Some Remarks on Spatial Correlation Function Models

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ABSTRACT

The method of optimal interpolation, which is widely used in meteorological data assimilation, relies very much on good approximations of spatial correlation functions. Therefore, many models for such functions have been developed. These models should fulfill certain mathematical constraints; particularly, they should be positive-definite functions. For the classes of homogeneous and isotropic processes, the positivity property and its consequences are reviewed. A special class of correlation models based on so-called spatial autoregressive processes is critically examined. It is shown that models of this type are not positive definite on the meteorological relevant spaces. Some other models taken from the literature are shown to lack this property also. Three strategies to obtain models that have the appropriate mathematical properties are outlined.

1. Introduction

In the last two decades, optimal or statistical interpolation has become a widespread tool in atmospheric data assimilation. Eliassen (1954) introduced this concept in meteorology; a comprehensive presentation of the method is given in Gandin (1965). One of the essential prerequisites to optimal interpolation is an approximate knowledge of spatial correlation or covariance functions. A large number of analytical functions for modeling the spatial correlations have been proposed and investigated (e.g., Buell 1972; Schlatter 1975; Julian and Thiébaux 1975; Thiébaux 1975; Thiébaux 1976; Thiébaux et al. 1986; Mitchell et al. 1990; Bartello and Mitchell 1992). A variant of optimal interpolation is optimal averaging (Vinnikov et al. 1990), a method in which no gridpoint values are approximated but some spatial mean value is estimated.

One of the strict mathematical restrictions on the choice of possible analytical forms of correlation functions is the requirement that such a function must be positive definite (Yaglom 1986) in order that all resulting covariance matrices are positive definite (Stuart and Ord 1987). Additionally, it may be required that their spectral behavior must resemble that found in observations, or that they must be differentiable at the origin and must be able to conform with geostrophy (Julian and Thiébaux 1975; Thiébaux 1975).

In an analogy to temporal autoregressive processes of order N, AR(N), with continuous time, Thiébaux (1976) introduces autoregressive models with distance instead of time as continuous parameter. Thiébaux's spatial AR(2) model is also cited as an example of a model for climatological background errors in Daley (1991). In this paper, some correlation function models—among them the spatial AR(2) model—are critically examined, and the conditions under which they are positive definite on meteorological relevant spaces are determined.

The paper is outlined as follows. Section 2 summarizes general mathematical properties of correlation functions and gives the spectral transformation formulas of isotropic homogeneous correlation functions. Section 3 reviews possible spatial analogs to autoregressive processes in time. Section 4 discusses whether certain correlation models are positive definite on the meteorological relevant spaces. Section 5 describes three feasible ways of obtaining models of correlation functions, and section 6 gives a summary of the paper.

2. Positive-definite functions on various spaces

The theory of isotropic homogeneous correlation functions is discussed in great detail in Yaglom (1986). Here we will review some important results for scalar random fields. For a random field $Y(\mathbf{x})$ defined on the n-dimensional Euclidian space $\mathbb{R}^n$, the correlation function is defined in the usual way as

$$B(\mathbf{x}_j, \mathbf{x}_k) = \frac{\langle [Y(\mathbf{x}_j) - \langle Y(\mathbf{x}_j)\rangle][Y(\mathbf{x}_k) - \langle Y(\mathbf{x}_k)\rangle]\rangle}{\{\langle [Y(\mathbf{x}_j) - \langle Y(\mathbf{x}_j)\rangle]^2\rangle \times \langle [Y(\mathbf{x}_k) - \langle Y(\mathbf{x}_k)\rangle]^2\rangle\}^{1/2}}, \quad (1)$$

where the $\langle \;\; \rangle$ denotes the expectation value and $\mathbf{x}_j$ and $\mathbf{x}_k$ are two points in $\mathbb{R}^n$. In many applications like ones in meteorology, the correlation function is as-

