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Financial modelling applying multivariate Lévy processes: New insights into estimation and simulation

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1. Introduction

Lévy processes, like the Generalized Hyperbolic (GH) process, the Normal Inverse Gaussian (NIG) process, or the Variance Gamma (VG) process are common and well-known methods for describing the behaviour of daily or intra-day stock returns, and are therefore an important part in derivative pricing and hedging (see [3–5]). The process types are well-covered in academic literature: the GH process (see [6–10]) as well as its two special cases the NIG process (see [11–14]) and the VG process (see [15–19]). All these univariate models covering only asset-specific behaviour have been applied to model single-asset returns and have been investigated extensively in the aforementioned literature concerning their modelling properties and fitting procedures.

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However, Harris [20] demonstrates that joint distributions for cross-sectional analyses are necessary to cover information arrival rates, which affect securities in different ways. Fama and French [21] identify a combination of a market factor and the usual proxies, such as size and book-to-market equity as a valid explanatory approach for cross-section of stock returns. Lin et al. [22] show that information traded in one market impacts returns of assets traded on another later-opening market. This means returns could be affected by market contagion. In their empirical study, Lo and Wang [23] find cross-sectional or components of common behaviour in trading various financial assets. All these findings indicate the requirements of a good framework to mathematically model the behaviour of asset prices. First, the model must be multivariate and cover a specific asset (aka idiosyncratic) component as well as a common market component. Second, it must simultaneously model in financial markets the often occurring fat tails for these two parts of asset components. Lévy processes perfectly fit these needs, as each of these two components can be modelled as a Lévy process, the sum of these two components is again a Lévy process, and they are ideal for handling fat tails. The accurate handling of the multivariate component is particularly essential in calculating the price of multivariate options.

Barndorff-Nielson et al. [24] develop a theoretical framework for a multivariate subordination approach for Lévy processes by taking into consideration specifications of an asset pricing model, as described above. For a basic overview of applying subordination and therewith associated time changes to Lévy processes, see e.g. [25,26], for an use in pricing average options see [27], in a multivariate framework for FX options see [28], and for a latest application of decoupled time-changed Lévy processes see [29]. Based on insights of earlier multivariate VG models by Madan and Seneta [16], Cont and Tankov [30], and Luciano and Schoutens [31], Semeraro [32] develops a multivariate VG model (so-called αVG model) through the subordination of a multivariate Brownian motion with independent multivariate gamma components. Fajardo and Farias [33] use concepts of a multivariate affine GH model from [7,34] for pricing multidimensional derivatives while Guillaume [35] applies a multivariate Sato model for pricing multivariate options. In effect, Guillaume [35] describes log-returns as a Sato time-changed Brownian motion, and the time change itself as a weighted sum of common and idiosyncratic components. Luciano and Semeraro [1,2] extend the insights gained from the use of the αVG model to the NIG, the Poisson, the CGMY, and the GH processes by additionally using a correlated Brownian motion for handling the correlation structure. They also calibrate respectively to a portfolio of international stocks and some major stock market indices. For an extension concerning the constraints of the αVG model see [36]. Luciano et al. [37] construct a framework for multivariate Lévy processes with an idiosyncratic asset and a common market parameter component for the GH, NIG, and VG processes. They model the so-called $\alpha\rho$GH, $\alpha\rho$NIG, and $\alpha\rho$VG processes as factor-based multivariate time-changed Brownian motions and test the fitting properties as well as in particular the dependence structure with an empirical data set. Finally, they identify the $\alpha\rho$GH model as the best multivariate model for their MSCI US Equity data set. Thereby, they demonstrate that the application of multivariate Lévy models works well. However, so far there is no evidence as to what is the best calibration procedure for these multivariate models nor on what is the best multivariate model, other than the one provided based on historical data in [1,2,37].

While a large set of literature exists on calibration of univariate Lévy processes (for an overview see [38,39]), detailed research on multivariate models is scarce. Research on fitting multivariate Lévy processes is limited to empirical data and the use of a single estimation method (see [1,2,37,40,41]). Based on a large simulation study, we close the literature gap on the best estimation method for multivariate Lévy models with a two-step estimation approach, as we can identify the best multivariate Lévy model concerning the modelling properties of idiosyncratic and market components. In contrast to previous literature we use several fitting methods, which can be directly derived from the properties of the Lévy process itself, and create a simulation setting for the $\alpha\rho$GH, $\alpha\rho$NIG, and $\alpha\rho$VG processes, within high-correlation and no-correlation samples. Seneta [42], Finlay and Seneta [38], or Rathgeber et al. [39] use various methods for fitting the univariate VG process, e.g. the simplified method of moments (SMoM), the method of moments (MoM), the maximum likelihood estimation (MLE), the empirical characteristic function (ECF) method, or the minimum χ^2 (χ^2) method. They determine that the MLE and the χ^2 methods work best for both simulated and empirical data. We apply all these methods for estimating the idiosyncratic components in a first step, and use the root-mean-square error (RMSE) to obtain common parameters affecting the correlation in a second step. Our simulation procedure has no direct say on the fit of Lévy models to financial market data, especially their linear and non-linear correlation, but it has some useful features compared to relying on pure empirical data. First, in contrast to the historical estimation and single method fitting by Luciano et al. [37], we know the exact ex-ante distribution, apply a broad range of fitting methods suggested in literature, and make clear statements on how the estimated parameters deviate from the original parameters. Second, our approach thereby allows analysis of the strengths and weaknesses of the respective models. As an overall result of this study and in line with Luciano et al. [37], we identify the $\alpha\rho$GH model as the best approach to handle correlations in a multivariate Lévy model. It performs best in the high as well as in the no-correlation scenario, while other models – especially the $\alpha\rho$NIG – have some weaknesses. Moreover, the MLE and χ^2 methods are the best estimation procedures. Accordingly, these two findings should be considered in multivariate option pricing and portfolio and risk management. Both as concerns the best ways to fit Lévy processes and as concerns the best fitting model among several Lévy alternatives, this work reaches the same conclusions of Luciano and Semeraro [1,2] and Luciano et al. [37]. This does not come as a surprise, because, with respect to the fit provided on single returns and their correlation matrix, examined here, the current paper adopts the same comparison criteria among alternative Lévy processes (Kolmogorov–Smirnov and Anderson–Darling for marginal distributions, RMSE and maximum absolute error between the model and data correlations) as Luciano et al. [37]. In that sense, it is a kind of empirical robustness check of Luciano and Semeraro [1,2], and part of Luciano et al. [37]. The main difference is that here we use simulated instead of real-market data. While we test more estimation methods than [1,2,37], we do not provide a measure of the ability of the Lévy models under scrutiny to fit non-linear dependence. Luciano et al. [37], to examine that ability, study the fit of the distribution of long only and long-short random portfolio returns. The distribution of portfolios' returns, which are sums of the returns of the single assets, indeed captures the non-linear dependence, so important in a Lévy context.

The remainder of this paper is structured as follows. The next section provides an overview of the theoretical background of multivariate Lévy processes. Section 3 introduces the simulation design and describes the simulation approach for multivariate Lévy processes as well as the various estimation methods for the process and the correlation parameters. Section 4 presents the simulation results and compares the high correlation sample to the non-correlated approach. Finally, Section 5 concludes the paper.

