Self similarity of dihedral tilings

J.-H. Eschenburg [a,*], H.J. Rivertz [b]

[a] *Institut für Mathematik, Universität Augsburg, D-86135 Augsburg, Germany*
[b] *Sør-Trøndelag University College, Trondheim, Norway*

1. Introduction

Tilings of euclidean plane with a dihedral (D_n-)symmetry for $n \neq 2, 3, 4, 6$ must be aperiodic, due to the crystallographic restriction: there is no translation preserving the tiling. However, there can be another type of ordering: self similarity. A tiling of the full plane $\mathbb{R}^2$ is called *self similar* if its vertex set V contains a subset V' which is a homothetic image of V, i.e. $V' = \lambda V$ for some $\lambda > 1$. It is our aim to show the following theorem:

* Corresponding author.
E-mail addresses: eschenburg@math.uni-augsburg.de (J.-H. Eschenburg), hans.j.rivertz@hist.no (H.J. Rivertz).

Theorem 1. *When $n \geqslant 5$ is a prime, there exist self similar planar tilings with D_n-symmetry.*

The case $n = 5$ consists of the two well known Penrose tilings with exact pentagon symmetry [3,1], while $n = 7, 11$ have been discussed in [2]. Pictures of the $n = 7$ tilings can be found in [2] and [4].

We construct the tilings using the projection method [1], see also [2]: Our tilings are obtained by orthogonal projection of a subset of the grid $\mathbb{Z}^n \subset \mathbb{R}^n$ onto a 2-dimensional affine subspace E; the projected subset is the intersection of $\mathbb{Z}^n$ with the so called *strip* $\Sigma_E = E + I^n$ with $I = (0, 1)$. The vertex set of the tiling is $V_E = \pi_E(\mathbb{Z}^n \cap \Sigma_E)$, and the tiles are projections of unit squares in $\mathbb{R}^n$ all of whose vertices belong to $\mathbb{Z}^n \cap \Sigma_E$. This tiling is well defined provided that E is *in general position* with respect to $\mathbb{Z}^n$, i.e. for every point of E at most 2 coordinates can be integers [1,5]. Assigning the n vertices of an n-gon to the standard unit vectors $e_1, \ldots, e_n$ of $\mathbb{R}^n$, we obtain a linear action of the dihedral group D_n onto $\mathbb{R}^n$. If $n = 2r + 1$ is odd, this representation decomposes into a 1-dimensional fixed space $\mathbb{R}d$ with $d = \sum_i e_i$ and irreducible 2-dimensional subrepresentations $E_1, \ldots, E_r$. We will choose E parallel to E_1, say $E = E_1 + a$ for some $a \in \mathbb{R}^n$. This tiling will have local D_n symmetry at many places. But in order to have global D_n symmetry we will choose $a = \frac{k}{n} d$ where $1 \leqslant k \leqslant n - 1$.

The self similarity will be caused by a self adjoint D_n-invariant integer matrix S ("inflation matrix") which is integer invertible on $W := d^\perp$ (i.e. there is another D_n-invariant symmetric integer matrix T with $ST = TS = I$ on W) and which has eigenvalues λ_i with $|\lambda_i| > 1$ on each 2-dimensional component E_i of W for $i \geqslant 2$. Then S acts as a contraction on E_1 and an expansion on the other E_i, and we have[1] $S(\Sigma) \supset \Sigma'$ where $\Sigma' = E' + I^n$ with $E' = S(E) = E_1 + Sa$. Projecting the grid points in Σ' onto E' yields the point set $V_{E'}$. By projecting the grid points in $S(\Sigma)$ onto E' we obtain a larger point set $S(V_E) \supset V_{E'}$. Since E_1 is an eigenspace of S, the set $S(V_E)$ is homothetic to V_E. When V_E is invariant under D_n, so is also $S(V_E)$ and $V_{E'}$. There are only finitely many of such tilings with full D_n-symmetry. Therefore, passing to a power of S if necessary, we can arrange for $V_{E'}$ and V_E to be homothetic. This reduces the proof of the theorem to the construction of such a matrix S.

The D_n-invariant tilings are not so special as it seems; in fact any tiling corresponding to $E_1 + a$ with $a \in d^\perp$ is almost isometric to any of the symmetric tilings, as will be explained in Theorem 2 below.