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sumed to be homogeneous, which means that it depends only on the difference vector $\mathbf{x}_j - \mathbf{x}_k$ of the two points $\mathbf{x}_j$ and $\mathbf{x}_k$ (Yaglom 1986, p. 323):

$$B(\mathbf{x}_j, \mathbf{x}_k) = B(\mathbf{x}_j - \mathbf{x}_k). \quad (2)$$

The meteorological fields themselves are in general not homogeneous since the single point variance $\langle [Y(\mathbf{x}_j) - \langle Y(\mathbf{x}_j)\rangle]^2\rangle$ and the covariance function in general depend on the position in space. Only the standardized fields $[Y(\mathbf{x}_j) - \langle Y(\mathbf{x}_j)\rangle]\{\langle [Y(\mathbf{x}_j) - \langle Y(\mathbf{x}_j)\rangle]^2\rangle\}^{-1/2}$ are often assumed to be homogeneous on horizontal or quasi-horizontal surfaces.

For a given function $B(\mathbf{x})$ (where $\mathbf{x} = \mathbf{x}_j - \mathbf{x}_k$ denotes the difference vector between two points) to be considered as a homogeneous correlation function of some random field, the function $B(\mathbf{x})$ must be positive definite. This means that for any integer m, any set of points $\mathbf{x}_1, \ldots, \mathbf{x}_m$, and any complex constants $c_1, \ldots, c_m$, the following sum must be nonnegative (Yaglom 1986, p. 327):

$$\sum_{j,k=1}^{m} B(\mathbf{x}_j - \mathbf{x}_k) c_j \bar{c}_k \geqslant 0, \quad (3)$$

where $\bar{c}_k$ denotes the complex conjugate of c_k. The positive definiteness is an immediate consequence of the definition of the correlation function for homogeneous random fields. It must not be confused with the ordinary positivity property of a function to take only positive values. Obviously, the latter is in general not true for a correlation function, and not every function taking only positive values is positive definite and can therefore be considered a correlation function.

For the property of a function to be positive definite, its Fourier transform must be a positive function in the ordinary sense, and vice versa. Strictly speaking, the Fourier transform of a homogeneous correlation function does not need to be an ordinary function, but can always be represented as a nonnegative spectral measure on the space of wave vectors. This is the essential content of Bochner's theorem (Yaglom 1986, p. 329). It generalizes the well-known Wiener Khinchin theorem from stationary temporal to homogeneous spatial processes.

Many of the correlation functions used in meteorology are furthermore assumed to be isotropic in horizontal or quasi-horizontal surfaces, with the consequence that the correlation function depends only on the Euclidian distance x between the points $\mathbf{x}_j$ and $\mathbf{x}_k$. For isotropic homogeneous random fields, the theoretical framework is much better known than for general homogeneous random fields. As there is no physical reason for meteorological fields to be homogeneous and isotropic on the globe, these assumptions may seem too restrictive. However, one should keep in mind that optimal interpolation is usually done with data from a small neighborhood of grid points, and optimal averaging is done within latitude belts and not on the whole globe. Within these smaller domains the observed correlation functions are in general reasonably homogeneous and isotropic on horizontal or quasi-horizontal surfaces. Deviations from isotropy are discussed; for example, in Thiébaux (1976).

For isotropic homogeneous fields on $\mathbb{R}^n$, the Fourier representation of the correlation function simplifies considerably. Provided the correlation function $B(x)$ decays rapidly enough for large distances x (Yaglom 1986, p. 357), its spectral representation reads

$$B(x) = (2\pi)^{n/2} \int_0^\infty \frac{J_{(n-2)/2}(kx)}{(kx)^{(n-2)/2}} k^{n-1} f(k) dk, \quad (4)$$

where J_ν is a Bessel function of the first kind and the spectral density $f(k)$ is a nonnegative function $f(k) \geqslant 0$ for all scalar wavenumbers $k = (k_1^2 + \cdots + k_n^2)^{1/2} \geqslant 0$. Conversely, $f(k)$ follows from $B(x)$ according to

$$f(k) = \frac{1}{(2\pi)^{n/2}} \int_0^\infty \frac{J_{(n-2)/2}(kx)}{(kx)^{(n-2)/2}} x^{n-1} B(x) dx. \quad (5)$$