2. Theoretical background

2.1. Multivariate Lévy processes

Following [1,2,37], we define the $\mathbb{R}^{n_A}$ valued time-changed process $(Y_t)_{t\geq 0}$, consisting of an idiosyncratic component Y^I and a common respectively (resp.) systematic component Y^ρ as

$$Y_t = \begin{pmatrix} Y^I_{1,t} + Y^\rho_{1_t} \\ \dots \\ Y^I_{n_A,t} + Y^\rho_{n_A,t} \end{pmatrix} = \begin{pmatrix} B^I_1(X_{IC,1,t}) + B^\rho_1(X_{MC,t}) \\ \dots \\ B^I_{n_A}(X_{IC,n_A,t}) + B^\rho_{n_A}(X_{MC,t}) \end{pmatrix}. \tag{1}$$

In general B is a factor-based time-changed Brownian motion, $X_{IC,j}$ ($j = 1, \dots, n_A$) and X_{MC} independent subordinators, which are responsible for the time change and are themselves again independent from B. n_A represents the number of assets in the multivariate framework. $X_{IC,j}$ drives the idiosyncratic component and X_{MC} the systematic component. Consequently, Luciano et al. [37] define Y as a time-changed multi-parameter process[1] B^I and a time-changed correlated multidimensional Brownian motion B^ρ. While B^I has independent time changes, B^ρ has a unique time change representing the systematic component. All in all, the construction by Luciano et al. [37] implies that the process j consists in a multivariate Lévy process model, of the sum of the individual idiosyncratic component and a common market component. The addition of the two components is possible due to the additivity of independent Lévy processes (see e.g. [43]). For a general overview on the corresponding Lévy triplets, see [37]. Taking the Lévy process $(G_t)_{t\geq 0}$ for assistance

$$G_t = (X_{IC,1,t} + \alpha_1 X_{MC,t}, \dots, X_{IC,n_A,t} + \alpha_{n_A} X_{MC,t}) \qquad \alpha_j > 0, \qquad j = 1, \dots, n_A, \tag{2}$$

the equality in law $\mathcal{L}$ holds, according to [2] (see theorem 5.1) and [37]. The marginal laws of Y result in

$$\begin{aligned} \mathcal{L}(Y_{j,t}) =& \mathcal{L}\left(\mu_j G_{j,t} + \sigma_j W(G_{j,t})\right) \\ \mathcal{L}(Y_{j,t}) =& \mathcal{L}\left(\mu_j X_{IC,j,t} + \sigma_j W_j(X_{IC,j,t}) + \mu_j \alpha_j X_{MC,t} + \sigma_j W^\rho_j(\alpha_j X_{MC,t})\right), \end{aligned} \tag{3}$$

where W^ρ is a standard correlated Brownian motion. Note that α_j works as a weighting factor for the multivariate component. Accordingly, assets with a high α_j are more affected by the market component. Furthermore, it can be seen in Eq. (3) that the first two summands are driven by the stochastic clock (subordinator) with idiosyncratic components and the last two summands by a stochastic clock with the market components.

To estimate the correlation structure $\rho_Y(\mathtt{i}, j)$ between the assets $\mathtt{i}$ and j ($\mathtt{i} = 1, \dots, n_A; j = 1, \dots, n_A$), we use the linear correlation approach

$$\rho_Y(\mathtt{i}, j) = \frac{\mu_\mathtt{i}\mu_j\alpha_\mathtt{i}\alpha_j Var(X_{MC}) + \rho_{\mathtt{i},j}\sigma_\mathtt{i}\sigma_j\sqrt{\alpha_\mathtt{i}}\sqrt{\alpha_j}E(X_{MC})}{\sqrt{Var(Y_\mathtt{i})Var(Y_j)}}, \tag{4}$$

as presented in [30] and adapted by Luciano et al. [37]. Cont and Tankov [30] state that in the most likely case of non-symmetric returns, the covariance of the returns does not only depend on the correlation of the Brownian motion ρ, but also on $\mu_\mathtt{i}\mu_j\alpha_\mathtt{i}\alpha_j Var(X_{MC})$. For a discussion of this correlation approach see e.g. [1,2]. In general, this representation of correlation structure is only possible as the Lévy processes are modelled as time-changed Brownian motions, where the basic processes follow a normal distribution and fulfil the requirements of Bravais–Pearson (e.g. interval-scaled and normally distributed marginal distributions). Otherwise, more complex concepts like copulas are necessary to describe the dependence structure. For details, refer to e.g. [44].

From a financial point of view and in light of the insights into market components by Harris [20], Fama and French [21] and Lin et al. [22], or Lo and Wang [23], this theoretical multivariate Lévy process setting by Luciano and Semeraro [1,2] and Luciano et al. [37] meets all prerequisites (jumps, market effects) for describing the behaviour of asset returns. For further and deeper information about the $\alpha\rho$ models, we also refer to these three papers. For the already well-known and well-discussed characteristics of the univariate processes (moments, probability density function, characteristic function), see Table 12 in the Appendix A and the presented literature in the first row of the table.

1 For a detailed overview of the multi-parameter Brownian motion see [24].

Table 1
Correlation matrix ρ for generating correlated normal random variables.

	1	2	3	4	5	6	7	8	9	10
1	1									
2	0.8014	1								
3	0.6623	0.8858	1							
4	0.6813	0.7007	0.8174	1						
5	0.7021	0.6713	0.6036	0.7402	1					
6	0.5970	0.7543	0.7631	0.8593	0.8274	1				
7	0.7040	0.9173	0.8935	0.6267	0.6379	0.7180	1			
8	0.8067	0.8510	0.7439	0.7730	0.7740	0.8379	0.7760	1		
9	0.6934	0.8223	0.7202	0.8338	0.8044	0.9328	0.6673	0.8388	1	
10	0.7631	0.9488	0.8237	0.7039	0.6988	0.7601	0.8155	0.8342	0.8530	1

This table presents the matrix ρ for generating correlated normal random variables within the high-correlation scenario.

2.2. The Generalized Hyperbolic process

Luciano and Semeraro [1], Luciano et al. [37] specify Y as a factor-based time changed process in the GH setting as a $\alpha\rho$GH process, denoted with the marginal parameters λ, δ_j, β_j, γ_j and the common market parameter m_p as

$$\mathcal{L}(Y_{j,t}) = \mathcal{L}(\beta_j G_{j,t} + W(G_{j,t})). \tag{5}$$

G is the Generalized Inverse Gaussian (GIG) subordinator and represents the marginal distributions for $X_{IC,j}$ resp. X_{MC} according to Luciano et al. [37] with

$$\begin{gathered} X_{IC,j} = X_{IC,j}^{GIG} + X_{IC,j}^{\Gamma}, \qquad \mathcal{L}(X_{IC,j}^{GIG}) = GIG\left(-\lambda, \delta_j, \frac{1}{\sqrt{\alpha_j}}\right), \qquad \mathcal{L}(X_{IC,j}^{\Gamma}) = \Gamma\left(\lambda - m_p, \frac{1}{2\alpha_j}\right), \\ \mathcal{L}(X_{MC}) = \Gamma\left(m_p, \frac{1}{2}\right), \qquad \mathcal{L}(X_{IC,j} + \alpha_j X_{MC}) = GIG\left(\lambda, \delta_j, \frac{1}{\sqrt{\alpha_j}}\right). \end{gathered} \tag{6}$$

For a detailed overview on the constraints of the parameters, see [37]. α_j is defined as $\sqrt{\alpha_j} = \sqrt{1/(\gamma_j^2 - \beta_j^2)}$ and it holds that $0 < m_p < \lambda$. To calibrate the $\alpha\rho$GH dependent structure between asset $\mathtt{i}$ and asset j, we apply the linear correlation

$$\rho_Y(\mathtt{i}, j) = \frac{4\beta_\mathtt{i}\frac{\delta_\mathtt{i}^2}{\zeta_\mathtt{i}^2}\beta_j\frac{\delta_j^2}{\zeta_j^2} + 2\rho_{\mathtt{i},j}\frac{\delta_\mathtt{i}}{\zeta_\mathtt{i}}\frac{\delta_j}{\zeta_j}}{\sqrt{Var(Y_\mathtt{i})Var(Y_j)}} m_p, \tag{7}$$

with

$$Var(Y_\mathtt{i}) = \frac{\delta_\mathtt{i}^2}{\zeta_\mathtt{i}}\frac{K_{\lambda+1}(\zeta_\mathtt{i})}{K_\lambda(\zeta_\mathtt{i})} + \beta_\mathtt{i}^2\frac{\delta_\mathtt{i}^4}{\zeta_\mathtt{i}^2}\left(\frac{K_{\lambda+2}(\zeta_\mathtt{i})}{K_\lambda(\zeta_\mathtt{i})} - \frac{K_{\lambda+1}^2(\zeta_\mathtt{i})}{K_\lambda^2(\zeta_\mathtt{i})}\right)$$

as presented in [37] and with $\zeta_j = \delta_j\sqrt{\gamma_j^2 - \beta_j^2}$. To aid in interpretation of results, it is useful to know that δ drives the scale, β the asymmetry (skewness), and a decreasing ζ resp. γ the fat tails (kurtosis) of the process. It is additionally worth noting that we set $\lambda = 1$ and use the Hyperbolic model in the simulation setting (see [6]).

