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Jost-Hinrich Eschenburg, H. J. Rivertz

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J.-H. Eschenburg [a,*], H.J. Rivertz [b]

[a] *Institut für Mathematik, Universität Augsburg, D-86135 Augsburg, Germany*
[b] *Sør-Trøndelag University College, Trondheim, Norway*

1. Introduction

Tilings of euclidean plane with a dihedral (D_n-)symmetry for $n \neq 2, 3, 4, 6$ must be aperiodic, due to the crystallographic restriction: there is no translation preserving the tiling. However, there can be another type of ordering: self similarity. A tiling of the full plane $\mathbb{R}^2$ is called *self similar* if its vertex set V contains a subset V' which is a homothetic image of V, i.e. $V' = \lambda V$ for some $\lambda > 1$. It is our aim to show the following theorem:

* Corresponding author.
  E-mail addresses: eschenburg@math.uni-augsburg.de (J.-H. Eschenburg), hans.j.rivertz@hist.no (H.J. Rivertz).

Theorem 1. *When $n \geqslant 5$ is a prime, there exist self similar planar tilings with D_n-symmetry.*

The case $n = 5$ consists of the two well known Penrose tilings with exact pentagon symmetry [3,1], while $n = 7, 11$ have been discussed in [2]. Pictures of the $n = 7$ tilings can be found in [2] and [4].

We construct the tilings using the projection method [1], see also [2]: Our tilings are obtained by orthogonal projection of a subset of the grid $\mathbb{Z}^n \subset \mathbb{R}^n$ onto a 2-dimensional affine subspace E; the projected subset is the intersection of $\mathbb{Z}^n$ with the so called *strip* $\Sigma_E = E + I^n$ with $I = (0, 1)$. The vertex set of the tiling is $V_E = \pi_E(\mathbb{Z}^n \cap \Sigma_E)$, and the tiles are projections of unit squares in $\mathbb{R}^n$ all of whose vertices belong to $\mathbb{Z}^n \cap \Sigma_E$. This tiling is well defined provided that E is *in general position* with respect to $\mathbb{Z}^n$, i.e. for every point of E at most 2 coordinates can be integers [1,5]. Assigning the n vertices of an n-gon to the standard unit vectors $e_1, \ldots, e_n$ of $\mathbb{R}^n$, we obtain a linear action of the dihedral group D_n onto $\mathbb{R}^n$. If $n = 2r + 1$ is odd, this representation decomposes into a 1-dimensional fixed space $\mathbb{R}d$ with $d = \sum_i e_i$ and irreducible 2-dimensional subrepresentations $E_1, \ldots, E_r$. We will choose E parallel to E_1, say $E = E_1 + a$ for some $a \in \mathbb{R}^n$. This tiling will have local D_n symmetry at many places. But in order to have global D_n symmetry we will choose $a = \frac{k}{n}d$ where $1 \leqslant k \leqslant n - 1$.

The self similarity will be caused by a self adjoint D_n-invariant integer matrix S ("inflation matrix") which is integer invertible on $W := d^\perp$ (i.e. there is another D_n-invariant symmetric integer matrix T with $ST = TS = I$ on W) and which has eigenvalues λ_i with $|\lambda_i| > 1$ on each 2-dimensional component E_i of W for $i \geqslant 2$. Then S acts as a contraction on E_1 and an expansion on the other E_i, and we have[1] $S(\Sigma) \supset \Sigma'$ where $\Sigma' = E' + I^n$ with $E' = S(E) = E_1 + Sa$. Projecting the grid points in Σ' onto E' yields the point set $V_{E'}$. By projecting the grid points in $S(\Sigma)$ onto E' we obtain a larger point set $S(V_E) \supset V_{E'}$. Since E_1 is an eigenspace of S, the set $S(V_E)$ is homothetic to V_E. When V_E is invariant under D_n, so is also $S(V_E)$ and $V_{E'}$. There are only finitely many of such tilings with full D_n-symmetry. Therefore, passing to a power of S if necessary, we can arrange for $V_{E'}$ and V_E to be homothetic. This reduces the proof of the theorem to the construction of such a matrix S.

The D_n-invariant tilings are not so special as it seems; in fact any tiling corresponding to $E_1 + a$ with $a \in d^\perp$ is almost isometric to any of the symmetric tilings, as will be explained in Theorem 2 below.