Though the correlation function and the corresponding spectral density of isotropic homogeneous random fields depend only on a scalar variable, namely, the Euclidian distance x and the scalar wavenumber k, respectively, the relations (4) and (5) connecting these functions depend on the dimension n of the space. As a consequence of this dependence, one finds that a positive-definite function $B(x)$ on $\mathbb{R}^n$ is also positive definite on $\mathbb{R}^{n-1}$, but not vice versa (Yaglom 1986, p. 354).

Isotropic and homogeneous random fields may be defined on more general manifolds than the Euclidian spaces $\mathbb{R}^n$ (Yadrenko 1983). For the sphere $\mathbb{S}^{n-1}$ embedded in the Euclidian space $\mathbb{R}^n$, there is a general representation of isotropic homogeneous correlation functions due to Schoenberg (1942):

$$B(\theta) = \sum_{m=0}^{\infty} f_m C_m^{(n-2)/2}(\cos\theta), \quad (6)$$

where the $C_m^\nu(x)$ are Gegenbauer's polynomials, the f_m are nonnegative constants satisfying $\sum_{m=0}^{\infty} f_m < \infty$, and θ is the spherical distance (or great circle distance) between two points on the sphere. If the function $B_1(x)$ is positive definite on the Euclidian space $\mathbb{R}^n$, then the function $B(\theta) = B_1[2a_e \sin(\theta/2)]$ is positive definite on the sphere $\mathbb{S}^{n-1}$ embedded in $\mathbb{R}^n$, where a_e denotes the radius of the sphere (Yaglom 1986, p. 389). In particular, if a function is positive definite on $\mathbb{R}^3$, it is also positive definite on $\mathbb{R}^2$ and $\mathbb{S}^2$ [with the correct substitution of the Euclidian distance by the spherical distance by means of $x = 2a_e \sin(\theta/2)$]. For the Euclidian spaces $\mathbb{R}^2$ and $\mathbb{R}^3$, the corresponding spectral transformation formulas (4) and (5) take the form

$$B(x) = 2\pi \int_0^\infty J_0(kx)kf(k)dk,$$

$$f(k) = \frac{1}{2\pi} \int_0^\infty J_0(kx)xB(x)dx \qquad (7)$$

for the plane $\mathbb{R}^2$, and

$$B(x) = 4\pi \int_0^\infty \frac{\sin(kx)}{x} kf(k)dk,$$

$$f(k) = \frac{1}{2\pi^2} \int_0^\infty \frac{\sin(kx)}{k} xB(x)dx \qquad (8)$$

for the three-dimensional space $\mathbb{R}^3$. On the usual sphere $\mathbb{S}^2$ the spectral representation (6) is given by

$$B(\theta) = \sum_{m=0}^{\infty} f_m P_m(\cos\theta),$$

$$f_m = \frac{2m+1}{2} \int_0^\pi P_m(\cos\theta)B(\theta)\sin\theta d\theta, \qquad (9)$$

where the $P_m(\cos\theta)$ are the Legendre polynomials. Recall that for the Euclidian spaces $\mathbb{R}^2$ and $\mathbb{R}^3$ it is necessary but not sufficient that the correlation function model is positive definite on the line $\mathbb{R}^1$.

Although each model of an isotropic homogeneous correlation function will depend only on a single variable, namely, an appropriate distance in the space where it is defined, its spectral representation and the answer to the question whether the function is positive definite depend very much on the underlying space. In other words, the same function may represent a proper correlation in one space but may fail to do so in some other space.

