Two-scale analysis of intermittency in fully developed turbulence

R. Badii(*) and P. Talkner

General Energy Research Department Paul Scherrer Institute
5232 Villigen, Switzerland

Abstract. – A self-affinity test for turbulent time series is applied to experimental data for the estimation of intermittency exponents. The method employs exact relations satisfied by joint expectations of observables computed across two different length scales. One of these constitutes a verification tool for the existence and the extent of the inertial range.

Introduction. – The velocity field $\mathbf{v}(\mathbf{x}, \mathbf{t})$ of a fluid in a regime of fully developed turbulence exhibits approximate self-affine properties upon a suitable rescaling of time t, space $\mathbf{x}$, and velocity $\mathbf{v}$ [1,2]. Indeed, longitudinal velocity differences computed across intervals of lengths ℓ and ℓ' are, on average, proportional to each other, provided that ℓ and ℓ' belong to the so-called inertial range $(\ell_{\min}, \ell_{\max})$, delimited by Kolmogorov's length $\ell_{\min}$ and by the outer length $\ell_{\max}$. The scaling factor γ depends on the ratio $r = \ell'/\ell < 1$. This behaviour underlies the existence of non-integer scaling exponents ζ_p for the moments

$$\langle d^p(\ell) \rangle \sim \ell^{\zeta_p} \tag{1}$$

of the longitudinal velocity difference $d(\ell) = v(x+\ell) - v(x)$, where both v and x are measured along a given spatial direction [3]. More important than the exponents' non-integer character is, however, their non-linear dependence on p. Experiments, in fact, show that

$$\zeta_p = p/3 + \tau_{p/3} \ , \tag{2}$$

where the former term [1] follows from strict self-affinity assumptions and the second [4, 5] accounts for the fluctuations of the energy dissipation $\varepsilon(\ell)$, averaged over an interval of length ℓ, as

$$\langle \varepsilon^p(\ell) \rangle \sim \ell^{\tau_p} \ . \tag{3}$$

(*) http://www1.psi.ch/~badii

In the present letter, we investigate this mechanism using the dependence of the scaling factor $\gamma(r)$ on the ratio r between the two scale lengths. This results in a very precise method for the evaluation of the exponent ζ_2. Extension of the self-affinity test to more general observables than d permits estimating ζ_p or τ_p for certain values of p. The reliability of the scaling laws (1), (3), which depends on the existence of an inertial range, is also tested with the introduction of a three-point correlation function.

Stochastic self-affinity test. – Let $V = \{v_1, v_2, \ldots, v_n\}$ be a scalar velocity time series originating from a turbulent fluid and $X_i(\ell)$ an observable formed with the values $\{v_i, \ldots, v_{i+\ell}\}$. Particularly relevant are observables having nearly self-affine graphs: *i.e.* nearly invariant upon rescaling of domain and range by different factors. A typical example is the velocity difference $d_i = d_i(\ell) = v_{i+\ell} - v_i$ across an interval L_i of length ℓ. By the conjectured self-affinity, the analogous quantity $X'_i = X_i(\ell')$, computed over the interval L'_i of length $\ell' = r\ell$ (fig. 1), is expected to satisfy a relation of the form

$$X(r\ell) \sim \gamma(r) X(\ell) , \tag{4}$$

where $r \in (0,1)$ and $\gamma(r) \in (0,1)$. Both ℓ and $r\ell$ are required to belong to the inertial range. Because of the stochastic features of the signal, arising from the activity of a huge number of degrees of freedom, eq. (4) can only hold in an average sense. Therefore, it is better replaced by the map

$$X'_i = \gamma X_i + q + f(X_i)\xi_i , \tag{5}$$

where q accounts for a non-vanishing expectation value of X, and $f(X_i)$ is an X-dependent amplitude for a "noise" term ξ_i. Notice that the physical-time index i is the same for all terms and that the "time" step of the map is represented by the scale transformation $\ell \to r\ell$. Averages will then be taken over all i. This is the discrete-time version of a recently proposed Langevin model of turbulence in which the pseudo-time was identified with the logarithm $\ln(1/r)$ of the lengths' ratio [6].