2. Dihedral tilings

Let D_n denote the group of all rotations and reflections of a regular n-gon (Dihedral group). It acts by certain permutations on the set of vertices of the n-gon which may

[1] More precisely, since S is not integer invertible on $\mathbb{R}d$, we might have to pass to a suitable power of S, see [2].

be identified with the standard basis of $\mathbb{R}^n$. Thus we obtain an orthogonal integer representation of D_n on $\mathbb{R}^n$. Let A be the generator of the rotation subgroup C_n; hence $Ae_i = e_{i+1}$ for $i = 1, \ldots, n$ modulo n. The eigenbasis for A is $v_\zeta = (1, \zeta, \ldots, \zeta^{n-1})$ where $\zeta^n = 1$; in fact we have $Av_\zeta = \bar{\zeta} v_\zeta$. If $n = 2r+1$ is odd, the only real eigenvalue is 1, corresponding to the eigenvector $d = \sum_i e_i$. The remaining eigenvalues come in complex conjugate pairs ζ_j, $\bar{\zeta}_j$ where $\zeta_j = e^{2j\pi i/n}$ and the real and imaginary parts of v_{ζ_j} span a 2-dimensional real subspace E_j on which A acts via rotation by the angle $2\pi j/n$.

From now on we assume that $n = 2r+1$ is a prime $\geqslant 5$.

Lemma 1. *$W = d^\perp$ is a rationally irreducible representation for D_n.*

Proof. The 2-dimensional subspaces E_j are inequivalent D_n-modules. Thus any D_n-module $W_1 \subset W$ is a sum of some of the E_j. On the other hand, a nonzero rational vector $v = \sum \lambda_\zeta v_\zeta \in W \cap \mathbb{Q}^n$ has only nonzero coefficients, $\lambda_\zeta \neq 0$ for all ζ. In fact, since v is rational and $v_\zeta \in \mathbb{K}^n$ where $\mathbb{K} = \mathbb{Q}(\zeta_1)$, we have $\lambda_\zeta \in \mathbb{K}$ for all ζ. Now $v \in \mathbb{Q}^n$ iff $v^\sigma = v$ for all $\sigma \in G_\mathbb{K}$ where $G_\mathbb{K}$ denotes the Galois group of $\mathbb{K}$ over $\mathbb{Q}$. Each $\sigma \in G_\mathbb{K}$ is of the type $\zeta \mapsto \zeta^k$ for $k \in \{1, \ldots, n-1\}$. Hence $v^\sigma = \sum_\zeta (\lambda_\zeta)^\sigma v_{\zeta^k}$, and $v^\sigma = v$ iff $(\lambda_\zeta)^\sigma = \lambda_{\zeta^k}$ for all ζ. Therefore, if $\lambda_\zeta \neq 0$ for some ζ, then also $\lambda_{\zeta^k} \neq 0$. Thus $\lambda_\zeta \neq 0$ for all ζ, and hence W_1 cannot miss any E_j if it contains a nonzero rational vector v. □

Lemma 2. *Let $n = 2r+1$ be a prime and $F = W \ominus E_1 = E_2 + \cdots + E_r$. Then the orthogonal projection $\pi_F(\mathbb{Z}^n)$ of the grid onto F is dense in F.*

Proof. The closure of $\pi_F(\mathbb{Z}^n)$ is a closed abelian subgroup of $(F, +)$ and hence of the type $V \oplus \Gamma$ where V is a linear subspace of F and Γ a lattice in $F \ominus V$. Then $\mathbb{Z}^n \subset E_1 + V + \Gamma$, and hence $S = (E_1 + V + \Gamma)/\mathbb{Z}^n$ is a proper closed subgroup of $T = \mathbb{R}^n/\mathbb{Z}^n$. But $\mathbb{Z}^n$ acts transitively on the set of cosets $\{E_1 + V + \gamma\colon\ \gamma \in \Gamma\}$ and therefore $S = \pi(E_1 + V)$ where $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n = T$ is the canonical projection. Hence S is connected, a proper subtorus of T. Since $\pi(E_1 + V)$ is a torus, $E_1 + V$ must contain a sublattice of $\mathbb{Z}^n$, and hence it is a “rational subspace”, that is it is spanned by rational vectors. Since E_1 is D_n-invariant, so is its rational hull, the smallest rational subspace containing E_1. Thus the rational hull of E_1 is W and therefore we have $E_1 + V = W$ and $V = F$. But V was the closure of the non-discrete part of $\pi_F(\mathbb{Z}^n)$. Since $V = F$, there is no discrete part and $\pi_F(\mathbb{Z}^n)$ is dense in F. □

Theorem 2. *Let $E = E_1 + a$ and $E' = E_1 + a'$ with $a, a' \in W$. Then the corresponding tilings are almost isometric, that is up to a translation the two tilings agree on the entire plane up to a set of errors spread over the entire plane with arbitrary small density everywhere.*