3. Spatial autoregressive processes

In Thiébaux (1976), correlation functions of continuous AR(1) and AR(2) processes in time are used as spatial correlation function models. The time parameter is identified as spatial distance with the intention to get models of isotropic homogeneous correlation functions. In Thiébaux et al. (1986) a similar AR(3) process was used to obtain spatial correlation functions; see also Thiébaux and Pedder (1987). The isotropic correlation function of the AR(2) process given in Thiébaux (1976) has the form

$$B(x) = \left[\cos(ax) + \frac{c}{a}\sin(ax)\right]\exp(-cx), \qquad \text{for } c > 0, \qquad (10)$$

where x is the great-circle distance between two points on the sphere, measured in units of 1000 km.

The simple analogy of processes in time and in space, which was used to derive the model (10), needs some comment. As clearly outlined in Yaglom (1987, p. 325), there are significant differences between a time variable and a spatial variable. Whereas for a time variable a distinction between past and future may be drawn, no such distinction is meaningful for spatial variables. A function in time that describes a process in nature is known to depend only on the past but not on the future (causality). However, this notion of causality has no analog in space. A function describing a process in one-dimensional space can depend on values from the left and right of a reference point. In higher dimensions even more freedom exists for the locations having influence on the value at a reference point.

As an example, a two-dimensional spatial generalization of a continuous autoregressive process in time may be described by a differential equation of the type

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - a^2\right)Y(x_1, x_2) = E(x_1, x_2), \qquad (11)$$

with spatial white noise $E(x_1, x_2)$ (Yaglom 1987, p. 334). In a discretized version of (11) all nearest neighbors of a given point x_1, x_2 contribute to the value of the random field at that point. They are known as nearest neighbor models in the literature. Model (11) is treated in Whittle (1954); more complicated models are investigated in Heine (1955). The correlation function of the random field $Y(x_1, x_2)$, which is a solution of (11), is given by $B(x) = axK_1(ax)$, where K_1 denotes the modified Bessel function of second kind or MacDonald function (Abramowitz and Stegun 1965), and $x = (x_1^2 + x_2^2)^{1/2}$.

Even if the model (10) and other correlation functions of AR(N) processes are derived from a too simple analogy between temporal and spatial stochastic processes, they might still be useful candidates for spatial correlation functions if they fulfill the conditions mentioned in section 2. In the following, it will be shown that the AR(2) correlation function (10) of Thiébaux (1976) is, however, not positive definite on the Euclidian space $\mathbb{R}^2$ and the sphere $\mathbb{S}^2$, at least not for the parameter values given in Thiébaux (1976).

4. Positive definiteness of some correlation models

In Julian and Thiébaux (1975), Thiébaux (1975), and Thiébaux (1976) the one-dimensional spectral transform of correlation models is compared with the one-dimensional spectrum obtained from observations as a possible aid to select the best model. In Thiébaux (1976) the one-dimensional spectral representation of the correlation function of the AR(2) model (10) is given and shown to be nonnegative for all wavenumbers. This is a necessary but not yet sufficient condition for the model (10) to be positive definite on the plane $\mathbb{R}^2$. In the following it shall be investigated whether the AR(2) model is positive definite on the plane $\mathbb{R}^2$ and the sphere $\mathbb{S}^2$ by means of the corresponding spectral transformations.

On the plane $\mathbb{R}^2$ the spectral density (7) of the AR(2) model (10) can be given in closed form (Prudnikov et al. 1992):

$$f(k) = \frac{1}{2\pi} R^{-3}\left[-\left(a + \frac{c^2}{a}\right)\sin\left(\frac{3}{2}\phi\right)\right], \quad (12)$$

with $R^4 = (k^2 - a^2 + c^2)^2 + 4a^2c^2$ and $\tan\phi = (-2ac)/(k^2 - a^2 + c^2)$ with $-\pi < \phi \leqslant \pi$. In Thiébaux (1976) some correlation models for 500-hPa geopotential are given: the parameters of the model (10) are $a = 1.4123$, $c = 0.5666$, and a multiplicative scale parameter $R_0 = 0.9625$. For these parameter values the corresponding two-dimensional spectral density (12) is shown in Fig. 1. For small wavenumbers it is negative; hence, the function (10) with the parameters of Thiébaux (1976) is not positive definite on the plane $\mathbb{R}^2$. It is positive definite on the plane if $a^2 < 3c^2$ holds, as a closer inspection of (12) reveals.