2. Dihedral tilings

Let D_n denote the group of all rotations and reflections of a regular n-gon (Dihedral group). It acts by certain permutations on the set of vertices of the n-gon which may

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[1] More precisely, since S is not integer invertible on $\mathbb{R}d$, we might have to pass to a suitable power of S, see [2].

be identified with the standard basis of $\mathbb{R}^n$. Thus we obtain an orthogonal integer representation of D_n on $\mathbb{R}^n$. Let A be the generator of the rotation subgroup C_n; hence $Ae_i = e_{i+1}$ for $i = 1, \ldots, n$ modulo n. The eigenbasis for A is $v_\zeta = (1, \zeta, \ldots, \zeta^{n-1})$ where $\zeta^n = 1$; in fact we have $Av_\zeta = \bar{\zeta}v_\zeta$. If $n = 2r + 1$ is odd, the only real eigenvalue is 1, corresponding to the eigenvector $d = \sum_i e_i$. The remaining eigenvalues come in complex conjugate pairs $\zeta_j, \bar{\zeta}_j$ where $\zeta_j = e^{2j\pi i/n}$ and the real and imaginary parts of v_{ζ_j} span a 2-dimensional real subspace E_j on which A acts via rotation by the angle $2\pi j/n$.

From now on we assume that $n = 2r + 1$ is a prime $\geqslant 5$.

Lemma 1. $W = d^\perp$ *is a rationally irreducible representation for* D_n.

Proof. The 2-dimensional subspaces E_j are inequivalent D_n-modules. Thus any D_n-module $W_1 \subset W$ is a sum of some of the E_j. On the other hand, a nonzero rational vector $v = \sum \lambda_\zeta v_\zeta \in W \cap \mathbb{Q}^n$ has only nonzero coefficients, $\lambda_\zeta \neq 0$ for all ζ. In fact, since v is rational and $v_\zeta \in \mathbb{K}^n$ where $\mathbb{K} = \mathbb{Q}(\zeta_1)$, we have $\lambda_\zeta \in \mathbb{K}$ for all ζ. Now $v \in \mathbb{Q}^n$ iff $v^\sigma = v$ for all $\sigma \in G_\mathbb{K}$ where $G_\mathbb{K}$ denotes the Galois group of $\mathbb{K}$ over $\mathbb{Q}$. Each $\sigma \in G_\mathbb{K}$ is of the type $\zeta \mapsto \zeta^k$ for $k \in \{1, \ldots, n-1\}$. Hence $v^\sigma = \sum_\zeta (\lambda_\zeta)^\sigma v_{\zeta^k}$, and $v^\sigma = v$ iff $(\lambda_\zeta)^\sigma = \lambda_{\zeta^k}$ for all ζ. Therefore, if $\lambda_\zeta \neq 0$ for some ζ, then also $\lambda_{\zeta^k} \neq 0$. Thus $\lambda_\zeta \neq 0$ for all ζ, and hence W_1 cannot miss any E_j if it contains a nonzero rational vector v. $\square$

Lemma 2. *Let* $n = 2r + 1$ *be a prime and* $F = W \ominus E_1 = E_2 + \cdots + E_r$. *Then the orthogonal projection* $\pi_F(\mathbb{Z}^n)$ *of the grid onto* F *is dense in* F.

Proof. The closure of $\pi_F(\mathbb{Z}^n)$ is a closed abelian subgroup of $(F, +)$ and hence of the type $V \oplus \Gamma$ where V is a linear subspace of F and Γ a lattice in $F \ominus V$. Then $\mathbb{Z}^n \subset E_1 + V + \Gamma$, and hence $S = (E_1 + V + \Gamma)/\mathbb{Z}^n$ is a proper closed subgroup of $T = \mathbb{R}^n/\mathbb{Z}^n$. But $\mathbb{Z}^n$ acts transitively on the set of cosets $\{E_1 + V + \gamma \colon \gamma \in \Gamma\}$ and therefore $S = \pi(E_1 + V)$ where $\pi : \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n = T$ is the canonical projection. Hence S is connected, a proper subtorus of T. Since $\pi(E_1 + V)$ is a torus, $E_1 + V$ must contain a sublattice of $\mathbb{Z}^n$, and hence it is a "rational subspace", that is it is spanned by rational vectors. Since E_1 is D_n-invariant, so is its rational hull, the smallest rational subspace containing E_1. Thus the rational hull of E_1 is W and therefore we have $E_1 + V = W$ and $V = F$. But V was the closure of the non-discrete part of $\pi_F(\mathbb{Z}^n)$. Since $V = F$, there is no discrete part and $\pi_F(\mathbb{Z}^n)$ is dense in F. $\square$

