Teachers' perspectives on mathematical argumentation, reasoning and justifying in calculus classrooms

Sabrina Bersch

University of Augsburg, Germany; sabrina.bersch@math.uni-augsburg.de

An interview study on mathematical argumentation (in a broad sense) was conducted with teachers of upper-secondary calculus classrooms. This paper describes the study's methods and its results. By using qualitative text analysis, four major categories were created to depict the current state of mathematical argumentation in calculus classrooms. Two dominant problem areas were revealed: Students' language difficulties and the heterogeneity of students. To address these problems, a learning environment was designed and evaluated in a follow-up study.

Keywords: Argumentation/Reasoning/Justifying, calculus, secondary school teachers, semi-structured interviews, learning environment.

Introduction and theoretical background

Mathematical argumentation, reasoning, justifying and proof indisputably constitute an important field of mathematical competencies. Nevertheless, the 1995 and 1999 TIMMS Video Studies found that reasoning did not occur frequently in mathematics classes of the examined countries (Hiebert et al 2003, p. 73-75). Since 2003, the *Bildungsstandards* set by the KMK[1] have functioned as an important framework for teaching mathematics in Germany. One of the process-related competences they specify is *Mathematisch Argumentieren* (approximately corresponding to *mathematical argumentation*). This term is used as an umbrella term for working with mathematical conjectures and statements by employing a range of argumentations, from arguments of plausibility through justifications to formal proofs (KMK 2012, p. 14). In the United States, the Principles and Standards for School Mathematics were published by the National Council of Teachers of Mathematics (NCTM) in 2000 as one of the first sets of standards for mathematics teaching. One of the Process Standards set by the NCTM is *Reasoning and Proof*, which also comprises reasoning, proving, using conjectures, argumentation and justification (NCTM, 2000). In this paper, *mathematical argumentation* is used in a broad sense, including all aspects used by the KMK and the NCTM. In addition, pre-formal or semi-formal mathematical activities of argumentation, reasoning and justifying are considered suitable for mathematics in school and useful, necessary steps to formal, deductive proving as an essential mathematical activity. The term *formal* is "referring to the standard language used to talk about mathematics, which encodes the meanings of mathematics" (Barwell 2016, p. 333). Mastering this standard language is considered its own learning item for students. Pericleous similarly states that "explanation, justification and argumentation […] provide a foundation for […] developing deductive reasoning" (2015, p. 226).

[1] The *Bildungsstandards* are Educational Standards set by the Conference of Ministers of Education and Cultural Affairs in Germany (KMK)

However, the level of formality and deductive reasoning that should be acquired in school is open to debate.

Teachers' perspectives on argumentation in class are of great importance, because teachers are responsible for providing learning environments and tasks for students (Buchbinder, 2017, p. 107). They have gained significant experience with students' processes of acquiring competencies. Yet, there is little research on argumentation from a teachers' perspective to date. The discussed study investigates the role and importance of mathematical argumentation in calculus classrooms, explores teachers' attitudes and ideas about mathematical argumentation and reveals problems and difficulties teachers face when training students' mathematical argumentation competencies. Interviews with 14 teachers of different schools teaching upper-secondary students in calculus were conducted and analysed using qualitative text analysis[2]. In a follow-up study, a learning environment was developed and evaluated based on the results of the interview study. In this paper, the interview study is described in detail including its methods and findings. The paper concludes with a short outlook on the follow-up study.

Methods

There were two main research questions: *(1) Which role does mathematical argumentation play in current calculus classrooms? (2) Which problems and difficulties do teachers face with regard to mathematical argumentation in calculus classrooms?* To answer these questions, semi-structured interviews were conducted with 14 upper-secondary school teachers. The interview manual had four parts with different thematic foci. The participants had been informed that the topic would be calculus teaching. However, the emphasis on mathematical argumentation was not mentioned before the second part of the interview, because the first part was about calculus teaching in general and argumentation was only focused on in the other three parts.