The decomposition (5) of $X_i(r\ell)$ into a contraction and a noise correction is evidently arbitrary. One might, in fact, speculate on the source of ξ as the cumulative effect of fluid structures of sizes larger than ℓ, on its probability distribution in dependence on f, on the absence of coupling between X_i and X_{i+j} for some j, and so on. Hence, the value of $\gamma(r)$ will depend on the form (5) and on the criterion chosen for its evaluation.

Without entering the discussion about the suitability of model (5) for turbulence, we present interesting results obtained with an appropriate choice of X and f. In particular, $\gamma(r)$ need

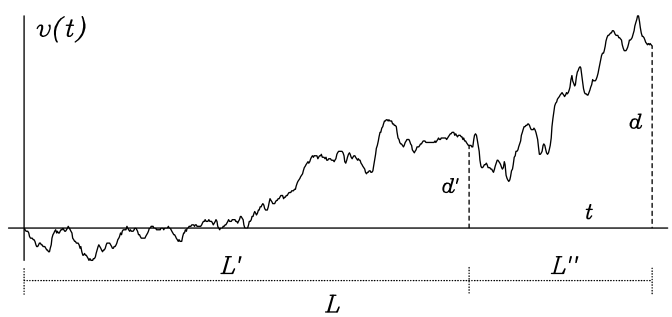

Fig. 1. – Graph of wind velocity $v(t)$ *vs.* time t from an experimental time series. The time intervals L, L', and L'', defined in the text, and the velocity differences d and d' are indicated.

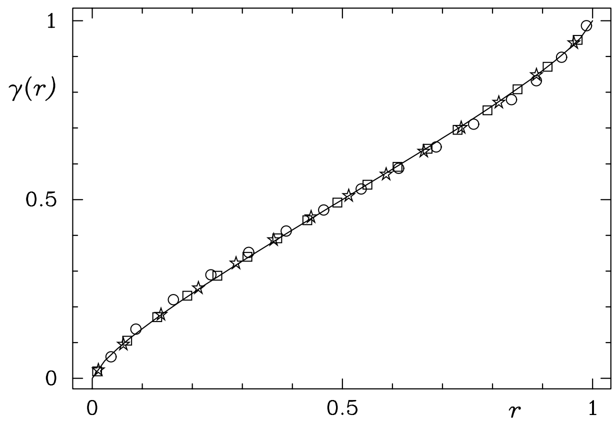

Fig. 2. – Curve $\gamma(r)$ *vs.* r estimated from three experimental time series [7] (only a few symbols have been drawn to avoid cluttering), and theoretical prediction from eq. (8) with $\zeta_2 = 0.69$ (solid line). The value of ℓ in the three cases was 600, 6000, and 250, respectively.

not satisfy a power law $\gamma(r) \sim r^z$, as might be intuitively expected, not even for observables X having expectations of the power law type $\langle X(\ell)\rangle \sim \ell^\zeta$. Our results have been obtained by minimizing the sum of the squared errors ξ_i, which yields

$$\gamma = \frac{S_{XX'}S_f - S_X S_{X'}}{S_{XX}S_f - S_X^2} \, , \tag{6}$$

where $S_{XX'} = \langle XX'/f^2$, $S_f = \langle 1/f^2$, $S_X = \langle X/f^2$, $S_{X'} = \langle X'/f^2$, and $S_{XX} = \langle X^2/f^2$.

Two-point correlation of velocity differences. – We first considered $X = d$ and $f = 1$. In this case, $q = 0$, since $\langle d(\ell)\rangle = 0 \; \forall \; \ell$, and

$$\gamma = \frac{\sum d(\ell)d(\ell')}{\sum d^2(\ell)} \, . \tag{7}$$

By setting $d(\ell) = d$ and $d(\ell') = d'$, for brevity, dd' can be rewritten as $[d^2 + (d')^2 - (d'')^2]/2$, where $d'' = d - d'$ is the velocity difference across the interval L'' of length $\ell'' = \ell - \ell'$ (fig. 1).