2.3. The Normal Inverse Gaussian process

Luciano and Semeraro [2], Luciano et al. [37] specify Y as a factor-based time changed process in the NIG setting as a $\alpha\rho$NIG process denoted with the marginal parameters δ_j, β_j, γ_j and the common market parameter m_p as

$$\mathcal{L}(Y_{j,t}) = \mathcal{L}\left(\beta_j\delta_j^2 G_{j,t} + \delta_j W(G_{j,t})\right). \tag{8}$$

G is the Inverse Gaussian (IG) subordinator and represents the marginal distributions for $X_{IC,j}$ resp. X_{MC} according to Luciano et al. [37]

$$\begin{gathered} \mathcal{L}(X_{IC,j}) \sim IG\left(1 - m_p\sqrt{\alpha_j}, \frac{1}{\sqrt{\alpha_j}}\right), \qquad \mathcal{L}(X_{MC}) \sim IG\left(m_p, 1\right), \\ \mathcal{L}(X_{IC,j} + \alpha_j X_{MC}) \sim IG\left(1, \frac{1}{\sqrt{\alpha_j}}\right), \end{gathered} \tag{9}$$

Table 2
Overview estimation methods.

Method	Formula	Description
Simplified method of moments (SMoM)	See notes on Table 12 in the Appendix A	We assume, as suggest by Seneta [42] and Finlay and Seneta [38], a symmetric case of the empirical distribution of the log-returns. Therefore, we can approximate μ resp. $\beta \approx 0$. This restriction implies $\mu^2 = \mu^3 = \mu^4 = 0$ resp. $\beta^2 = \beta^3 = \beta^4 = 0$ and simplifies the moments enormously. By means of the empirical moments M_i^E, with $i = 1, \ldots, 4$, we can directly calculate the parameters of the process.
Method of moments (MoM)	$\hat{\eta} = \underset{\eta}{\operatorname{argmin}} \sum_{i=1}^{4} \left(\frac{M_i^E - M_i(\eta)}{M_i^E} \right)^2$	In accordance with [45], we apply the least-square method and minimise the relative deviation between the i-th sample moment M_i^E, calculated from the empirical data set and the i-th moment M_i, with $i = 1, \ldots, 4$.
Maximum likelihood estimation (MLE)	$\mathcal{LL}\left(\hat{\eta}\right) = \underset{\eta}{\operatorname{argmax}} \sum_{i=1}^{T} \ln\left(f_X(x \mid \eta)\right)$	Aldrich [46] or Seneta [42] use the log-likelihood function $\mathcal{LL}(\eta) = \ln(\mathcal{L}(\eta))$ for the maximisation and estimation of the process parameter set.
Empirical characteristic function (ECF)	$\Phi_{ECF}(\omega) = \frac{1}{T} \sum_{j=1}^{T} e^{i\omega X_j}$ $\hat{\eta} = \underset{\eta}{\operatorname{argmin}} \int_{-\infty}^{+\infty} \mid \Phi_{X,\eta}(\omega) - \Phi_{ECF}(\omega) \mid^2 e^{-\omega^2} d\omega$	Yu [47] matches the characteristic function derived from the Lévy process model $\Phi_{X,\eta}(\omega)$ with the empirical characteristic function obtained from empirical data. We first calculate an estimator for the empirical characteristic function $\Phi_{ECF}(\omega)$ using the T observed log-returns $X = (x_1, \ldots, x_T)$, where $i = \sqrt{-1}$ and ω are the evaluation points and then set up the optimisation problem.
Minimum χ^2 (χ^2)	$\tilde{O}_i(\eta) = T \int_{B_{i-1}}^{B_i} f_{X,\eta}(x) dx$ $\hat{\eta} = \underset{\eta}{\operatorname{argmin}} \sum_{i=1}^{I} \frac{\left(O_i - \tilde{O}_i(\eta)\right)^2}{O_i}$	Berkson [48] and Finlay and Seneta [38] minimise the relative difference between the observed and expected numbers of log-returns in the determined intervals. The basis is T observed log-returns $X = (x_1, \ldots, x_T)$ and the determination of I intervals with a vector $B := (B_i)_{i=0,\ldots,I}$ as right borders of the intervals. The number of the expected observations $\tilde{O}_i(\eta)$ in any interval is dependent on the parameter set η. Finlay and Seneta [38] suggest that the log-returns are divided into 1% sample quantile bands.

This table presents the five estimation methods used in order to calibrate the idiosyncratic components of the Lévy process models. η represents the parameter set of the respective process.

with $0 < m_p < \frac{1}{\sqrt{\alpha_j}}$, $\gamma_j > 0$, $\delta_j > 0$, $-\gamma_j < \beta_j < \gamma_j$, and the definition for α_j as

$$\sqrt{\alpha_j} = \frac{1}{\delta_j\sqrt{\gamma_j^2 - \beta_j^2}}. \tag{10}$$

To calibrate the $\alpha\rho$NIG dependent structure between asset $\mathtt{i}$ and asset j, we apply the linear correlation

$$\rho_Y(\mathtt{i}, j) = \frac{\beta_\mathtt{i}\frac{\delta_\mathtt{i}^2}{\zeta_\mathtt{i}^2}\beta_j\frac{\delta_j^2}{\zeta_j^2} + \rho_{\mathtt{i},j}\frac{\delta_\mathtt{i}}{\zeta_\mathtt{i}}\frac{\delta_j}{\zeta_j}}{\sqrt{\gamma_\mathtt{i}^2\delta_\mathtt{i}(\gamma_\mathtt{i}^2 - \beta_\mathtt{i}^2)^{-\frac{3}{2}}\gamma_j^2\delta_j(\gamma_j^2 - \beta_j^2)^{-\frac{3}{2}}}} m_p, \tag{11}$$

as presented in [37]. Again, it holds, as in the $\alpha\rho$GH model, that $\zeta_j = \delta_j\sqrt{\gamma_j^2 - \beta_j^2}$. For interpretation of the results, it is useful to know that δ drives the scale, β the asymmetry (skewness), and a decreasing ζ resp. γ the fat tails (kurtosis) of the process.

2.4. The Variance Gamma process

Luciano and Semeraro [2], Luciano et al. [37] specify Y as a factor-based time changed process in the VG setting as a $\alpha\rho$VG process, denoted with the marginal parameters $\mu_j, \alpha_j, \sigma_j > 0$ and the common market parameter m_p as

$$\mathcal{L}(Y_{j,t}) = \mathcal{L}(\mu_j G_{j,t} + \sigma_j W(G_{j,t})). \tag{12}$$

As the VG process is driven by μ, we obtain the same formula as in Eq. (3). G is the Gamma subordinator and represents the marginal distributions for $X_{IC,j}$ resp. X_{MC} according to [37] with the Gamma laws

$$\mathcal{L}(X_{IC,j}) = \Gamma\left(\frac{1}{\alpha_j} - m_p, \frac{1}{\alpha_j}\right), \quad \mathcal{L}(X_{MC}) = \Gamma\left(m_p, 1\right), \quad \mathcal{L}(X_{IC,j} + \alpha_j X_{MC}) = \Gamma\left(\frac{1}{\alpha_j}, \frac{1}{\alpha_j}\right), \tag{13}$$

and with the constraint $0 < \alpha_j < \frac{1}{m_p}$.

To calibrate the $\alpha\rho$VG dependent structure between asset $\mathtt{i}$ and asset j, we apply the linear correlation

$$\rho_Y(\mathtt{i}, j) = \frac{\mu_\mathtt{i}\alpha_\mathtt{i}\mu_j\alpha_j + \rho_{\mathtt{i},j}\sigma_\mathtt{i}\sqrt{\alpha_\mathtt{i}}\sigma_j\sqrt{\alpha_j}}{\sqrt{(\sigma_\mathtt{i}^2 + \mu_\mathtt{i}^2\alpha_\mathtt{i})(\sigma_j^2 + \mu_j^2\alpha_j)}} m_p, \tag{14}$$

as presented in [37]. It is obvious that the correlation between assets increases with a rising market parameter m_p and vice versa. If $\rho_{\mathtt{i},j} = 0$, the result is the αVG case as discussed in [32]. Once more, for the interpretation of the results later, it is useful to know that σ drives the scale, μ the asymmetry (skewness), and α the fat tails (kurtosis) of the process.