14 upper-secondary school teachers, 5 female and 9 male, were chosen from 7 different schools (6 in Bavaria, Germany; 1 in Hesse, Germany), teaching different subjects in addition to mathematics. Their age ranged from 30 to 64 years with teaching experience from 4 to 36 years.

The analysis of the interviews used a combination of methods of qualitative content analysis (Mayring, 2015) and thematic qualitative text analysis (Kuckartz, 2014). First, a selection criterion was applied to detect all passages of the interviews concerning the topic of mathematical argumentation. Then, major categories were created deductively according to the interview guidelines and the research questions. After applying them to the data, they were further differentiated into subcategories inductively using the codes of each major category. Processes of subsumption and clustering were used to establish the final category system with various levels for the analysis.

[2] A rough overview of the interview study is being published in (Scheffler, 2018) and parts of the results have been published in (Scheffler, 2017).

Results

Figure 1 presents an overview of the category system. The major categories are *Understanding Concepts, Current Implementation in Class, Positive Aspects* and *Problems and Difficulties*. The numbers in brackets state the numbers of respondents (out of 14) whose statements contained segments for the respective subcategories. Each category is described separately in the following.

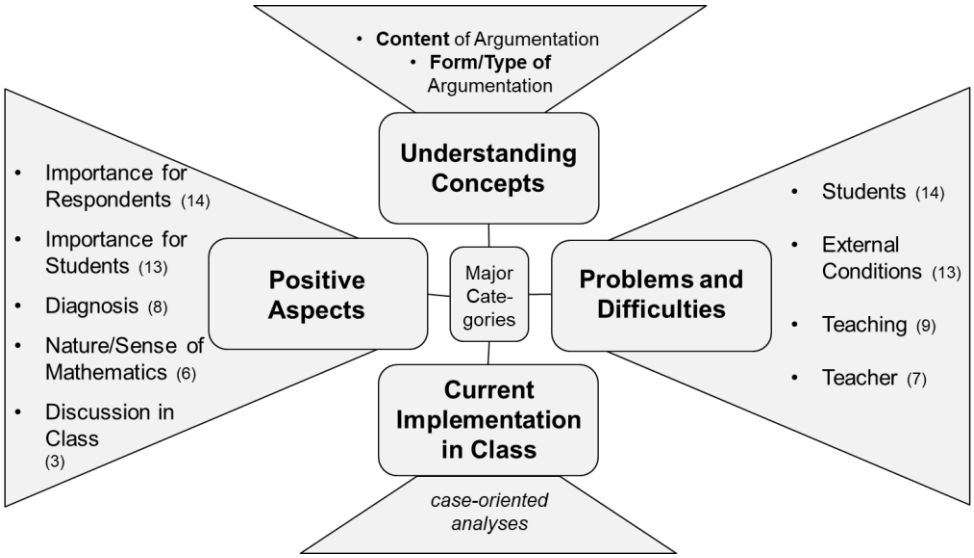

Figure 1: Major categories of the qualitative text analysis of the interviews

Understanding Concepts

The major category *Understanding Concepts* comprises segments from which it can be concluded what teachers mean when speaking of *argumentation, reasoning, justifying* or *proving*[3]. Segments within this category were categorized throughout all parts of the interviews, because the participants were not directly asked about their understanding of the terms. Various ideas could be found and subdivided into two subcategories: Ideas about the *Content of Argumentation* and ideas about the *Form/Type of Argumentation*. The teachers' statements do not only contain ideas about their actual teaching but also about their general understandings of argumentation. Both subcategories demonstrated a wide range of understandings. The most frequent opportunities for mathematical argumentation mentioned were situations in which students needed to justify their approaches when dealing with any mathematical exercise or task or justify certain mathematical theorems, rules or formulas. In addition, it was described that students reason when working with properties of various functions or when modelling mathematically. More generally, the teachers stated that mathematical relations or issues can be used for mathematical argumentation in class.

