^{n-1})^2 \subset G_2(\mathbb{C}^{2n})\}$
BDI	O_{p+q}/O_pO_q	pq	p	$\{\mathbb{R}^p \subset \mathbb{R}^{p+q}\}$	$\{\mathbb{RP}^{p-1} \subset \mathbb{RP}^{p+q-1}\}$
	$O_{4n}/O_{4p}O_{4n-4p}$			$\{\mathbb{HH}^p \subset \mathbb{HH}^n\}$	$\{G_4(\mathbb{R}^{4p}) \subset G_4(\mathbb{R}^{4n})\}$
	O_{2n}/O_nO_n	n^2	$n4$	$\{\mathcal{C}^n \subset \tilde{\mathcal{C}}\mathbb{C}^n\}$	$\{(\mathbb{RP}^{n-1})^2 \subset G_2(\mathbb{R}^{2n})\}$
	$O_{4n}/O_{2n}O_{2n}$	$4n^2$	$2n$	$\{\tilde{\mathcal{C}}\mathbb{CC}^n \subset \mathbb{HH}^n\}$	$\{G_2(\mathbb{R}^{2n})^2 \subset G_4(\mathbb{R}^{4n})\}$
$DIII$	O_{2n}/U_n	$n(n-1)$	$[\frac{n}{2}]$	$\{\mathbb{C}^n \subset \tilde{\mathcal{C}}\mathbb{C}^n\}$	$\{\mathbb{CP}^{n-1} \subset G_2(\mathbb{R}^{2n})\}$
	$O_{4n}/O_{2n}O_{2n}$	$4n^2$	$2n$	$\{\tilde{\mathcal{C}}\mathbb{CC}^n \subset \mathbb{HH}^n\}$	$\{G_2(\mathbb{R}^{2n})^2 \subset G_4(\mathbb{R}^{4n})\}$
CI	Sp_n/U_n	$n(n-1)$	n	$\{\mathbb{C}^n \subset \mathbb{H}^n\}$	$\{\mathbb{CP}^{n-1} \subset \mathbb{HP}^{n-1}\}$
CII	Sp_{p+q}/Sp_pSp_q	$4pq$	p	$\{\mathbb{H}^p \subset \mathbb{H}^{p+q}\}$	$\{\mathbb{HP}^{p-1} \subset \mathbb{HP}^{p+q-1}\}$

6. Isotropic Grassmannians

Any orthogonal map A on $\mathbb{R}^n$ is determined by its graph $W = \{(x, Ax) : x \in \mathbb{R}^n\} \subset \mathbb{R}^{2n}$. Since $|Ax|^2 = |x|^2$, this is an *isotropic* subspace of $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ with the quadratic form $\langle v, sv\rangle$ for $s = \begin{pmatrix} I_n & \\ & -I_n \end{pmatrix}$, meaning $\langle W, sW\rangle = 0$ or more precisely, $sW = W^\perp$. Such a space is called (maximal) isotropic. Hence O_n can be considered as the *isotropic Grassmannian* $I(\mathbb{R}^{2n})$, consisting of all isotropic n-dimensional subspaces of $\mathbb{R}^{2n}$. Likewise, the (orthogonal) automorphism group of $\mathbb{A}^n$ (where $\mathbb{A}$ is as in Table 2) can be viewed as $I(\mathbb{A}^{2n})$, containing the submodules $W \cong \mathbb{A}^n$ such that $W = (sW)^\perp$ for some symmetric involution $s \in \mathrm{Aut}\ \mathbb{A}^n$ with eigenspaces of equal dimension (paracomplex structure) which commutes with the scalars in $\mathbb{A}$.