2.5. The price process model

Finally, we integrate the Lévy processes in the well-known price process setting and define a n_A-dimensional price process $S = \{S_t, t \geq 0\}$

$$S_t = S_0 e^{ct + Y_t}, \tag{15}$$

where c represents the constant drift of the price process. It should be taken into account that c has the same interpretation for all three models later on.

3. Simulation and research design

We use the characteristics of a simulation study to identify the best fitting method, as well as the best model for multivariate Lévy processes. In general, we estimate realistic parameters for each idiosyncratic process component for our simulation study taken from an empirical data set. In the next step, we apply these parameters to a predefined correlation matrix, which remains the same in all three settings, in order to simulate. The fitting quality of the simulations are verified via statistical methods as well as classical distribution tests. In detail, we conduct the following procedure:

1. Overall simulation setting and idiosyncratic parameters:

To test each model, we simulate 1000 runs with a length of $T = 1000$ observations for $n_A = 10$ assets. This is in line with [38]. To acquire realistic idiosyncratic parameters for the simulation of each process, we apply the MSCI US equity indices from 02.01.2009 to 31.05.2013. This data is also used by Luciano et al. [37] in their empirical multivariate Lévy process study. To avoid bias in the subsequent re-estimations of the parameters and preference of a single method, we apply the mean of the empirical estimation of the MLE and χ^2 methods. This choice is based on results of Finlay and Seneta

[38] and Rathgeber et al. [39], who find these methods work best for simulated and empirical data in a univariate case. The original parameters for each process used for the simulation can be seen in Table 3 ($\alpha\rho$GH model), Table 4 ($\alpha\rho$NIG model), and Table 5 ($\alpha\rho$VG model) in the respective parameter row. To make the results more suitable to interpretation, the assets and their corresponding parameters for the simulation are sorted in the following way. We calculate the empirical kurtosis for the MSCI US equity indices and define this order: asset 1 has the highest empirical kurtosis and asset 10 has the lowest one.

2. Market component:
To obtain in-depth insights into the relationship between correlation effects and the common market parameter m_p, we generate two scenarios. First, a high-correlation scenario is created where m_p is specified as 98% of its theoretical maximum value in each model. The theoretical maximum value is a result of the chosen original parameters and the bounds of m_p described for each model in Sections 2.2 to 2.4. We choose 98% in order to have enough "space" to identify possible estimation errors. Furthermore, in this case we predefine a highly correlated positive semi-definite matrix ρ (see Table 1), which is used for the generation of correlated normal random variables resp. the correlated Brownian motion increments. These random variables are an essential part of including common market behaviour in the multivariate Lévy processes. Second, we create a no-correlation scenario by setting $m_p = 0.0001$ and using an identity matrix implying zero correlation for the Brownian components. The no-correlation scenario can be seen as an "univariate case", since the market factor is neglected in this instance. This comparison should aid in an understanding of the strengths and weaknesses of modelling Y as a factor-based time-changed Brownian motion.

3. Simulation algorithm:
Subsequently, we provide a short overview on the simulation algorithm for the price process S_t, which is adapted and expanded by an earlier version of Luciano et al. [37]:

- Generate T random variables $x_{IC,j}$ for $j = 1, \ldots, n_A$ and T random variables x_{MC}, which follow the specified distributions. For Gamma and IG random variables, we use the algorithms suggested by Cont and Tankov [30] and for GIG random variables the non-universal rejection methods proposed by Devroye [49].
- Generate normal $\mathcal{N}(0, 1)$ and normal correlated $\mathcal{N}(0, \rho)$ random variables with length T.
- Calculate the increments $y_{j,t}$ of the multivariate process according to Eq. (3) and finally aggregate them into $Y_{j,t} = Y_{j,t-1} + y_{j,t}$ with $t = 1, \ldots, T$; note: $W(b) \stackrel{d}{=} \sqrt{b}W(1)$.
- Calculate the price process S_t.

4. Re-estimation procedure:
After the simulation of the three multivariate Lévy processes, we conduct a two-step re-estimation procedure. First, we use the five estimation methods shown in Table 2, the SMoM, MoM, MLE, ECF, and χ^2, for re-estimating the idiosyncratic parameters. This re-estimation is equivalent to the estimation procedure in a univariate model. Second, the market parameter m_p as well as the matrix $\rho_Y(\mathtt{i}, j; m_p, \rho_{\mathtt{i},j})$ are obtained via the root-mean-square error (RMSE)

$$RMSE(m_p, \rho) = \sqrt{\frac{2}{n_A(n_A-1)} \sum_{\mathtt{i}=1}^{n_A} \sum_{j>\mathtt{i}} \left(\rho_Y^{emp}(\mathtt{i}, j) - \rho_Y(\mathtt{i}, j; m_p, \rho_{\mathtt{i},j})\right)^2}, \tag{16}$$

where $\rho_Y^{emp}(\mathtt{i}, j)$ represents the correlation within the simulated asset returns. Hereby, we apply the methodology of Higham [50], using weighted Frobenius norms and convex analysis, in order to obtain the nearest correlation matrix of ρ with positive-semidefinite properties. This approach is also suggested by Luciano et al. [37].

5. Evaluation of fitting quality:
After this re-estimation, we use various measures to evaluate the quality of the fittings in line with our two-step estimation approach. It is sufficient to compare the estimated parameters to the original parameters in a simulation study by the relative mean absolute deviation[2] (RMAD)

$$RMAD_i = \frac{1}{n_S} \sum_{j=1}^{n_S} \frac{|\,\widehat{\eta}_{i,j} - \eta_i\,|}{|\,\eta_i\,|}, \tag{17}$$

where $\widehat{\eta}_{i,j}$ is the re-estimated i-th parameter in the parameter set of the respective process in the j-th simulation run, η_i the i-th original parameter in the parameter set, and n_S the number of simulations. The RMADs for all simulated samples and parameters can finally be aggregated to the ARMAD

$$ARMAD = \frac{1}{n_P} \sum_{i=1}^{n_P} RMAD_i, \tag{18}$$

[2] Finlay and Seneta [38] use in their work the mean absolute deviation. However, parameters of the Lévy processes have different orders of magnitude. To increase comparability, we take the relative version.

with n_P as the number of parameters in η. For an application in literature of these two measures, see e.g. [38]. RMAD and ARMAD are used to identify stable models. A lower RMAD resp. ARMAD indicates that the model performs more regular. Moreover, note that to look at the mean of the (relative) deviations in a simulation setting makes only sense to test if the model works and deviations have a mean value of about zero. However, these results do not bring in-depth insights into the behaviour of the models. Moreover, we measure the fitting of the correlation matrices with the RMSE between the empirical and the estimated $\rho\alpha$ model correlations. For a better understanding of fitting errors and their origins, we also calculate the RMSE between the estimated ρ and the original matrix presented in Table 1. Finally, we conduct distribution tests by the Kolmogorov–Smirnov test (KS test) (see [51]) and the Anderson–Darling test (AD test) (see [52,53]).

4. Simulation results

Analysis of the multivariate Lévy processes focuses on the simulation results in two different manners, in line with characteristics of the multivariate Lévy process model and of the simulation design. First, we examine the fitting properties of the idiosyncratic components via the RMAD, ARMAD, and the KS test as well as the AD test. Second, we evaluate the handling of the correlation matrices of the three models via RMSE and the fitting of the market parameter m_p via the RMAD. Based on this procedure, we compare insights of the high and no-correlation scenarios and finally make a decision on which multivariate model is best.

Initially, the RMAD statistics of the three models are analysed. In doing so, we compare the high-correlation scenario (see Tables 3, 4, and 5) with the no-correlation scenario (see tables 13, 14, and 15 in the Appendix B), among others. It is important to note that the assets are sorted according to rules defined in Section 3. Overall, no huge differences appear in the fitting errors between the high-correlation and the no-correlation scenarios for the $\alpha\rho$VG model. This fact implies that this model is able to handle both extreme scenarios. Furthermore, the $\alpha\rho$VG model is able to handle high and low kurtosis samples without significant differences in fitting quality. We see some outliers for the asymmetry parameter μ with regard to assets with a high kurtosis, and some problems with the fitting of a very small drift parameter c. In consideration of the various estimation methods, an outperformance by the MLE and the χ^2 methods is noticeable, as again the MLE fits

Table 3
RMAD results for $\alpha\rho$GH model — high-correlation scenario.