On the sphere $\mathbb{S}^2$, a numerical integration of the spectral representation (9) gives negative expansion coefficients f_m for $m \leqslant 4$; therefore, the AR(2) model (10) is not positive definite on the sphere, at least not for the parameter values given in Thiébaux (1976).

In Table 4.1 of Thiébaux and Pedder (1987) other correlation models for 500-hPa geopotential height are listed. The one-dimensional spectrum was compared with the spectra obtained from observations, and some conditions were imposed on the correlation function in order to make it conform with geostrophy. However, it was not tested whether the functions are positive definite on the plane or the sphere. The second entry of Table 4.1 of Thiébaux and Pedder (1987) gives the correlation model

$$B(x) = [c\cos(ax) + R_0 - c]\exp(-bx^2), \quad (13)$$

with the parameter values $a = 1.252$, $b = 0.151$, $c = 0.778$, and $R_0 = 0.98$. The result of a numerical integration of the two-dimensional spectral transform (7) is shown in Fig. 2. Evidently, the model (13) is not positive definite on the plane $\mathbb{R}^2$ and is therefore not positive definite on any $\mathbb{R}^n$ with $n \geqslant 2$. On the sphere $\mathbb{S}^2$, the spectral expansion (9) was carried out numerically for some orders m, and gave negative expansion coefficients for $m \leqslant 3$. Hence, the correlation model (13) is not positive definite on the Euclidian space $\mathbb{R}^2$ or on the sphere $\mathbb{S}^2$. For both models discussed so far, the spectral transform becomes negative for wavenumbers $k < k_0$ [$k_0 = 0.85$ for the AR(2) model on $\mathbb{R}^2$ and $k_0 = 0.75$ for the second model on $\mathbb{R}^2$]. If the correlation model is used only for distances smaller than $2\pi/k_0$, the resulting correlation matrices are positive definite as required.

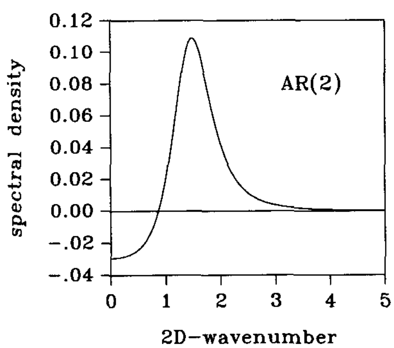

FIG. 1. Spectral density $f(k)$ on the two-dimensional Euclidian space as obtained from the expansion (7) of the AR(2) model (10) as function of the scalar wavenumber $k = (k_1^2 + k_2^2)^{1/2}$ (in units of 0.001 km^{-1}). The parameter values are taken from Thiébaux (1976).

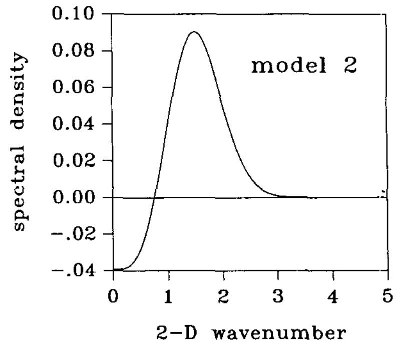

FIG. 2. Spectral density $f(k)$ on the two-dimensional Euclidian space as obtained from the expansion (7) of the correlation model (13) as function of the scalar wavenumber $k = (k_1^2 + k_2^2)^{1/2}$ (in units of 0.001 km^{-1}). The parameter values are taken from Thiébaux and Pedder (1987), Table 4.1, the second model.