[3] In the interviews, I used the German terms *Argumentieren* and *Begründen* (approximately corresponding to *argumentation* and *justification* in English) as synonyms and avoided the term *Beweis* (*proof)* as it has negative connotations for some teachers. For the category *Understanding Concepts*, all segments of the interviews were used which showed understandings of any of the terms *Argumentieren, Begründen* or *Beweisen*.

In the subcategory *Form/Type of Argumentation*, it is striking that most teachers talked about formal proving but mostly commented on the lacking feasibility of using proofs in class. Other ways of mathematical argumentation mentioned by several teachers were justifying using calculation, explaining or elucidating, verbal justification and justification supplemented by sketches. This results in a varied field of teachers' understandings of how argumentative competencies can play a role in calculus classrooms and what mathematical content can be used for these purposes. These findings correspond to the broad understanding of the term used in the German KMK *Bildungsstandards* and in the NCTM standards.

Current Implementation in Class

Descriptions of what the teachers actually do in class concerning mathematical argumentation are collected in the subcategory *Current Implementation in Class*. Each case was analysed separately by summarising and abstracting the main ideas. The following overall tendencies about the current state of mathematical argumentation and proof in calculus classrooms could be found:

- Tasks in which students are asked to give reasons play a significant role.
- Formal proving, theoretical justifying and systematic derivations only occur occasionally, with most argumentations and justifications being informal, oral and not written.
- Teachers reason and justify more than their students.
- Argumentation and reasoning seem to be opposed to standard techniques which are trained mainly for the final examinations.

These practical tendencies are based on teachers' attitudes towards argumentation and reasoning in calculus classrooms. These attitudes are connected to the reasons teachers have to train their students' argumentative competencies. These reasons can be deduced from positive remarks about argumentation collected in the major category *Positive Aspects*. On the other hand, *Problems and Difficulties* with argumentation in calculus classrooms concern reasons why teachers use fewer opportunities for mathematical argumentation in their classes than they ideally should.

Positive Aspects

The positive statements about mathematical argumentation in calculus classrooms can be divided into 5 subcategories: Segments showing that mathematical argumentation is important for the teachers themselves (1) or for the students (2), segments explaining that mathematical argumentation is a good way for diagnosing students' skills (3), segments in which teachers state mathematical argumentation to be an essential part of mathematics (4) and segments in which teachers express that employing mathematical argumentation results in good discussions in class (5). The most interesting results can be found in the subcategory *Importance for Students* (2) which includes segments in which teachers explain how mathematical argumentation in class has positive effects for the students. In their opinion, mathematical argumentation is crucial for the students' content-related competence. It is also considered important for the students' future in mathematics and beyond. Teachers point to students who really like reasoning and to more proficient students who can demonstrate their skills with justification tasks.

Problems and Difficulties

Nevertheless, there are many problems and difficulties with mathematical argumentation in calculus classrooms. As it has been explained above, there is a reluctance to use formal proofs for different reasons which are not focused on in the study. For this reason, remarks stating difficulties and problems specifically with formal proofs were not coded in the major category *Problems and Difficulties.* Emphasis was put on argumentation in general. As Figure 1 and Figure 2 show, 4 subcategories could be created inductively in the major category *Problems and Difficulties*. All teachers mentioned problems and difficulties concerning the *Students* and nearly all teachers have problems with the *External Conditions* they face. In addition, there are problems and difficulties in the area of *Teaching* and difficulties for the *Teachers* themselves. Notably, difficulties for the *Teachers* themselves all deal with grading mathematical argumentation tasks. Problems with *Teaching* arise because teachers do not consider reasoning and justifying tasks suitable for examinations, and training standard calculation techniques has priority in their teaching. *External Conditions* that cause most problems for teachers are the restricted time available for teaching and the requirements of the centrally organised final examinations.