	1	2	3	4	5	6	7	8	9	10
δ										
Parameter	0.0010	0.0027	0.0037	0.0011	0.0045	0.0021	0.0025	0.0016	0.0052	0.0044
SMoM	1,532.23	203.38	136.81	1,071.94	83.57	427.49	168.78	595.00	148.73	200.16
MoM	461.29	85.52	69.14	336.61	53.24	143.87	75.65	191.65	66.10	78.92
MLE	135.80	47.26	44.60	106.96	43.29	62.60	52.49	71.07	47.42	48.61
ECF	880.75	206.00	128.66	677.50	97.99	291.38	194.33	398.26	85.06	114.69
χ^2	126.77	51.60	47.21	103.14	46.00	63.47	56.80	69.92	50.21	51.19
β										
Parameter	0.54	−13.02	−11.21	−5.50	−8.30	−4.95	−12.82	−4.55	−5.28	−6.70
SMoM	279.12	55.57	55.98	56.19	61.21	61.44	56.13	64.84	58.82	55.72
MoM	306.51	53.09	53.63	57.77	61.46	63.74	54.78	68.60	60.56	55.24
MLE	353.06	34.95	39.53	48.89	59.62	61.97	40.80	67.74	56.60	41.41
ECF	1,275.84	86.79	99.14	227.15	142.57	246.31	95.39	274.98	232.88	177.29
χ^2	402.08	39.67	44.18	55.61	64.30	68.37	46.57	76.36	61.76	46.54
γ										
Parameter	68.86	143.99	135.77	91.41	159.67	100.48	180.08	102.8	90.25	87.15
SMoM	23.48	19.27	18.60	22.82	11.87	21.74	16.58	22.66	19.97	20.15
MoM	7.90	7.74	8.15	8.34	8.11	7.59	7.65	7.97	8.89	7.99
MLE	3.32	4.33	4.63	3.40	6.88	3.68	4.75	3.45	4.27	4.08
ECF	15.76	22.26	19.85	18.13	16.37	18.10	21.60	18.41	14.88	15.26
χ^2	3.46	5.95	5.81	3.74	8.39	4.18	7.43	3.74	4.52	4.28
c										
Parameter	0.0008	0.0020	0.0018	0.0019	0.0011	0.0019	0.0014	0.0016	0.0019	0.0026
SMoM	85.59	37.39	40.88	40.36	48.59	32.02	36.70	34.82	43.40	40.67
MoM	91.54	35.99	39.43	40.07	48.06	32.59	35.88	35.61	43.52	40.02
MLE	78.65	18.77	22.92	25.09	37.79	24.50	21.09	27.12	33.58	23.92
ECF	318.21	58.29	70.87	143.84	106.37	118.72	61.58	134.88	148.26	113.62
χ^2	86.30	20.68	24.92	27.22	39.57	26.03	23.31	29.97	35.57	25.87

This table presents RMAD fitting results for each marginal parameter (δ, β, γ, c) of the 10 assets in the $\alpha\rho$GH model for the high-correlation scenario. The parameter row describes input for the simulation obtained from the MSCI US Equity dataset. All RMAD values are indicated in %.

Table 4
RMAD results for $\alpha\rho$NIG model — high-correlation scenario.

	1	2	3	4	5	6	7	8	9	10
δ										
Parameter	0.0113	0.0091	0.0100	0.0113	0.0098	0.0110	0.0083	0.0112	0.0153	0.0158
SMoM	334.44	91.49	87.79	205.52	67.17	166.95	64.00	136.88	82.73	79.21
MoM	207.98	75.42	76.39	163.79	59.25	139.41	51.70	117.43	78.28	74.02
MLE	209.16	87.62	83.59	168.77	64.18	146.31	61.01	125.38	80.60	76.01
ECF	143.59	79.52	81.23	137.55	64.20	128.71	54.45	115.97	74.90	70.22
χ^2	198.77	81.08	75.17	154.06	57.40	132.22	55.33	111.58	73.86	69.53
β										
parameter	−1.55	−13.98	−11.04	−5.21	−6.80	−5.37	−13.44	−4.46	−5.79	−7.36
SMoM	454.87	88.85	101.02	259.07	162.05	243.90	91.11	274.01	126.60	96.01
MoM	249.31	56.75	73.86	166.02	132.11	172.03	59.57	203.44	122.59	91.51
MLE	255.03	52.86	66.93	158.98	114.41	158.45	55.17	186.42	110.36	82.42
ECF	346.98	61.97	75.96	154.05	121.29	157.59	64.38	185.08	118.54	92.98
χ^2	277.53	65.30	75.41	159.07	124.11	157.82	65.82	182.46	115.63	86.43
γ										
Parameter	22.82	84.27	79.00	43.34	99.39	50.27	119.63	55.47	53.90	53.83
SMoM	341.77	125.87	118.59	304.96	71.37	252.64	64.31	206.39	108.37	99.62
MoM	228.50	107.41	106.32	252.04	63.59	217.64	51.89	182.49	106.04	96.95
MLE	211.93	123.15	115.65	258.40	69.31	227.26	62.15	193.51	108.52	99.10
ECF	150.54	109.69	110.22	214.31	68.50	202.25	54.29	179.20	100.30	90.64
χ^2	193.92	113.82	103.89	236.03	61.51	205.69	56.28	172.97	99.13	90.00
c										
Parameter	0.0012	0.0021	0.0017	0.0019	0.0009	0.0020	0.0014	0.0016	0.0021	0.0028
SMoM	211.33	52.08	66.26	106.38	108.23	94.03	58.09	104.77	83.67	60.33
MoM	239.14	110.79	137.81	119.03	233.64	108.41	106.54	153.31	100.80	72.70
MLE	155.16	31.95	44.35	84.03	76.86	68.26	35.56	79.53	72.57	54.79
ECF	173.87	38.61	51.03	79.57	82.00	68.47	43.22	78.82	77.74	60.60
χ^2	168.71	38.92	49.38	84.41	81.53	67.99	41.82	78.33	76.08	56.69

This table presents RMAD fitting results for each marginal parameter (δ, β, γ, c) of the 10 assets in the $\alpha\rho$NIG model for the high-correlation scenario. The parameter row describes input for the simulation obtained from the MSCI US Equity dataset. All RMAD values are indicated in %.

a fractional part better than the χ^2 method does. For example, asset 1 has a very high kurtosis (see e.g. parameter α in the $\alpha\rho$VG model) and the errors for σ and α are very small, but quite large for the other two parameters for the MLE and χ^2 methods. This fact indicates that although the correctness of c and β suffers, there is no problem handling high kurtosis and variance-driving parameters. Furthermore, in this case, the χ^2 method works better for the kurtosis-driving parameter α than the MLE method does. Again, these observations are valid for both correlation scenarios. All in all, these results confirm the findings of Finlay and Seneta [38] and Rathgeber et al. [39] for the univariate case, thereby emphasising the high quality of the enhanced multivariate version.

However, there are huge differences between the high and no-correlation settings within the $\alpha\rho$NIG model: the factor-based time-changed approach to model multivariate Lévy processes is not able to handle high-correlations and no-correlations without significant impact on the fitting quality of the idiosyncratic parameters for the $\alpha\rho$NIG model. Particularly asset 1 as well as some other assets have unacceptably high RMAD values in the high-correlation scenario. A look at asset 10 shows that the $\alpha\rho$NIG model works significantly better for the asset with the lowest kurtosis. To conclude, the $\alpha\rho$NIG model has major problems handling high-correlations with a high kurtosis. In contrast to the other models, we do not notice a huge outperformance of the SMoM, MoM, and ECF methods by the MLE and χ^2 methods. RMAD values for the no-correlation scenario decrease and are in an acceptable range. Furthermore, the MLE and χ^2 methods outperform again. These two observations show that the $\alpha\rho$NIG model performs at least for a no-correlation scenario. In combination with the problems discussed, this observation raises general questions concerning the flexibility and modelling of idiosyncratic components in the $\alpha\rho$NIG model.