Proof. Let $E = E_1 + a$ and $E' = E_1 + a'$ with $a, a' \in W$; in fact we may assume $a, a' \in F = W \ominus E_1$. Since $\pi_F(\mathbb{Z}^n)$ is dense in F, we may approximate $a' - a \in F$ by some $\pi_F(z_\epsilon)$ with $z_\epsilon \in \mathbb{Z}^n$. Hence $E'' = E + z_\epsilon$ is arbitrary close to E'. Consequently,

$\Sigma_{E'}$ and $\Sigma_{E''}$ are almost equal which means that most lattice points admissible for E' are also admissible for E'' and vice versa. Deviations occur only for the grid points contained in $(\Sigma_{E'} \setminus \Sigma_{E''}) \cup (\Sigma_{E''} \setminus \Sigma_{E'})$. Let $C = I^n$ be the open unit cube, $C_k = \{x \in C\colon \sum x_i = k\}$ for $k = 1, \dots, n-1$ and $V_i = \pi_F(C_k)$. Then $V_k + a + z$ is arbitrary close to $V_k + a'$, and its intersection is a set of arbitrarily small measure ϵ. The ratio $\epsilon / \operatorname{vol}(V_k)$ measures the probability for a grid point z with $\sum z_i = k$ to behave differently for E' and E''. $\square$

Remark. For the D_n-symmetric tilings we put $a = \frac{k}{n} d$. This seems to violate the condition of Theorem 2 since $a \notin W$. However, we may replace a by (say) $a' = a - ke_1 \in W$; the translation by $ke_1 \in \mathbb{Z}^n$ does not change the geometry of the tiling.

3. General position

We must convince ourselves that the affine space $E_1 + a$ with $a = \frac{k}{n} d$ is in general position, i.e. for any $x \in E_1$ at most two coordinates of $x + a$ can be integers. We are using the complex basis v, $\bar{v}$ for $E_1^c = E_1 \otimes \mathbb{C}$ with $v = (\zeta, \zeta^2, \dots, \zeta^n)$ and $\zeta = \zeta_1 = e^{2\pi i/n}$. Then any element $x \in E_1$ can be written in the form $x = cv + \bar{c}\bar{v}$ for some complex number c. We have to show that not more than two components of such a vector can be an integer.

More generally we assume for two different k, l

$$c\zeta^k + \bar{c}\bar{\zeta}^k = p, \qquad c\zeta^l + \bar{c}\bar{\zeta}^l = q$$

for some nonzero rational numbers p, q. Multiplying the first equation by ζ^l and the second by ζ^k and substracting we obtain

$$\bar{c} = \frac{p\zeta^l - q\zeta^k}{\zeta^{l-k} - \bar{\zeta}^{l-k}}, \qquad c = \frac{-p\bar{\zeta}^l + q\bar{\zeta}^k}{\zeta^{l-k} - \bar{\zeta}^{l-k}}.$$

Now suppose that we have a third such equation:

$$c\zeta^j + \bar{c}\bar{\zeta}^j = r$$

where j, k, l are different modulo n and $r \in \mathbb{Q} \setminus \{0\}$. Plugging in our results for $\bar{c}$ and c we obtain

$$\left(-p\bar{\zeta}^l + q\bar{\zeta}^k\right)\zeta^j + \left(p\zeta^l - q\zeta^k\right)\bar{\zeta}^j = r\left(\zeta^{l-k} - \bar{\zeta}^{l-k}\right)$$

which is a rational linear dependence of certain powers of ζ or rather their imaginary parts, the corresponding sine values:

$$p\left(\zeta^{l-j} - \bar{\zeta}^{l-j}\right) + q\left(\zeta^{j-k} - \bar{\zeta}^{j-k}\right) + r\left(\zeta^{k-l} - \bar{\zeta}^{k-l}\right) = 0. \tag{$*$}$$

Since n is prime, the powers $\zeta, \zeta^2, \dots, \zeta^{n-1}$ are linearly independent over the rationals, thus p, q, r must vanish, unless two of the three numbers $l - j$, $j - k$, $k - l$ are equal up to sign, say $l-j=j-k$. Then the trivial relation $p=-q$, $r=0$ remains, but this is excluded by the assumption $p,q,r\neq 0$.

4. Construction of the inflation matrix

Consider the symmetric integer matrix $S_k=A^k+A^{-k}$ sending each e_j onto $e_{j+k}+e_{j-k}$ (the indices have to be taken mod n). Being D_n-invariant it keeps the E_j invariant, and the eigenvalue on E_j is $\lambda_{jk}=\zeta^{jk}+\bar\zeta^{jk}$. We look for an integer matrix S, a polynomial in the S_k, which is invertible on $W=d^\perp$ such that $|\lambda_i|>1$ for all eigenvalues λ_i with $i\geqslant 2$.