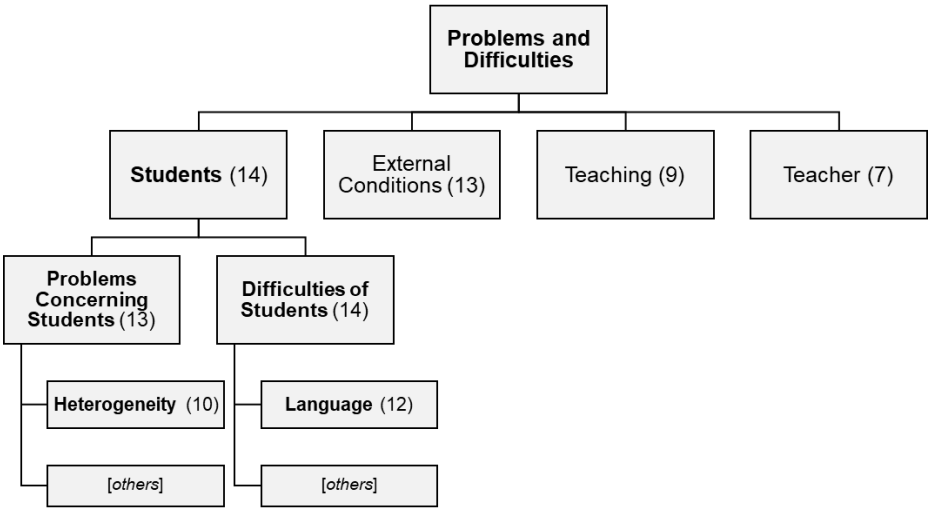

Figure 2: Part of the category system with a focus on *Problems and Difficulties*

The subcategory *Students* contains by far the most difficulties and problems that were mentioned by the teachers. It is further subdivided into *Problems Concerning Students* and *Difficulties of Students.* Due to their size, these subcategories were further subdivided:

Firstly, by far the largest subcategory of the subcategory *Problems Concerning Students* is *Heterogeneity*. Segments in this subcategory are about problems that arise because of students' different performance levels. While teachers are of the opinion that low achieving students have serious problems with mathematical argumentation tasks, such tasks are seen as a particular challenge for high achieving students. Consequently, teachers do not know how to cope with the great span and often decide not to use justification tasks in class. An example segment within the subcategory *Heterogeneity* is the following:

> First of all, I often think that these justification tasks are only accessible for a part of the students so that another part of the students is left behind by these justification tasks. And for them, it is

> important to do tasks in which they can use their learnt strategies. So, I would not use 45 minutes just for training argumentation, because after some time I would sit there just talking to five students and the other 20 are looking into the air (Interview 8, paragraph 58, own translation).

Other *Problems Concerning Students* result from students' aversion to argumentation amongst others.

Secondly, within the other subcategory, *Difficulties of Students*, a dominant subcategory evolved as well: *Language*. This subcategory contains segments dealing with problems students have with, for example, terminology, formulations, and especially writing down argumentations and justifications. An example segment within the subcategory *Language* is the following:

> And of course language, that's an important point, whether mathematical language or German language, stringing two sentences together. What is given? So, what can be concluded? That is what causes most problems (Interview 9, paragraph 56, own translation).

Apart from language problems, there are other issues students have problems with when working on argumentation tasks: the general validity of mathematical statements, mathematical precision and accuracy, recognizing the expectations and technical contents amongst others.

The interview study showed that teachers have a wide range of ideas about which aspects of mathematical argumentation exist and their attitude towards argumentation in calculus classrooms is positive to a large extent. However, teachers state that there is little formal argumentation and proof in their classrooms. Training standard techniques is far more important than training argumentation competencies. In addition, many varied problems and difficulties concerning the training of argumentation competencies could be gathered. As *Heterogeneity* and *Language* could be found as being dominant problem areas, developing a proposal for facing these problems was the aim of a follow-up study.