A correlation model that is widely used is given by a Gaussian function

$$B(x) = \exp(-ax^2), \quad (14)$$

with $a > 0$; for example, Buell (1972), Schlatter (1975), or Lorenc (1981). It is well known that the function (14) is positive definite on every Euclidian space $\mathbb{R}^n$ (Yaglom 1986). It is also positive definite on the sphere $\mathbb{S}^2$ if the Euclidian distance x is properly replaced by $2a_e\sin(\theta/2)$, leading to

$$B(\theta) = \exp[-4\tilde{a}\sin^2(\theta/2)], \quad (15)$$

where the new constant is $\tilde{a} = aa_e^2$. For small θ this correlation model (15) becomes

$$B(\theta) = \exp(-\tilde{a}\theta^2); \quad (16)$$

the function that is actually used; for example, in Schlatter (1975), where $\theta < 0.22$, and in Lorenc (1981), where $\theta < 0.1$. It is interesting to know whether this correlation model (16) is positive definite on the whole sphere. The spectral expansion (9) cannot be carried out in closed form, but some approximations are possible. For large $\tilde{a}$, $\tilde{a} \gg 1$, the spectral expansion coefficients (9) are obtained as (see the Appendix)

$$f_m \approx \frac{2m+1}{2}\left[\frac{1}{2\tilde{a}} - \frac{m(m+1)}{8\tilde{a}^2}\right]. \quad (17)$$

The spectral coefficients are in fact nonnegative as long as $\tilde{a} > m(m+1)/4$ holds; however, if this condition is violated, the approximation (17) is no longer valid. Then both $\tilde{a}$ and the order m are large and the Legendre polynomial may be approximated by a Bessel function, resulting in the spectral coefficients result as (see the Appendix)

$$f_m \approx \frac{2m+1}{4\tilde{a}} \exp\left(-\frac{m^2}{4\tilde{a}}\right), \qquad (18)$$

which are always nonnegative. Hence, for large $\tilde{a}$ the spectral coefficients f_m are always nonnegative, and the correlation model (16) is therefore positive definite on the sphere. For small $\tilde{a}$ the spectral coefficient f_2 is obtained as $f_2 \approx -10\tilde{a}/9$ (see the Appendix) and is thus negative for all positive values of $\tilde{a}$. Numerical integration gives negative spectral coefficients for values of $\tilde{a} < 1$. In correlation models of geopotential height (Schlatter 1975; Lorenc 1981; Thiébaux and Pedder 1987) the values of $\tilde{a}$ range from 25 to 82; in this parameter range the correlation model (16) can be assumed to be positive definite from the results discussed already. For smaller values of $\tilde{a}$ the model (15) should be used on the sphere. It is known to be positive definite on the whole sphere for all nonnegative parameter values.

5. Some strategies to obtain positive-definite functions

Three strategies to get positive-definite functions as possible correlation models on a given space are outlined in the following.

First, one may take one of the parametric models that are known to be positive definite on a given space. In Yaglom (1986) and Yadrenko (1983) such correlation function models for the Euclidian spaces and the sphere $\mathbb{S}^2$ are given. In Table 1 some functions that are positive definite on the Euclidian space $\mathbb{R}^2$ are listed. In Table 2 some functions that are positive definite on the sphere $\mathbb{S}^2$ are given. All the functions listed in Table 1 are also positive definite on $\mathbb{R}^3$ except for the second last model, for which a stronger constraint $\alpha \geqslant \sqrt{3}\omega_0$ holds, and the last model, for which $\nu \geqslant 1/2$ in $\mathbb{R}^3$. As already mentioned, all positive-definite functions on $\mathbb{R}^3$ are also positive definite on the sphere $\mathbb{S}^2$, if the Euclidian distance x is properly replaced by the spherical distance θ (great circle distance): $x = 2a_e \sin(\theta/2)$ (Yaglom 1986, p. 389).