We may pose this structure into the framework of the previous section by introducing the algebra $\mathcal{C}\mathbb{A} = \mathcal{C} \otimes \mathbb{A}$ with the usual tensor multiplication, i.e. $s \in \mathcal{C}$ commutes with the scalars of $\mathbb{A}$. Then $V = \mathbb{A}^{2n} = (\mathcal{C}\mathbb{A})^n$ is a free $\mathcal{C}\mathbb{A}$-module which splits into the eigenspaces of s as $V = V_+ \oplus V_-$. Any automorphism of $(\mathcal{C}\mathbb{A})^n$ preserves these eigenspaces which are both isomorphic to $\mathbb{A}^n$, thus

$$\text{Aut}\,(\mathcal{C}\mathbb{A})^n \cong (\text{Aut}\ \mathbb{A}^n)^2. \tag{6.1}$$

Moreover, $\mathbb{A}^n$ is canonically embedded into $(\mathcal{C}\mathbb{A})^n$ as the fixed set of a balanced automorphims: the paracomplex conjugation on $\mathcal{C}\mathbb{A}$ and $(\mathcal{C}\mathbb{A})^n$,

$$\tilde{\kappa}(a + sb) = a - sb, \tag{6.2}$$

which anticommutes with s and thus interchanges $V_\pm$. Since $(\mathcal{C}\mathbb{A})^n = \mathbb{A}^n + s\mathbb{A}^n$, any automorphism of $\mathbb{A}^n$ extends uniquely to an automorphism of $(\mathcal{C}\mathbb{A})^n$, and the inclusion of the (orthogonal) automorphims groups Aut $\mathbb{A}^n \subset$ Aut $(\mathcal{C}\mathbb{A})^n$ is the diagonal embedding Aut $(\mathbb{A}^n) \subset (\text{Aut}\ \mathbb{A}^n)^2$.

Now we have for the isotropic Grassmannians:

$$\begin{aligned} I(\mathbb{A}^{2n}) &= \{\mathbb{A}^n \subset (\mathcal{C}\mathbb{A})^n\} \\ &= \{\mathbb{AP}^{n-1} \subset (\mathbb{AP}^{n-1})^2\} \\ &= (\text{Aut}\ \mathbb{A}^n)^2/\ \text{Aut}\ \mathbb{A}^n \\ &= \text{Aut}\ \mathbb{A}^n \end{aligned} \tag{6.3}$$

Clearly, $\mathcal{C}\mathbb{R} = \mathcal{C}$. Moreover, $\mathcal{C}\mathbb{C} \cong \mathbb{C}\mathbb{C}$ where we identify s with $S = i\hat{i}$ and $\hat{i}$ with $-si$. Therefore U_n and O_n did already appear in Table 4, No. 3 and 10 while $Sp_n = \{\mathbb{H}^n \subset (\mathcal{C}\mathbb{H})^n\}$ is new.

7. Exceptional spaces

Table 6 contains the exceptional spaces. It is partially taken from Huang and Leung [4,5] who include the exceptional spaces in their tables. We are also using D. Leung's lists of self-reflective subspaces, [6], p. 173-175, together with the list of polars by Chen and Nagano, [2], p. 294. By $G^\#_{n/2}(\mathbb{R}^n)$ we denote the manifold of oriented balanced splittings. A balanced splitting of $\mathbb{R}^n$ is a set $\{W, W^\perp\}$ where $W \subset \mathbb{R}^n$ is an $n/2$-dimensional subspace (n even). An orientation of W induces an orientation of $W^\perp$. Any balanced splitting $\{W, W^\perp\}$ can carry two possible orientations, thus $G^\#_{n/2}(\mathbb{R}^n)$ is a two-fold covering of the space $\bar{G}_{n/2}(\mathbb{R}^n)$ of all splittings $\{W, W^\perp\}$. The usual unoriented Grassmannian $G_{n/2}(\mathbb{R}^n)$ is another two-fold covering of $\bar{G}_{n/2}(\mathbb{R}^n)$ which is not diffeomorphic to $G^\#_{n/2}(\mathbb{R}^n)$.

8. Concluding remarks

Table 6 has given a strong motivation for this paper. The four exceptional spaces with dimensions 16, 32, 64, 128 can be considered in a certain sense as "projective planes" (*Rosenfeld planes*) over $\mathbb{A} = \mathbb{O}, \mathbb{OC}, \mathbb{OH}, \mathbb{OO}$ where $\mathbb{O}$ denotes the

Table 6.