Theorem 2. *Let* $E = E_1 + a$ *and* $E' = E_1 + a'$ *with* $a, a' \in W$. *Then the corresponding tilings are almost isometric, that is up to a translation the two tilings agree on the entire plane up to a set of errors spread over the entire plane with arbitrary small density everywhere.*

Proof. Let $E = E_1 + a$ and $E' = E_1 + a'$ with $a, a' \in W$; in fact we may assume $a, a' \in F = W \ominus E_1$. Since $\pi_F(\mathbb{Z}^n)$ is dense in F, we may approximate $a' - a \in F$ by some $\pi_F(z_\epsilon)$ with $z_\epsilon \in \mathbb{Z}^n$. Hence $E'' = E + z_\epsilon$ is arbitrary close to E'. Consequently,

$\Sigma_{E'}$ and $\Sigma_{E''}$ are almost equal which means that most lattice points admissible for E' are also admissible for E'' and vice versa. Deviations occur only for the grid points contained in $(\Sigma_{E'} \setminus \Sigma_{E''}) \cup (\Sigma_{E''} \setminus \Sigma_{E'})$. Let $C = I^n$ be the open unit cube, $C_k = \{x \in C : \sum x_i = k\}$ for $k = 1, \ldots, n-1$ and $V_i = \pi_F(C_k)$. Then $V_k + a + z$ is arbitrary close to $V_k + a'$, and its intersection is a set of arbitrarily small measure ϵ. The ratio $\epsilon / \operatorname{vol}(V_k)$ measures the probability for a grid point z with $\sum z_i = k$ to behave differently for E' and E''. $\quad \square$

Remark. For the D_n-symmetric tilings we put $a = \frac{k}{n} d$. This seems to violate the condition of Theorem 2 since $a \notin W$. However, we may replace a by (say) $a' = a - ke_1 \in W$; the translation by $ke_1 \in \mathbb{Z}^n$ does not change the geometry of the tiling.

3. General position

We must convince ourselves that the affine space $E_1 + a$ with $a = \frac{k}{n} d$ is in general position, i.e. for any $x \in E_1$ at most two coordinates of $x + a$ can be integers. We are using the complex basis $v, \bar{v}$ for $E_1^c = E_1 \otimes \mathbb{C}$ with $v = (\zeta, \zeta^2, \ldots, \zeta^n)$ and $\zeta = \zeta_1 = e^{2\pi i / n}$. Then any element $x \in E_1$ can be written in the form $x = cv + \bar{c}\bar{v}$ for some complex number c. We have to show that not more than two components of such a vector can be an integer.

More generally we assume for two different k, l

$$c\zeta^k + \bar{c}\bar{\zeta}^k = p, \qquad c\zeta^l + \bar{c}\bar{\zeta}^l = q$$

for some nonzero rational numbers p, q. Multiplying the first equation by ζ^l and the second by ζ^k and substracting we obtain

$$\bar{c} = \frac{p\zeta^l - q\zeta^k}{\zeta^{l-k} - \bar{\zeta}^{l-k}}, \qquad c = \frac{-p\bar{\zeta}^l + q\bar{\zeta}^k}{\zeta^{l-k} - \bar{\zeta}^{l-k}}.$$

Now suppose that we have a third such equation:

$$c\zeta^j + \bar{c}\bar{\zeta}^j = r$$

where j, k, l are different modulo n and $r \in \mathbb{Q} \setminus \{0\}$. Plugging in our results for $\bar{c}$ and c we obtain

$$\left(-p\bar{\zeta}^l + q\bar{\zeta}^k\right)\zeta^j + \left(p\zeta^l - q\zeta^k\right)\bar{\zeta}^j = r\left(\zeta^{l-k} - \bar{\zeta}^{l-k}\right)$$

which is a rational linear dependence of certain powers of ζ or rather their imaginary parts, the corresponding sine values:

$$p\left(\zeta^{l-j} - \bar{\zeta}^{l-j}\right) + q\left(\zeta^{j-k} - \bar{\zeta}^{j-k}\right) + r\left(\zeta^{k-l} - \bar{\zeta}^{k-l}\right) = 0. \qquad (*)$$

Since n is prime, the powers $\zeta, \zeta^2, \ldots, \zeta^{n-1}$ are linearly independent over the rationals, thus p, q, r must vanish, unless two of the three numbers $l - j$, $j - k$, $k - l$ are equal up

to sign, say $l - j = j - k$. Then the trivial relation $p = -q$, $r = 0$ remains, but this is excluded by the assumption $p, q, r \neq 0$.