TABLE 1. List of positive-definite functions on the Euclidian space $\mathbb{R}^2$ together with constraints on the parameters from Yaglom (1986). The $K_\nu(x)$ denote modified Bessel functions of the third kind (MacDonald functions); J_ν are Bessel functions of the first kind.

Model on $\mathbb{R}^2$	Constraints
$\exp(-\alpha x)$	$\alpha > 0$
$(\alpha x)^\nu K_\nu(\alpha x)$	$\alpha > 0,\ \nu \geqslant 0$
$\exp(-\alpha x^2)$	$\alpha > 0$
$\exp(-\alpha x^m)$	$\alpha > 0,\ 0 < m \leqslant 2$
$1/(\alpha^2 + x^2)^\nu$	$\alpha > 0,\ \nu > 0$
$\exp(-\alpha x)\cos(\omega_0 x)$	$\omega_0 > 0,\ \alpha \geqslant \omega_0$
$(\alpha x)^{-\nu} J_\nu(\alpha x)$	$\alpha > 0,\ \nu \geqslant 0$

TABLE 2. List of positive-definite functions on the sphere $\mathbb{S}^2$ together with constraints on the parameters from Yadrenko (1983). The $P_m(x)$ are Legendre polynomials.

Model on $\mathbb{S}^2$	Constraints
$B(\theta) = (1 - 2q\cos\theta + q^2)^{-1/2}$	$0 < q < 1$
$B(\theta) = \frac{1}{4\pi}\frac{1-q^2}{[1 - 2q\cos(\theta) + q^2]^{3/2}}$	$0 < q < 1$
$B(\theta) = P_m(\cos\theta)$	

Any linear combination of positive-definite functions with nonnegative constants is also a positive-definite function (Yaglom 1986, p. 58). These parametric models have some disadvantages, however. Most of them do not change sign, whereas observed correlation functions, especially the ones of climatological background errors, often do so; for example, Schlatter (1975) and Julian and Thiébaux (1975). Furthermore these models have only a few free parameters, and are therefore not sufficiently flexible to fit arbitrary observed correlations. Some of the listed models are positive definite only if the parameters fulfill certain conditions, which means that they require the use of fit procedures with restrictions.

A second strategy may be to expand the observed correlation function into a set of spectral basis functions (4) or (6); for example, Rutherford (1972), Hollingsworth and Lönnberg (1986), Lönnberg and Hollingsworth (1986), and Bartello and Mitchell (1992). The resulting expansion coefficients should be nonnegative, but "due to scatter in the data," as stated in Bartello and Mitchell (1992), sometimes negative values may occur with negligible small amplitudes.

The last and very pragmatic method consists of simply taking some analytical functions that fit the observed correlations well, but for which the positive definiteness is not proven (e.g., Buell 1972) for a list of candidate functions. Then it may be tested numerically whether they are—with the parameter values as obtained from the fit—positive definite on the appropriate space like $\mathbb{R}^2$ or $\mathbb{S}^2$ and therefore positive definite on any subset of the corresponding space, too. Alternatively, one can determine the eigenvalues of each resulting covariance matrix and see whether they are nonnegative (Lorenc 1981; Weber 1992); this procedure gives additional information about possible algorithmical instabilities (Lorenc 1981) due to eigenvalues very close to zero.

6. Conclusions

The important property of positive definiteness for a correlation function model was reviewed. A method of testing this property by means of spectral transformation on the underlying spaces was discussed. It was outlined that the spatial AR(2) model of Thiébaux (1976) was derived from a too simple analogy between temporal and spatial processes. As a consequence, the correlation function of the spatial AR(2) model with parameters as given in Thiébaux (1976) is not positive definite on spaces relevant for meteorological and other geophysical applications. Nonetheless, the model may still be useful; given some array of stations and grid points, it may still yield only positive-definite covariance matrices. The reduced model (Thiébaux et al. 1986; Mitchell et al. 1990) with $a \to 0$, especially, may still be practical in applications. But the positive definiteness should, at least numerically, be tested. This can be done either with the appropriate spectral representation on the respective spaces or with direct calculation of the eigenvalues of the covariance matrix, which is obtained by the restriction of the covariance function on the array of observation locations or on a discrete lattice in space.