Follow-up study: Development and evaluation of a learning environment

To address the dominant problem areas found in the interview study, students' *Language* difficulties and the *Heterogeneity* of students, a calculus learning environment[4] with justification tasks was designed and given to 15 teachers for application and subsequent evaluation.[5] Language support is provided by a toolbox in two versions, based on ideas of Meyer and Prediger (2012), among others. To cope with the students' heterogeneity, potential for differentiation is given by a task structure orientated towards Bruder and Reibold's concept of *Blütenaufgaben*[6] (2011). To support students who have problems with argumentation in general on the one hand and students

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[4] The term *learning environment* is used for a large task with several subtasks embedded in a lesson plan together with instructions and additional material bound together by one central idea. This is based mainly on (Hirt & Wälti, 2008).

[5] The design principles of the learning environment and first results of study 2 have been published in (Scheffler, 2018).

[6] *Blütenaufgaben* (literal translation: *blossom tasks*) open like flower heads, which means their subtasks have different requirement levels and vary from closed to open-ended tasks. The subtasks are independent, though (Bruder & Reibold, 2011).

with problems concerning language on the other hand, there is a prepended worked-out example. A study of Reiss et al indicates that "self-explaining heuristic worked-out examples are a qualified instrument for improving students' achievement on reasoning and proof in the mathematics classroom" (2008, p. 463).

The learning environment was evaluated using written interviews. The analysis of these interviews showed that the learning environment is suitable for differentiation and the language support works if a suitable version of the learning environment is chosen. As a result, it is important that teachers have distinct diagnostic competencies to be able to support their students.

Discussion and Conclusions

This paper presented an interview study with teachers about mathematical argumentation in upper-secondary calculus classrooms. The qualitative and explorative character of the study provided an insight into current practices of argumentation in calculus teaching. The results might be used to generate possible hypotheses which could be examined quantitatively to learn more from the teachers' perspective. Whether the results can be transferred to other sections of mathematics teaching in upper-secondary school, is debatable. The complexity of calculus in comparison to stochastics and analytic geometry indicate that automatic transfer is not possible. What could be shown is that the interviewed teachers have a wide understanding of mathematical argumentation. They include different aspects of argumentation and reasoning in their calculus classrooms, but they hesitate to incorporate justifications in a written way or let students do so. They are also reluctant to use formal argumentations such as proofs, which is a bit surprising because teachers spoke of the upper-secondary level. This, however, can be justified as long as pre- or semi-formal mathematical argumentation is seen as pre-stage to proving, interested students are able to encounter formal arguments as well, and a realistic and representative view of mathematics is conveyed. Although the KMK *Bildungsstandards* have set a framework for teaching mathematics on an upper-secondary level in Germany, argumentation does not seem to play a role in mathematics teaching as much as it ideally should. Teachers basically have a positive attitude towards training argumentative competencies in their calculus teaching, but they also face a wide range of problems and difficulties. Two dominant problem areas could be found: Students have difficulties with language, especially when writing down their justifications, and teachers have problems dealing with the heterogeneity of their students. To work on these problems, a learning environment with differentiating character and language support has been developed and evaluated in a follow-up study. It could be shown that taking action is possible and that it is important for teachers to choose suitable teaching material for their students. More material should be developed to assist teachers and hence to help students develop argumentative competencies. It is a good basis that the interview study suggests that teachers consider argumentation in calculus classrooms important.

References

Barwell, R. (2016). Formal and informal mathematical discourses. Bakhtin and Vygotsky, dialogue and dialectic. *Educational Studies in Mathematics, 92*(3), 331-345.

Bruder, R., & Reibold, J. (2011). Differenzierung im Mathematikunterricht [Differentiation in Mathematics Classes]. In M. Eisenmann & T. Grimm (Eds.), *Heterogene Klassen:*

Differenzierung in Schule und Unterricht (pp. 118–136). Baltmannsweiler, Germany: Schneider-Verlag Hohengehren.