For the $\alpha\rho$GH model, we realise more or less the same results for the RMAD as for the $\alpha\rho$VG model: well-fitting RMAD results for the MLE and χ^2 methods and problems with the asymmetry parameter β and location parameter c for the original high kurtosis cases. Additionally, the error for the variance parameter δ increases, while simultaneously γ and the corresponding kurtosis are more adequately modelled. A RMAD of about 3% for γ for the MLE and χ^2 methods provides a nearly perfect modelling possibility. As a consequence, if the target is the lowest deviation for the kurtosis-driving parameter, the $\alpha\rho$GH model would properly fit. Differences between high and no-correlation scenarios are not visible.

Table 5
RMAD results for $\alpha\rho$VG model — high-correlation scenario.

	1	2	3	4	5	6	7	8	9	10
σ										
Parameter	0.0208	0.0103	0.0112	0.0156	0.0099	0.0142	0.0091	0.0140	0.0167	0.0172
SMoM	3.17	10.94	15.84	2.74	31.40	5.21	3.64	3.72	14.66	11.76
MoM	3.62	9.77	14.53	2.76	29.79	4.64	3.41	3.42	13.92	10.66
MLE	3.01	10.04	14.71	2.69	29.94	4.80	3.47	3.51	14.16	10.93
ECF	3.86	10.36	15.01	2.85	30.38	4.94	3.61	3.70	14.17	11.13
χ^2	4.64	11.34	16.28	3.16	31.45	6.04	4.67	4.48	14.82	11.76
μ										
Parameter	0.00045	−0.00116	−0.00132	−0.00117	−0.00114	−0.00115	−0.00067	−0.00100	−0.00123	−0.00185
SMoM	289.26	68.95	72.09	84.26	103.24	85.16	91.78	99.78	111.78	72.93
MoM	246.10	78.16	82.04	89.92	117.15	91.02	97.50	106.22	118.09	81.50
MLE	131.67	48.13	53.03	46.77	85.46	53.33	56.83	58.97	75.40	48.81
ECF	330.76	77.88	78.27	90.21	107.07	89.38	97.64	105.65	111.83	69.70
χ^2	153.77	47.80	51.04	50.22	77.82	55.50	59.21	61.96	77.54	51.08
α										
Parameter	1.2926	0.8035	0.7540	1.0070	0.6476	0.8766	0.9197	0.9255	0.7607	0.7942
SMoM	22.81	26.60	28.23	24.70	28.62	25.13	24.25	24.50	25.82	27.10
MoM	20.98	25.43	26.24	23.81	26.46	24.32	23.94	23.40	25.08	25.35
MLE	7.54	11.51	11.99	9.12	15.02	10.75	10.09	9.80	12.20	11.17
ECF	26.04	32.17	36.54	27.84	37.46	31.01	25.46	28.91	35.31	35.92
χ^2	7.11	13.52	14.99	10.80	18.52	12.77	11.69	11.37	14.87	13.82
c										
Parameter	0.0004	0.0018	0.0017	0.0018	0.0014	0.0021	0.0012	0.0017	0.0016	0.0024
SMoM	311.12	37.95	45.10	50.55	60.08	43.03	48.67	55.28	73.31	49.42
MoM	262.52	43.34	51.52	54.14	69.58	46.13	51.85	58.98	77.74	55.22
MLE	95.90	22.23	27.74	22.54	39.39	22.86	25.56	26.89	42.66	27.41
ECF	375.05	44.97	51.22	54.99	68.56	46.26	52.81	59.15	74.73	49.24
χ^2	105.29	23.48	28.75	23.33	41.09	23.45	25.91	27.30	45.51	30.40

This table presents RMAD fitting results for each marginal parameter (σ, μ, α, c) of the 10 assets in the $\alpha\rho$VG model for the high-correlation scenario. The parameter row describes input for the simulation obtained from the MSCI US Equity dataset. All RMAD values are indicated in %.

Table 6
Aggregated ARMAD results for high-correlation scenario.

	SMoM	MoM	MLE	ECF	χ^2
$\alpha\rho$GH	150.27	73.01	45.52	184.71	49.07
$\alpha\rho$NIG	146.32	129.15	112.89	109.08	109.89
$\alpha\rho$VG	55.36	55.51	30.45	61.30	32.46

This table presents the aggregated ARMAD fitting results for the high-correlation scenario. In this context aggregated is the mean of RMAD over all estimated parameters and assets. All ARMAD values are indicated in %.

Furthermore, we observe an acceptable performance of the SMoM. Limitations and insights for this method are explained in [54] and also valid for the multivariate case.

To sum up, a comparison of the models by the RMAD shows that the $\alpha\rho$GH and $\alpha\rho$VG models work much more accurately than the $\alpha\rho$NIG model does. However, all three models have problems fitting asset 1 with a kurtosis, that is extremely high compared to the other assets. The models should be calibrated by the use of the MLE or χ^2 methods. We renounce to go into detail for the SMoM, MoM, and ECF methods as results are significantly worse than of the other two methods. To complete, we aggregate the RMAD to the ARMAD by using the mean over all parameters and assets (see Table 6 and table 16 in the Appendix B). The ARMAD results clarify the results of the RMAD. The MLE and χ^2 methods significantly outperform the other three methods and the $\alpha\rho$VG model works better than the $\alpha\rho$GH model. While results for the $\alpha\rho$GH and $\alpha\rho$VG models are stable for high and no-correlation scenarios, ARMAD results illustrate the weakness of the idiosyncratic parameter estimations in the high-correlation scenario of the $\alpha\rho$NIG model. While ARMAD statistics reach over 100% for all estimation methods in the high-correlation scenario, it decreases to about 35% in the no-correlation scenario for the MLE method. For comparison only, ARMAD statistics for the $\alpha\rho$GH model resp. $\alpha\rho$VG model are about 45% resp. 35% in both correlation scenarios.

The results of the KS and AD tests show different findings (for the high-correlation scenario, see Table 7 resp. Table 8; for the no-correlation scenario, see table 17 resp. table 18 in the Appendix B). While the $\alpha\rho$GH model works better than the

Table 7
KS test results — high-correlation scenario.

	SMoM	MoM	MLE	ECF	χ^2
$\alpha\rho$GH					
mean_{dev}	0.0414	0.0372	0.0304	0.0458	0.0323
std_{dev}	0.0096	0.0193	0.0200	0.0157	0.0200
1%	85.99	86.69	89.23	68.83	88.06
5%	56.92	78.43	82.13	46.90	80.39
10%	38.22	69.44	76.96	34.32	74.77
15%	28.60	62.88	73.53	27.36	70.86
20%	21.20	55.82	70.59	22.04	67.28
$\alpha\rho$NIG					
mean_{dev}	0.0204	0.1317	0.0169	0.0300	0.0186
std_{dev}	0.0148	0.1095	0.0038	0.0349	0.0087
1%	99.69	24.28	100.00	92.00	99.34
5%	99.29	18.28	100.00	89.41	99.20
10%	98.73	15.16	100.00	87.07	99.13
15%	98.15	13.31	99.99	85.01	99.08
20%	97.26	11.74	99.97	82.96	98.95
$\alpha\rho$VG					
mean_{dev}	0.0796	0.0647	0.0405	0.0752	0.0450
std_{dev}	0.2465	0.0898	0.0291	0.1041	0.0387
1%	71.62	72.08	74.62	67.48	70.78
5%	64.50	65.05	66.73	58.47	62.69
10%	58.29	59.05	61.91	51.00	57.50
15%	53.74	54.47	58.65	45.73	54.09
20%	48.51	49.30	55.35	40.35	50.70

This table presents KS test results aggregated from all simulations (10,000) for each multivariate Lévy process for the high-correlation scenario. We focus on the mean (mean_{dev}) and the standard deviation (std_{dev}) of the maximal deviation of the KS test. 1%, 5%, 10%, 15%, and 20% indicate fitting rates (that means percentages of simulations in which we could not reject the null hypothesis that log-returns follow the Lévy process) for the respective significance levels.

Table 8
AD test results — high-correlation scenario.