TABLE I. – *Exponents* ζ_2 *and* ζ_6 *for three experimental time series* [7], *estimated from eqs.* (8), (14), (15). *The largest interval length was in the ranges* $[400, 600]$, $[2000, 6000]$ *and* $[150, 300]$ *in the three cases, respectively. The values within parentheses refer to* ESS *estimates.*

Data set	ζ_2	ζ_6
Wind 1	0.685 ± 0.005 (0.7 ± 0.015)	1.835 ± 0.03 (1.81 ± 0.06)
Wind 2	0.685 ± 0.005 (0.7 ± 0.02)	1.84 ± 0.04 (1.83 ± 0.07)
Jet	0.675 ± 0.005 (0.69 ± 0.015)	1.835 ± 0.03 (1.8 ± 0.05)

Recalling eq. (1), one obtains

$$\gamma = \frac{1 + r^{\zeta_2} - (1-r)^{\zeta_2}}{2} \ . \tag{8}$$

No prefactors appear since L and L' have the same origin $t = i$, in the same way as L and L'' both end at $t = i + \ell$. Figure 2 shows a fit curve $\gamma(r)$ *vs.* r (eq. (8)) for three data sets [7]: two (Wind 1 and 2) refer to atmospheric turbulence and one (Jet) to a laboratory experiment. In the outer regions, $r \ll 1$ and $1 - r \ll 1$, one of the intervals L' or L'' falls in the dissipation range and the scaling (1) is no longer expected to hold. The exponent ζ_2 may then deviate from its inertial-range value. The values of ζ_2, estimated for $r \in (0.1, 0.9)$, are listed in table I (consistent results have been obtained for gaseous helium data provided by J. Peinke).

The curve $\gamma(r)$ is left-right symmetric under the transformation $r \to 1 - r$ and satisfies the relation $\gamma(r) + \gamma(1-r) = 1$ (since exchanging ℓ' with ℓ'' is tantamount to exchanging r with $1 - r$). Notice that $\gamma(r)$ is not a power r^z of r, as eq. (1) might instead suggest, and that the exponent appearing in its expression (8) is not close to 1/3 (the rescaling exponent of Kolmogorov's 1941 approach) but is exactly ζ_2, because of the quadratic nature of the chi-square estimate. This agrees with recent estimates made with n-point correlation functions [8] in which p extrema of the difference velocity vectors are let coalesce: then, ζ_p appears, possibly in connection with ζ_n, for certain geometries (the disposition of the intervals of fig. 1 has not been considered in [8]).

The value of ζ_2 is quite insensitive to changes in the largest-interval length ℓ. Hence, a further average over several values of $\ell \in (\ell_{\min}, \ell_{\max})$ can be taken to improve the statistics. The "inverse cascade", from small to large scales, can be investigated as well, by taking $X = d'$ and $X' = d$, thus obtaining a ratio $\gamma'(r) = \gamma(1/r)$.

Three-point correlation and inertial range. – The two-scale approach further yields a relation which can be used to test the existence and the extent of the inertial range, frequently defined as the interval $(\ell_{\min}, \ell_{\max})$ in which eq. (1) holds with $\zeta_3 = 1$ [1]. By expanding the cube $\langle (d'')^3 \rangle = \langle (d - d')^3 \rangle$, dividing each term by $\langle d^3 \rangle$, and recalling eq. (1), one obtains

$$1 = r^{\zeta_3} + (1-r)^{\zeta_3} + 3\Gamma(r) \ , \tag{9}$$

where $\Gamma(r)$ is the three-point correlation function

$$\Gamma(r) = \langle d d' d'' \rangle / \langle d^3 \rangle \ . \tag{10}$$

If $\zeta_3 = 1$, eq. (9) is identically satisfied with $\Gamma(r) = 0$. Thus, the vanishing of $\Gamma(r)$ is an indication of a good realization of the ideal turbulent cascade.

The connection with eq. (5) is established by writing $dd'd'' = d^2 d' - d(d')^2$ and noticing that the choices $(X, f) = (d^2, \sqrt{d})$ and $(X, f) = ((d')^2, \sqrt{d'})$ yield the contraction rates $\gamma_2 = \langle d(d')^2 \rangle / \langle d^3 \rangle$ and $\gamma_2' = \langle d' d^2 \rangle / \langle (d')^3 \rangle$, respectively. Hence, Γ is related to the difference between the rates γ_2 and γ_2' of direct- and inverse-cascade stochastic models (5) for the observable d^2 and a multiplicative noise (γ_2' being actually multiplied by r^{ζ_3}).