Two other correlation models were analyzed and seen to be positive definite only for a restricted range of parameter values. In summary, one finds that there is still a lack of flexible correlation models on the Euclidian two-dimensional space or on the sphere, or even for limited regions of these spaces.

More detailed models that determine the random fields themselves, such as the one given by (11), for example, provide still another approach for obtaining mathematically consistent correlation functions. Further, they allow simulations of these fields and provide the basis for a variety of statistical tests.

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APPENDIX

Spectral Transform of the Gaussian Model on the Sphere

Approximations of the spectral transform (9) of the correlation model (16) on the sphere $\mathbb{S}^2$ for both large and small $\tilde{a}$ are presented in the following. The integrals to solve are given by

$$f_m = \frac{2m+1}{2} \int_0^\pi e^{-\tilde{a}\theta^2} P_m(\cos\theta) \sin\theta d\theta. \quad \text{(A1)}$$

For $\tilde{a} \gg 1$, only contributions from small θ are important and the integrand is dominated by the exponential decay as long as m is not too large. Using the sum representation of the Legendre polynomials (Abramowitz and Stegun 1965, p. 775),

$$P_m(x) = \sum_{k=0}^{[m/2]} c_k x^{m-2k},$$

$$\text{with} \quad c_k = \frac{1}{2^m}(-1)^k \binom{m}{k}\binom{2m-2k}{m}, \quad \text{(A2)}$$

expanding $x^{m-2k} = \cos^{m-k}\theta \approx 1 - (m-2k)\theta^2/2$, and extending the upper limit of integration to infinity, the spectral coefficients become

$$f_m \approx \frac{2m+1}{2} \sum_{k=0}^{[m/2]} c_k \left[\frac{1}{2\tilde{a}} - \frac{1}{2\tilde{a}^2}\frac{m-2k}{2}\right]. \quad \text{(A3)}$$

The first term of the right-hand side of (A3) can be summed up by using $\sum_{k=0}^{[m/2]} c_k = P_m(1) = 1$; the second sum can be found in Prudnikov et al. (1986, p. 632), resulting in

$$f_m \approx \frac{2m+1}{4\tilde{a}}\left[1 - \frac{m(m+1)}{4\tilde{a}}\right]. \quad \text{(A4)}$$

If both $\tilde{a}$ and m are large, the decay of the integrand in (A1) is dominated by the behavior of the Legendre polynomial, and not the exponential, as assumed above. For large m and small θ the Legendre polynomial can be approximated by $P_m(\cos\theta) \approx J_0(m\theta)$ (Abramowitz and Stegun 1965, p. 787), where J_0 is a Bessel function of the first kind. The spectral coefficients are given in this limit by

$$f_m \approx \frac{2m+1}{2} \int_0^\infty \exp(-\tilde{a}\theta^2) J_0(m\theta)\theta d\theta$$

$$= \frac{2m+1}{4\tilde{a}} \exp\left(-\frac{m^2}{4\tilde{a}}\right) \quad \text{(A5)}$$

(Gradshteyn and Rizhik 1980, p. 717), and are thus always positive.

For small $\tilde{a}$ the exponential function in the integrand of (A1) can be expanded as $\exp(-\tilde{a}\theta^2) \approx 1 - \tilde{a}\theta^2$. For the lowest values of m the resulting integrals give

$$f_0 \approx 1 + \left(2 - \frac{\pi^2}{2}\right)\tilde{a}, \quad f_1 \approx \frac{3}{8}\pi^2\tilde{a}, \quad f_2 \approx -\frac{10}{9}\tilde{a} \quad \text{(A6)}$$

for the spectral coefficients.

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