	SMoM		MoM		MLE		ECF		χ^2	
	mean	std.	mean	std.	mean	std.	mean	std.	mean	std.
$\alpha\rho$GH	2.5016	1.2585	2.4769	3.6810	2.2938	3.2052	3.1678	2.6018	2.5593	3.7375
$\alpha\rho$NIG	2.5901	157.9620	78.7109	150.9662	0.2792	0.1347	5.7212	42.9809	0.4883	2.5446
$\alpha\rho$VG	8.8613	28.8762	7.1642	19.1515	2.9511	3.8327	9.5558	24.1767	3.7060	6.7603

This table presents the AD test statistic for each estimation method and each multivariate Lévy process for the high-correlation scenario. We focus on the mean and the standard deviation (std.) over all 10,000 simulations (1000*10).

Table 9
RMAD results parameter m_p — high-correlation scenario.

	m_p original	SMoM	MoM	MLE	ECF	χ^2
$\alpha\rho$GH	0.9800	2.04	1.87	1.52	2.04	1.53
$\alpha\rho$NIG	0.2515	515.23	498.69	662.67	236.83	514.14
$\alpha\rho$VG	0.7582	18.16	16.59	6.49	17.84	6.67

This table presents RMAD for the parameter m_p in each estimation method for all multivariate Lévy processes for the high-correlation scenario. The second column presents the parameter m_p used to simulate the respective process. All RMAD values are indicated in %.

$\alpha\rho$VG model and again the MLE and χ^2 methods outperform the other methods, the $\alpha\rho$NIG model shows aberrant results for the high-correlation scenario. For the $\alpha\rho$GH and $\alpha\rho$VG models results are stable for both correlation scenarios with fitting rates of about 90% and 70% for the KS test, which is a very conservative test. The mean of the maximal deviations shows very good fittings for the cumulative distribution function (CDF) of the $\alpha\rho$GH model. Although the test only regards the maximum deviation between the empirical and the estimated CDF, the mean of these maximum deviations is only about 0.03 for the estimations by both the MLE method as well as the χ^2 method. For the $\alpha\rho$VG model, results for these deviations are slightly higher. The smaller AD test statistics demonstrate the superiority of the $\alpha\rho$GH model to the $\alpha\rho$VG model in handling fat tails in combination with extreme correlation scenarios. The small standard deviations of AD test statistics over all simulation runs for the MLE and χ^2 methods reveal a good stability of these two estimation methods. Simultaneously, the big standard deviations indicate instability for the other methods in estimating the parameters of the

Table 10
RMSE results ρ_Y matrix — high-correlation scenario.

	SMoM		MoM		MLE		ECF		χ^2	
	mean	std.	mean	std.	mean	std.	mean	std.	mean	std.
$\alpha\rho$GH	0.1213	0.014	0.0350	0.0231	0.0126	0.0053	0.1096	0.0260	0.0139	0.0078
$\alpha\rho$NIG	0.3126	0.0168	0.3062	0.0168	0.3156	0.0136	0.2796	0.0313	0.3042	0.0210
$\alpha\rho$VG	0.0932	0.0825	0.0799	0.0446	0.0295	0.0219	0.0637	0.0360	0.0313	0.0297

This table presents RMSE fitting results for each estimation method and each multivariate Lévy process for the high-correlation scenario. The RMSE is calculated between the estimated and the simulated correlation matrices ρ_Y. We focus on the mean RMSE of the 1000 simulation runs and the standard deviation (std.).

Table 11
RMSE results ρ matrix — high-correlation scenario.

	SMoM		MoM		MLE		ECF		χ^2	
	mean	std.	mean	std.	mean	std.	mean	std.	mean	std.
$\alpha\rho$GH	0.2041	0.0158	0.0787	0.0351	0.0406	0.0116	0.1649	0.0279	0.0427	0.0119
$\alpha\rho$NIG	0.2300	0.0530	0.2450	0.0453	0.2772	0.0411	0.2187	0.0468	0.2621	0.0434
$\alpha\rho$VG	0.1690	0.0401	0.1606	0.0384	0.1553	0.0401	0.1712	0.0399	0.1687	0.0393

This table presents RMSE fitting results for each estimation method and each multivariate Lévy process for the high-correlation scenario. The RMSE is calculated between the estimated and the original correlation matrix ρ. We focus on the mean RMSE of the 1000 simulation runs and the standard deviation (std.).

models. In contrast, the $\alpha\rho$NIG model shows higher, nearly perfect fitting rates and smaller AD test statistics, although the RMAD is quite weak. This fact reveals that a systematic overestimation of the parameters can distort the KS and AD tests, calling into question results of these tests in the high-correlation scenario for the $\alpha\rho$NIG model. To figure out the reasons behind, we have a look at the mean of the deviations of the idiosyncratic parameters and notice an overestimation for γ and δ up to 100%, which explains the collapse of the model. For the no-correlation scenario these deviations go back again to the expected mean value of about zero. The MoM is the only method which indicates a bad fitting (fitting rate of about 25% at a 1% significance level) and points to be careful in dealing with the results of this model. The systematic overestimation of parameters increases the (higher) central moments in such a way that a matching to the empirical moments is impossible. From a model testing perspective, these results suggest at least one robustness test to avoid an overconfidence in a particular model. In the no-correlation setting, the KS and AD tests produce good results as we already have a quite acceptable RMAD. By comparing our simulation results with the empirical results by Luciano et al. [37], we see the same effects concerning the KS and AD tests. Luciano et al. [37] find good test statistics for these two tests, but the correlation handling of the $\alpha\rho$NIG model is unsatisfactory, although the idiosyncratic components seem correct from a statistical point of view. However, we know that the RMAD for the idiosyncratic parameters of the $\alpha\rho$NIG model are insufficient, requiring caution with the interpretation and reliability of these results.

Next, we therefore evaluate the market parameter m_p. Note that m_p is estimated via RMSE based on the input of idiosyncratic parameters from the five estimation methods. Therefore, we examine m_p with regard to these methods. The RMAD results (see Table 9) demonstrate very good fittings for the $\alpha\rho$GH model and acceptable fittings for the $\alpha\rho$VG model. Again, the MLE and χ^2 methods generate the best results for both models, whereas results for the $\alpha\rho$GH model are quite good for all estimation methods. Generally in this case, the $\alpha\rho$GH model has the advantage that the bound for m_p only depends on $\lambda = 1$ and not on possible estimation biased idiosyncratic parameters like in the other two models. The accuracy of m_p in the $\alpha\rho$NIG model is unacceptable in the high-correlation case. The quite large deviations for m_p within the $\alpha\rho$NIG model can be explained as follows. We fix m_p for the simulation on the basis of the theoretical bounds of the model (see conditions in Eq. (9)). These bounds come from the original parameters, and are limited by asset 1 and especially their value for γ. However, as Table 4 shows, we have a very high RMAD for γ. A detailed look at the simulated data reveals a considerable overestimation of all parameters, leading to a higher bound for m_p within each run in the re-estimation of the parameters. As such misspecification of the idiosyncratic parameters also leads to errors in the common market parameters. All these findings are not valid for the no-correlation models (see table 19 in the Appendix B), in which we face an enormous, definitely non-acceptable overestimation of the market parameter m_p. This fact is a result of the lower importance of m_p in the no-correlation setting and therefore the values of m_p have only a very small influence on the correlation structure. But, this RMAD also demonstrates that parameters estimated via the χ^2 method and used in a second step for the calibration for m_p generate better results than the MLE and other methods.

Finally, the RMSE (see Tables 10 and 11 resp. tables 20 and 21 in the Appendix B) for the log-return correlations ρ_Y and for the correlation matrix ρ (again results are based on the input of idiosyncratic parameters obtained via the five estimation methods) complete the picture of the preferable $\alpha\rho$GH model. The $\alpha\rho$GH model handles correlations best in the high-correlation scenario and is marginally lower in fitting than the $\alpha\rho$VG model in the no-correlation scenario for ρ and ρ_Y. While the MLE method sometimes has a smaller mean, the χ^2 method sometimes returns a lower standard deviation and the other way round. The other three methods provide much worse results. The $\alpha\rho$NIG model provides

Table 12
Overview characteristics Lévy processes.