Type	Space	Dim.	Rank	"Linear algebra"	Symmetric spaces
$E\,I$	E_6/Sp_4	42	6	"$\{\hat{C}\mathbb{H}^3 \subset \mathbb{O}\mathbb{C}^3\}$"	$\{G_2(\mathbb{H}^4)/\mathbb{Z}_2 \subset \mathbb{O}\mathbb{C}\mathbb{P}^2\}$
$E\,II$	E_6/SU_6Sp_1	40	4	"$\{\mathbb{H}\mathbb{C}^3 \subset \mathbb{O}\mathbb{C}^3\}$"	$\{G_2(\mathbb{C}^6) \subset \mathbb{O}\mathbb{C}\mathbb{P}^2\}$
$E\,III$	$E_6/Spin_{10}U_1$	32	2	"$\{\mathbb{O}\mathbb{C}^1 \subset \mathbb{O}\mathbb{C}^3\}$"	$\mathbb{O}\mathbb{C}\mathbb{P}^2$
				"$\{\mathbb{O}\mathbb{C}^2 \subset \mathbb{O}\mathbb{C}^3\}$"	$\{G_2^{\rm or}(\mathbb{R}^{10}) \subset \mathbb{O}\mathbb{C}\mathbb{P}^2\}$
$E\,IV$	E_6/F_4	26	2	"$\{\mathbb{O}^3 \subset \mathbb{O}\mathbb{C}^3\}$"	$\{\mathbb{O}\mathbb{P}^2 \subset \mathbb{O}\mathbb{C}\mathbb{P}^2\}$
$E\,V$	E_7/SU_8	70	7	"$\{\hat{C}\mathbb{H}\mathbb{C}^3 \subset \mathbb{O}\mathbb{H}^3\}$"	$\{G_4(\mathbb{C}^8)/\mathbb{Z}_2 \subset \mathbb{O}\mathbb{H}\mathbb{P}^2\}$
$E\,VI$	$E_7/Spin_{12}Sp_1$	64	4	"$\{\mathbb{O}\mathbb{H}^1 \subset \mathbb{O}\mathbb{H}^3\}$"	$\mathbb{O}\mathbb{H}\mathbb{P}^2$
				"$\{\mathbb{O}\mathbb{H}^2 \subset \mathbb{O}\mathbb{H}^3\}$"	$\{G_4^{\rm or}(\mathbb{R}^{12}) \subset \mathbb{O}\mathbb{H}\mathbb{P}^2\}$
				"$\{\mathbb{H}\mathbb{H}^3 \subset \mathbb{O}\mathbb{H}^3\}$"	$\{G_4^{\rm or}(\mathbb{R}^{12}) \subset \mathbb{O}\mathbb{H}\mathbb{P}^2\}$
$E\,VII$	E_7/E_6U_1	54	3	"$\{\mathbb{O}\mathbb{C}^3 \subset \mathbb{O}\mathbb{H}^3\}$"	$\{\mathbb{O}\mathbb{C}\mathbb{P}^2 \subset \mathbb{O}\mathbb{H}\mathbb{P}^2\}$
$E\,VIII$	$E_8/Spin_{16}$	128	8	"$\{\mathbb{O}\mathbb{O}^1 \subset \mathbb{O}\mathbb{O}^3\}$"	$\mathbb{O}\mathbb{O}\mathbb{P}^2$
				"$\{\mathbb{O}\mathbb{O}^2 \subset \mathbb{O}\mathbb{O}^3\}$"	$\{G_8^{\#}(\mathbb{R}^{16}) \subset \mathbb{O}\mathbb{O}\mathbb{P}^2\}$
				"$\{\hat{C}\mathbb{H}\mathbb{H}^3 \subset \mathbb{O}\mathbb{O}^3\}$"	$\{G_8^{\#}(\mathbb{R}^{16}) \subset \mathbb{O}\mathbb{O}\mathbb{P}^2\}$
$E\,IX$	E_8/E_7Sp_1	112	4	"$\{\mathbb{O}\mathbb{H}^3 \subset \mathbb{O}\mathbb{O}^3\}$"	$\{\mathbb{O}\mathbb{H}\mathbb{P}^2 \subset \mathbb{O}\mathbb{O}\mathbb{P}^2\}$
$F\,I$	F_4/Sp_3Sp_1	28	4	"$\{\mathbb{H}^3 \subset \mathbb{O}^3\}$"	$\{\mathbb{H}\mathbb{P}^2 \subset \mathbb{O}\mathbb{P}^2\}$
$F\,II$	$F_4/Spin_9$	16	1	"$\{\mathbb{O}^1 \subset \mathbb{O}^3\}$"	$\mathbb{O}\mathbb{P}^2$
				"$\{\mathbb{O}^2 \subset \mathbb{O}^3\}$"	$\{S^8 = \mathbb{O}\mathbb{P}^1 \subset \mathbb{O}\mathbb{P}^2\}$
$G\,I$	G_2/SO_4	8	2	$\{\mathbb{H} \subset \mathbb{O}\}$	