Four curves $\Gamma(r)$ *vs.* r, computed from the data set "Wind 2" for different values of ℓ, are displayed in fig. 3 with two fits made with eq. (9) using $\zeta_3 = 1.05$ and $\zeta_3 = 0.97$. Although $|\Gamma| \ll 1$, it is evidently not zero: a further average over a range of values of ℓ seems necessary to recover the exact result $\zeta_3 = 1$. In fact, the form of $\Gamma(r)$ reflects both fluctuations and systematic (logarithmic) corrections to the power law (1). Analogous deviations may affect other moments as well. It is, however, difficult to quantify them since no exact theoretical estimates are available which could be formulated in a similar way to eq. (9).

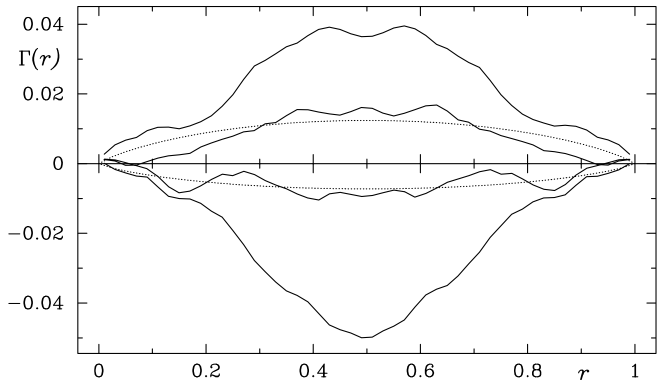

Fig. 3. – Curves $\Gamma(r)$ *vs.* r estimated from the data set "Wind 2", for $\ell = 1000, 1125, 1250$, and 1500 (solid lines, from top to bottom). The middle curves are compared with two fits (dotted lines) made with eq. (9) using $\zeta_3 = 1.05$ (above) and $\zeta_3 = 0.97$ (below).

Energy dissipation. – The stochastic self-affinity relation (5) can be further applied to the energy dissipation $E(\ell)$ which, in the interval $L_i = [i+1, i+\ell]$, is usually computed as

$$E_i(\ell) = c \sum_{j=i+1}^{i+\ell} (v_j - v_{j-1})^2 \equiv \ell \varepsilon_i(\ell) \; , \tag{11}$$

where the constant c depends on the viscosity ν and on the sampling time $\mathrm{d}t$, and $\varepsilon(\ell)$ is the quantity appearing in eq. (3). Setting $(X, f) = (\varepsilon(\ell), 1)$, yields the least-square rate

$$\gamma_e = \frac{\langle \varepsilon\varepsilon' \rangle - \langle \varepsilon \rangle \langle \varepsilon' \rangle}{\langle \varepsilon^2 \rangle - \langle \varepsilon \rangle^2} \; , \tag{12}$$

which, unfortunately, cannot be written as a function of r only, as in eqs. (8) or (9), since different powers of ε appear. It is, however, possible to rewrite it as

$$\gamma_e = \gamma_s (1 + \sigma^{-2}) - \rho\sigma^{-2} \; , \tag{13}$$

where $\gamma_s = \langle \varepsilon\varepsilon' \rangle / \langle \varepsilon^2 \rangle$, $\rho = \langle \varepsilon' \rangle / \langle \varepsilon \rangle \approx 1$ (to within 10^{-4} for our data), and $\sigma^2(\ell) = \langle \varepsilon^2 \rangle / \langle \varepsilon \rangle^2 - 1$ is closely related to the flatness of the probability distribution of $(\partial v/\partial x)^2$ [9]: for $\ell \to 0$, $\sigma^2 + 1$ scales as $Re_{\mathrm{T}}^{1.5\tau_2}$, where Re_{T} is the Taylor-Reynolds number [10,9]. The rate γ_s can be expressed in terms of r alone by noticing that $E = E' + E''$ (with analogous notation to that used for the velocity differences) and assuming the validity of eqs. (2) and (3): in fact,