4. Construction of the inflation matrix

Consider the symmetric integer matrix $S_k = A^k + A^{-k}$ sending each e_j onto $e_{j+k} + e_{j-k}$ (the indices have to be taken mod n). Being D_n-invariant it keeps the E_j invariant, and the eigenvalue on E_j is $\lambda_{jk} = \zeta^{jk} + \bar{\zeta}^{jk}$. We look for an integer matrix S, a polynomial in the S_k, which is invertible on $W = d^\perp$ such that $|\lambda_i| > 1$ for all eigenvalues λ_i with $i \geqslant 2$.

Let ζ be any primitive n-th unit root where $n = 2r + 1$. For every positive integer $a \leqslant r$ we have $\frac{1-\zeta^a}{1-\zeta} = 1 + \zeta + \cdots + \zeta^{a-1}$. This number is invertible in the ring $\mathbb{Z}[\zeta]$ since its inverse $\frac{1-\zeta}{1-\zeta^a}$ is a sum of powers of ζ^a; note that ζ is itself a power of ζ^a. Multiplying by ζ^{-b} with $b = \frac{a-1}{2} \in \mathbb{F}_n$ in the field $\mathbb{F}_n = \mathbb{Z}/n\mathbb{Z}$ makes this sum of powers symmetric, hence real, and we obtain an invertible element $\xi_a \in \mathbb{Z}[\zeta + \bar{\zeta}]$,

$$\xi_a := \zeta^{-b} \frac{1 - \zeta^a}{1 - \zeta} = \sum_{j=-c}^{c} \zeta^j \quad \text{where } c = \begin{cases} \frac{a-1}{2} & \text{if } a - 1 \text{ is even,} \\ \frac{n-a+1}{2} & \text{if } a - 1 \text{ is odd.} \end{cases} \tag{1}$$

For any $a \in \{2, \ldots, r\}$ we consider the matrix

$$U_a = \sum_{j=-c}^{c} A^j = I + S_1 + \cdots + S_c. \tag{2}$$

Putting $\zeta = \zeta_j$, the eigenvalue of U_a on E_j will be ξ_a in (1) which now more precisely will be called ξ_{ja}. Since all ξ_a are invertible in $\mathbb{Z}[\zeta + \bar{\zeta}]$, all U_a are integer invertible on $W = d^\perp$. Our inflation matrix will be of the form

$$S = U_2^{t_2} U_3^{t_3} \cdots U_r^{t_r} \tag{3}$$

for suitable powers $t_2, \ldots, t_r \in \mathbb{Z}$. The eigenvalue of S on E_j is

$$\lambda_j = (\xi_{j2})^{t_2} \cdots (\xi_{jr})^{t_r}. \tag{4}$$

We want $|\lambda_j| > 1$ for all $j = 2, \ldots, r$. Taking logarithms of absolute values we obtain

$$t_2 \log |\xi_{j2}| + \cdots + t_r \log |\xi_{jr}| = \log |\lambda_j| \tag{5}$$

for $j = 2, \ldots, r$. This is viewed as a system of $r - 1$ linear equations for the unknowns $t_2, \ldots, t_r$ with coefficient matrix $X = (\log |\xi_{jk}|)_{j,k \geqslant 2}$. By [6, Theorem 8.2, p. 145], the determinant of X is nonzero (the so called regulator). Thus the solution of (5) is $\vec{t} = X^{-1} \vec{l}$ where $\vec{l} = (l_2, \ldots, l_r)^T$ with $l_j = \log |\lambda_j|$. In particular, near any "positive" vector $\vec{l}$ with

$l_2, \ldots, l_r > 0$ there is a positive vector $\vec{l'}$ such that $\vec{t} = X^{-1}\vec{l'}$ is a rational vector. Multiplying by the product (or the lcm) of the denominators we may assume that $\vec{t}$ is an integer vector. Thus the eigenvalues λ_j of $S = U_2^{t_2} U_3^{t_3} \cdots U_r^{t_r}$ satisfy $|\lambda_j| > 1$ for $j = 2, \ldots, r$ and hence S is an inflation matrix. This finishes the proof of Theorem 1.

Acknowledgment

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