	GH	NIG	VG
Source	Barndorff-Nielsen [6], Eberlein and Keller [9], Schoutens [55], Scott et al. [56], Rathgeber et al. [54]	Barndorff-Nielsen [11], Rydberg [14]	Madan and Seneta [15], Madan and Seneta [16], Finlay and Seneta [38]
SMoM	The calculation of the SMoM for the GH process is more difficult as for the other two processes due to the modified Bessel function within the moments. Therefore, we apply and refer to the approach of Rathgeber et al. [54]	$\hat{\delta} = \sqrt{\frac{3M_2^E}{M_4^E-3}}$ $\hat{\gamma} = \frac{\hat{\delta}}{M_2^E}$ $\hat{\beta} = \frac{1}{3}M_3^E\hat{\delta}^{\frac{1}{2}}\hat{\gamma}^{\frac{3}{2}}$ $\hat{c} = M_1^E - \frac{\hat{\delta}\hat{\beta}}{\hat{\gamma}}$	$\hat{\sigma} = \sqrt{M_2^E}$ $\hat{\alpha} = \frac{M_4^E}{3} - 1$ $\hat{\mu} = \frac{M_3^E\hat{\sigma}}{3\hat{\alpha}}$ $\hat{c} = M_1^E - \hat{\mu}$
M_i	$M_1 = \frac{\delta^2}{\zeta}\beta\frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)} + c$ $M_2 = \frac{\frac{\delta^2}{\zeta}K_{\lambda+1}(\zeta)+\left(\frac{\delta^2}{\zeta}\right)^2\beta^2K_{\lambda+2}(\zeta)}{K_\lambda(\zeta)}$ We refer for M_3 and M_4 to the derivations of the moments of the GH process by Scott et al. [56]	$M_1 = \frac{\delta\beta}{\sqrt{\gamma^2-\beta^2}} + c$ $M_2 = \frac{\gamma^2\delta}{(\gamma^2-\beta^2)^{\frac{3}{2}}}$ $M_3 = \frac{3\beta}{\gamma\sqrt{\delta}(\gamma^2-\beta^2)^{\frac{1}{4}}}$ $M_4 = 3\left(1+\frac{\gamma^2+4\beta^2}{\delta\gamma^2\sqrt{\gamma^2-\beta^2}}\right)$	$M_1 = c + \mu$ $M_2 = \sigma^2 + \mu^2\alpha$ $M_3 = \frac{2\mu^3\alpha^2+3\sigma^2\mu\alpha}{(\sigma^2+\mu^2\alpha)^{\frac{3}{2}}}$ $M_4 = 3 + \frac{3\sigma^4\alpha+12\sigma^2\mu^2\alpha^2+6\mu^4\alpha^3}{(\sigma^2+\mu^2\alpha)^2}$
$f_X(x)$	$\frac{(\gamma^2-\beta^2)^{\frac{\lambda}{2}}}{\sqrt{2\pi}\gamma^{\lambda-\frac{1}{2}}\delta^\lambda K_\lambda(\zeta)}(\delta^2+(x-c)^2)^{(\lambda-\frac{1}{2})/2}C$ with $C = K_{\lambda-\frac{1}{2}}(\gamma\sqrt{\delta^2+(x-c)^2})e^{\beta(x-c)}$	$\frac{\delta\gamma}{\pi}e^{\delta\sqrt{\gamma^2-\beta^2}+\beta(x-c)}\frac{K_1\left(\gamma\sqrt{\delta^2+(x-c)^2}\right)}{\sqrt{\delta^2+(x-c)^2}}$	$\frac{2}{\sigma\sqrt{2\pi}\alpha^{\frac{1}{\alpha}}\Gamma\left(\frac{1}{\alpha}\right)}e^{\mu\frac{x-c}{\sigma^2}}\left(\frac{\lvert x-c\rvert}{\sqrt{\frac{2\sigma^2}{\alpha}+\mu^2}}\right)^{\frac{1}{\alpha}-\frac{1}{2}}C$ with $C = K_{\frac{1}{\alpha}-\frac{1}{2}}\left(\frac{\lvert x-c\rvert\sqrt{\frac{2\sigma^2}{\alpha}+\mu^2}}{\sigma^2}\right)$
$\Phi_X(u)$	$e^{iuc}\left(\frac{\gamma^2-\beta^2}{\gamma^2-(\beta+iu)^2}\right)^{\frac{\lambda}{2}}\frac{K_\lambda\left(\delta\sqrt{\gamma^2-(\beta+iu)^2}\right)}{K_\lambda\left(\delta\sqrt{\gamma^2-\beta^2}\right)}$	$e^{-\delta\left(\sqrt{\gamma^2-(\beta+iu)^2}-\sqrt{\gamma^2-\beta^2}\right)+iuc}$	$e^{iuc}\left(1-i\mu\alpha u+\frac{\sigma^2\alpha u^2}{2}\right)^{-\frac{1}{\alpha}}$

This table presents the simplified moments (SMoM), the moments (mean, variance, skewness and kurtosis) M_i with $i = 1, \ldots, 4$, the probability density functions (PDF) $f_X(x)$, and the characteristic function (CF) $\Phi_X(u)$ for the GH, NIG, and VG processes. $K(\cdot)$ represents the modified Bessel function of the third kind. For the GH process, we set $\lambda = 1$.

bad results for nearly all correlation matrices and scenarios, which demonstrates its unfitness to handle correlations. To sum up, the findings reveal that the $\alpha\rho$GH is the best model, as it particularly shows the lowest errors in correlation matrices, works accurately in the high-correlation as well as in the no-correlation scenarios, reveals good distribution test results, and also provides good results for the idiosyncratic parameters.

5. Conclusion

We provided an overview and a general robustness check on modelling and fitting multivariate Lévy processes utilising a factor-based time changed framework introduced by Luciano and Semeraro [1,2], and Luciano et al. [37], with the help of a large simulation study in order to provide insights to models being able to describe cross-sectional effects in returns which has been identified by for example [20–23]. Consequently, we chose the simulation approach to answer which fitting method works best for multivariate Lévy models and which model is best for modelling correlations. To this end, we analysed the idiosyncratic and common market parameters of $\alpha\rho$GH, $\alpha\rho$NIG, and $\alpha\rho$VG models with a two-step estimation approach, various parameter estimation methods, and different statistical evaluation methods. Furthermore, a comparison of a high correlation and a no correlation scenario, which can be regarded as a univariate model, reveals the strength and weaknesses of the models. As all these models can be useful for pricing multivariate options our insights build a basis for the selection of the best stochastic process for the underlyings.

Concerning the superiority of the MLE and the χ^2 method, we can verify the findings of Finlay and Seneta [38] and Rathgeber et al. [39] for the univariate VG process. We can also confirm its validity for the multivariate $\alpha\rho$GH, $\alpha\rho$NIG and $\alpha\rho$VG models. In so doing, we extent the insights of Luciano et al. [37], who only applied the MLE method for an empirical data set. Moreover, we found out that the fitting quality of the idiosyncratic parameters does not depend on the correlation for the $\alpha\rho$VG and $\alpha\rho$GH models, while the fitting of the common parameter gets significantly inferior respectively unacceptable for the $\alpha\rho$NIG model. This result means the $\alpha\rho$NIG model is not able to handle high correlations. A comparison with results of the no-correlation scenario for the $\alpha\rho$NIG highlights these findings. Furthermore, we notice especially within the $\alpha\rho$NIG case, and in general in the $\alpha\rho$VG and $\alpha\rho$GH cases that bad idiosyncratic estimation results can distort market results. Based on the results of the RMAD, ARMAD, RMSE, AD, and KS tests, we identify the $\alpha\rho$GH as the best multivariate Lévy process model. All in all, these results extend the empirical findings by Luciano et al. [37] on the $\alpha\rho$GH, $\alpha\rho$NIG, and $\alpha\rho$VG models. Altogether, we back-tested and confirmed already existing results of Luciano et al. [37]. The main difference is that we tested the models with simulated data, and therefore know ex-ante the exact parameters of the respective simulated model, while they used real market data. We used different methods, while Luciano et al. [37] used MLE. Based on insights from these simulations, we suggest using the $\alpha\rho$GH model to handle correlation effects or market components in issues dealing with, for example, the description of cross-sectional asset returns or multivariate risk models. Topics for further research could be modification of the two-step estimation approach to a one-step, via use of multivariate distributions within a large simulation framework, or by using option prices to evaluate the models with implicit parameters.

Appendix A

See Table 12.

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