$$\gamma_s = \; 1 + r^{\zeta_6} - (1-r)^{\zeta_6} \; /(2r) \; , \tag{14}$$

for the direct cascade and

$$\gamma_s' = r\Big\{ 1 + r^{-\zeta_6} \; 1 - (1-r)^{\zeta_6} \; \Big\}/2 \tag{15}$$

for the inverse one (defined by $X = \varepsilon'$ and $X' = \varepsilon$). The results, confirmed by a direct fit for $\tau_2 = \zeta_6 - 2$ from the second moment of ε as in eq. (3), appear in table I, together with "extended self-similarity" (ESS) estimates [11]. The three data sets, in spite of their quite different origin, yield nearly the same value of $\tau_2 \approx -0.165$ from both eq. (14) and eq. (15) [12]. Similar consistency has been found for ζ_2. It may be noted that our results satisfy the relation $\zeta_2 = 2/3 - \tau_2/9$, arising from the lognormality assumption [4,5].

Conclusions. – The approximate self-affinity of turbulent time series has been used to extract intermittency exponents from correlation functions of observables depending on two scale lengths. The same approach led to the deduction of a three-point correlation which is expected to vanish in the inertial range. Accurate results have been obtained for experimental data of quite different origin, recorded both in nature and in the laboratory.

***

The authors are indebted to M. Furger, R. Weber and J. Peinke, who kindly provided their data. They also benefitted from discussions with R. Friedrich and J. Peinke.

REFERENCES

[1] Kolmogorov A. N., *Dokl. Akad. Nauk SSSR*, **30** (1941) 301.

[2] Mandelbrot B. B., *J. Fluid Mech.*, **62** (1974) 331.

[3] Temporal displacements at a fixed position are equivalent to spatial displacements at a fixed time by Taylor's hypothesis: see Taylor G. I., *Proc. R. Soc. London, Ser. A*, **164** (1938) 476.

[4] Obukhov A. M., *J. Fluid Mech.*, **13** (1962) 77.

[5] Kolmogorov A. N., *J. Fluid Mech.*, **13** (1962) 82.

[6] Friedrich R. and Peinke J., *Phys. Rev. Lett.*, **78** (1997) 863; Friedrich R. and Peinke J., *Physica*, **D 102** (1997) 147.

[7] First two sets (Wind 1 and 2): wind velocity recorded in the atmosphere at 3 and 30 kHz, respectively, at a height of 25 m above ground, using a hot-film anemomenter with a spatial resolution of 2 mm (M. Furger and R. Weber, Paul Scherrer Institute, Switzerland). The average and rms velocities were 3.7 and 1.6 m/s, respectively, the integral length 100 m, the Kolmogorov length 0.45 mm, and the Taylor-Reynolds number 10000. Third set (Jet): air velocity in an axisymmetric free jet (dry air in air), recorded at 8 kHz with a hot-wire anemometer (resolution 1 mm), placed at a distance of 125 nozzle diameters from the source, along the axis of the jet (Ch. Renner, J. Peinke, University of Bayreuth, Germany). The Reynolds number at the nozzle was 27000. At the probe, the mean velocity was 2.25 m/s, the integral length 67 mm, the Kolmogorov length 0.25 mm and the Taylor-Reynolds number 200. Each set contains 10^7 points.

[8] L'Vov V. and Procaccia I., *Phys. Rev. E*, **54** (1996) 6268; Fairhall A. L., Dhruva B., L'Vov V., Procaccia I. and Sreenivasan K., *Phys. Rev. Lett.*, **79** (1997) 3174.

[9] Grossmann S. and Lohse D., *Europhys. Lett.*, **21** (1993) 201.

[10] Wyngaard J. C. and Tennekes H., *Phys. Fluids*, **13** (1970) 1962.

[11] Benzi R., Ciliberto S., Baudet C., Tripiccione R., Massaioli F. and Succi S., *Phys. Rev. E.*, **48** (1993) R29.

[12] The "commonly accepted value" for ζ_6 lies in the range $(1.6, 1.9)$: see Arneodo A. *et al.*, *Europhys. Lett.*, **34** (1996) 411, fig. 1.