Let ζ be any primitive n-th unit root where $n=2r+1$. For every positive integer $a\leqslant r$ we have $\frac{1-\zeta^a}{1-\zeta}=1+\zeta+\cdots+\zeta^{a-1}$. This number is invertible in the ring $\mathbb{Z}[\zeta]$ since its inverse $\frac{1-\zeta}{1-\zeta^a}$ is a sum of powers of ζ^a; note that ζ is itself a power of ζ^a. Multiplying by ζ^{-b} with $b=\frac{a-1}{2}\in\mathbb{F}_n$ in the field $\mathbb{F}_n=\mathbb{Z}/n\mathbb{Z}$ makes this sum of powers symmetric, hence real, and we obtain an invertible element $\xi_a\in\mathbb{Z}[\zeta+\bar\zeta]$,

$$\xi_a:=\zeta^{-b}\frac{1-\zeta^a}{1-\zeta}=\sum_{j=-c}^{c}\zeta^j\quad\text{where } c=\begin{cases}\frac{a-1}{2} & \text{if } a-1 \text{ is even,}\\ \frac{n-a+1}{2} & \text{if } a-1 \text{ is odd.}\end{cases}\tag{1}$$

For any $a\in\{2,\dots,r\}$ we consider the matrix

$$U_a=\sum_{j=-c}^{c}A^j=I+S_1+\cdots+S_c.\tag{2}$$

Putting $\zeta=\zeta_j$, the eigenvalue of U_a on E_j will be ξ_a in (1) which now more precisely will be called ξ_{ja}. Since all ξ_a are invertible in $\mathbb{Z}[\zeta+\bar\zeta]$, all U_a are integer invertible on $W=d^\perp$. Our inflation matrix will be of the form

$$S=U_2^{t_2}U_3^{t_3}\cdots U_r^{t_r}\tag{3}$$

for suitable powers $t_2,\dots,t_r\in\mathbb{Z}$. The eigenvalue of S on E_j is

$$\lambda_j=(\xi_{j2})^{t_2}\cdots(\xi_{jr})^{t_r}.\tag{4}$$

We want $|\lambda_j|>1$ for all $j=2,\dots,r$. Taking logarithms of absolute values we obtain

$$t_2\log|\xi_{j2}|+\cdots+t_r\log|\xi_{jr}|=\log|\lambda_j|\tag{5}$$

for $j=2,\dots,r$. This is viewed as a system of $r-1$ linear equations for the unknowns $t_2,\dots,t_r$ with coefficient matrix $X=(\log|\xi_{jk}|)_{j,k\geqslant 2}$. By [6, Theorem 8.2, p. 145], the determinant of X is nonzero (the so called regulator). Thus the solution of (5) is $\vec t=X^{-1}\vec l$ where $\vec l=(l_2,\dots,l_r)^T$ with $l_j=\log|\lambda_j|$. In particular, near any "positive" vector $\vec l$ with

$l_2, \ldots, l_r > 0$ there is a positive vector $\vec{l'}$ such that $\vec{t} = X^{-1}\vec{l'}$ is a rational vector. Multiplying by the product (or the lcm) of the denominators we may assume that $\vec{t}$ is an integer vector. Thus the eigenvalues λ_j of $S = U_2^{t_2} U_3^{t_3} \cdots U_r^{t_r}$ satisfy $|\lambda_j| > 1$ for $j = 2, \ldots, r$ and hence S is an inflation matrix. This finishes the proof of Theorem 1.

Acknowledgment

It is our pleasure to thank Jürgen Ritter for calling our attention to the number theoretic results which we need.

References

[1] N.G. de Bruijn, Algebraic theory of Penrose non-periodic tilings of the plane, Indag. Math. 43 (1981) 27–37.
[2] J.-H. Eschenburg, H.J. Rivertz, Self similar symmetric planar tilings, J. Geom. 87 (2007) 55–75.
[3] R. Penrose, Pentaplexity – a class of nonperiodic tilings of the plane, Math. Intelligencer 2 (1979) 32–37.
[4] H.J. Rivertz, Symmetric 7-Penrose (“Heprose”) Tilings, http://www.math.uni-augsburg.de/~eschenbu/.
[5] D. Stern, Penrose type tilings, Diplom thesis, Augsburg, 2003, http://www.math.uni-augsburg.de/~eschenbu/.
[6] L.C. Washington, Introduction to Cyclotomic Fields, second ed., Springer, 1997.