octonian algebra. However, $\mathbb{A}^3$ is not a module over $\mathbb{A}$, by lack of associativity. Therefore there are no submodules in $\mathbb{A}^3$, and the inclusion sets in quotation marks do not really exist. But the last column of the table does make sense: All exceptional symmetric spaces (but the last one) are spaces of reflective submanifolds in Rosenfeld planes. In order to see the analogy we have represented the classical spaces in the same fashion.

However, the relation between the last two columns is not as strict as we would wish. E.g. in the E VI case, the inclusion set "$\{\mathbb{H}\mathbb{H}^3 \subset \mathbb{O}\mathbb{H}^3\}$" should be the same as $\{\mathbb{H}\mathbb{H}\mathbb{P}^2 \subset \mathbb{O}\mathbb{H}\mathbb{P}^2\}$. But we have $\mathbb{H}\mathbb{H}\mathbb{P}^2 = G_4(\mathbb{R}^{12})$ (cf. Table 3) while the reflective submanifold of $\mathbb{O}\mathbb{H}\mathbb{P}^2$ is $G_4^{\rm or}(\mathbb{R}^{12})$, according to Leung and Chen-Nagano [2,6]. Further, by analogy, one would think that the projective line over $\mathbb{A} = \mathbb{K}\mathbb{L}$ is $G_k^{\rm or}(\mathbb{R}^{k+l})$. This is true for $\mathbb{A} = \mathbb{K}\mathbb{R}$ where $\mathbb{A}\mathbb{P}^1 = S^k$ and it holds also for $\mathbb{A} = \mathbb{C}\mathbb{C}, \mathbb{C}\mathbb{H}$; note that $G_2^{\rm or}(\mathbb{R}^4) = S^2 \times S^2$ and $G_2^{\rm or}(\mathbb{R}^6) = G_2(\mathbb{C}^4)$. But $\mathbb{H}\mathbb{H}\mathbb{P}^1 = G_4(\mathbb{R}^8)$ is different: it is the unoriented Grassmannian (cf. Table 3). On the other hand, the oriented Grassmannian $G_4^{\rm or}(\mathbb{R}^8)$ is a reflective submanifold of $\mathbb{O}\mathbb{H}\mathbb{P}^2$ (cf. Table 6). Further, the reflective submanifolds corresponding to "$\{\mathbb{O}\mathbb{L}^2 \subset \mathbb{O}\mathbb{L}^3\}$" for $\mathbb{L} = \mathbb{C}, \mathbb{H}, \mathbb{O}$ are $G_2^{\rm or}(\mathbb{R}^{10})$, $G_4^{\rm or}(\mathbb{R}^{12})$ and $G_8^{\#}(\mathbb{R}^{16})$, respectively. Again, the $\mathbb{H}$-case is different. For the submanifolds corresponding to $\hat{C}\mathbb{H}^3$, $\hat{C}\mathbb{H}\mathbb{C}^3$, $\hat{C}\mathbb{H}\mathbb{H}^3$ we have no geometric interpretation yet. We expect that Vinberg's formula ([1, p. 192]) which holds for $\mathbb{A} = \mathbb{K}\mathbb{L}$ with $\mathbb{K}, \mathbb{L} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ can be extended to the non-associative algebras $\mathbb{A} = \hat{C}\mathbb{H} \subset \mathbb{O}\mathbb{C}$, $\hat{C}\mathbb{H}\mathbb{C} \subset \mathbb{O}\mathbb{H}$, $\hat{C}\mathbb{H}\mathbb{H} \subset \mathbb{O}\mathbb{O}$.

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