Buchbinder, O. (2017). Supporting classroom implementation of proof-oriented tasks: Lessons from teacher researcher collaboration. In T. Dooley & G. Gueudet (Eds.), *Proceedings of the Tenth Congress of the European Society for Research in Mathematics Education (CERME 10, February 1-5, 2017)* (pp. 107–114). Dublin, Ireland: DCU Institute of Education and ERME.

Hirt, U., & Wälti, B. (2008). *Lernumgebungen im Mathematikunterricht: Natürliche Differenzierung für Rechenschwache bis Hochbegabte* [Learning Environments in Mathematics Classes: Natural Differentiation for Mathematically Impaired to Highly Gifted Students]. Seelze, Germany: Kallmeyer.

Hiebert, J., Gallimore, R., Garnier, H., Givvin, K. B., Hollingsworth, H., Jacobs, J., et al. (2003). *Teaching Mathematics in Seven Countries: Results From the TIMMS 1999 Video Study*. Washington, DC: National Centre for Education Statistics.

Kuckartz, U. (2014). *Qualitative Text Analysis: A Guide to Methods, Practice and Using Software*. London, UK: SAGE Publications.

Kultusministerkonferenz (KMK). (2012). Bildungsstandards im Fach Mathematik für die allgemeine Hochschulreife [Educational Standards in Mathematics for Gernal Matriculation Standard]. Retrieved from http://www.kmk.org/fileadmin/Dateien/veroeffentlichungen_beschluesse/2012/2012_10_18-Bildungsstandards-Mathe-Abi.pdf

Mayring, P. (2015). Qualitative Content Analysis: Theoretical Background and Procedures. In A. Bikner-Ahsbahs, C. Knipping, & N. Presmeg (Eds.), *Advances in Mathematics Education. Approaches to Qualitative Research in Mathematics Education: Examples of Methodology and Methods* (pp. 365–380). Dordrecht, Heidelberg, New York, London: Springer.

Meyer, M., & Prediger, S. (2012). Sprachenvielfalt im Mathematikunterricht: Herausforderungen, Chancen und Förderansätze [Language Diversity in Mathematics Classrooms: Challenges, Chances and Promotional Approaches]. *Praxis der Mathematik in der Schule*, *54*(45), 2–9.

National Council of Teachers of Mathematics (NCTM). (2000). Executive Summary: Principles and Standards for School Mathematics. Retrieved from https://www.nctm.org/uploadedFiles/Standards_and_Positions/PSSM_ExecutiveSummary.pdf

Pericleous, M. (2015). How is proving constituted in Cypriot classroom? In K. Krainer & N.'a. Vondrová (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education (CERME9, 4-8 February 2015)* (pp. 226–227). Prague, Czech Republic: Charles University in Prague, Faculty of Education and ERME.

Reiss, K. M., Heinze, A., Renkl, A., & Groß, C. (2008). Reasoning and proof in geometry. Effects of a learning environment based on heuristic worked-out examples. *ZDM Mathematics Education, 40* (3), 455–467.

Scheffler, S. (2017). Argumentieren im Analysisunterricht: Erkenntnisse aus Lehrerinterviews [Argumentation in Calculus Classrooms: Results of an Interview Study with Teachers]. In U. Kortenkamp & A. Kuzle (Eds.), *Beiträge zum Mathematikunterricht 2017* (pp. 833–836).

Münster: WTM-Verlag. Retrieved from https://eldorado.tu-dortmund.de/bitstream/2003/36627/1/BzMU-2017-SCHEFFLER.pdf

Scheffler, S. (2018). Mathematisch Argumentieren im Analysisunterricht [Mathematical Argumentation in Calculus Classrooms]. In Fachgruppe Didaktik der Mathematik der Universität Paderborn (Eds.), *Beiträge zum Mathematikunterricht 2018* (pp. 1571–1574). Münster: WTM-Verlag.