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Stochastic resonance in the semiadiabatic limit

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Abstract. Periodically driven Markovian processes are treated in the semiadiabatic limit. In this limit all but one characteristic times of the undriven system are fast compared to the driving frequency. The proposed theory is based on the Floquet representation of the conditional probability and makes use of the instantaneous bases of eigenfunctions of the corresponding forward and backward master operators. The methodology is exemplified with the standard Brownian motion model of stochastic resonance. Quantitative agreement is reached with known results in the fully adiabatic limit and in the regime of linear response theory.

1. Introduction

Various phenomena of physics, chemistry, and other natural and engineering sciences can be properly modelled by stochastic processes that are driven by an external periodic forcing. As long as the periodic driving force is weak, linear response theory describes the output of such systems and provides a wealth of information about the dynamics of the undriven system [1, 2]. Countless experimental techniques are based on this mechanism. Dielectric relaxation, resonance absorption, nuclear magnetic resonance and dynamic light scattering are but a few examples. In the past two decades, however, nonlinear effects have attracted much interest. The interaction of irregular perturbations as given by thermal noise with a periodic forcing mediated by a nonlinear dynamical system can produce surprising effects. The amplification of a weak periodic signal by means of noise, known as stochastic resonance [3, 4] was originally suggested as the essential mechanism by which the small secular variations of the earth's orbital parameters may lead to recurrent ice ages. In the past two decades many other realizations of stochastic resonance have been found [5].

In a spatially periodic system which lacks symmetry within a unit cell, an external periodic field may produce a direct current. Such rectifying devices are particular examples of so-called ratchets [6] which have been proposed as models for transport in living cells.

Coming back to stochastic resonance, one can understand it as a synchronization of the random transitions between two metastable states of a system with an external periodic driving

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force. As a rule of thumb, the matching of these antagonistic processes is best if, in the absence of periodic driving, the two metastable states are symmetric and their common mean waiting time is just half the period of the driving force. According to this simple picture, at the most favourable moment when the barrier as seen from the initial state is lowest, the system hops from the initial state to the other one and, half a period later, back again. Typically, the waiting time of the metastable state is of Arrhenius type and therefore much longer than all other intrinsic time-scales of the system, except the period of the external forcing. Hence, the dynamics of the driven system is governed by two distinct slow processes while all other time-scales of the system are much faster. In a naive adiabatic treatment of the system, one would assume that at each instant of time the system is in its equilibrium state at the instantaneous value of the external forcing. In the interesting regime of stochastic resonance this approximation does not hold since the populations of the metastable states cannot follow the external forcing fast enough and hence are not in equilibrium.

The purpose of this paper is to present a *semiadiabatic theory* that can allow for other slow processes than the external driving. The semiadiabatic theory as presented here applies for Markovian systems. It is a perturbation theory for the Floquet states of the master operator that governs the time-evolution of the conditional probability of the process. The zeroth order of this perturbation theory is given by the naive adiabatic theory and the unperturbed states are determined by the eigenfunctions of the instantaneous master operator. The smallness parameter of this perturbation theory turns out to be the ratio between the external driving frequency and a typical fast relaxation rate of the system. The proposed theory is therefore not restricted to small driving fields and in particular also works for low temperatures, i.e. weak noise, for which linear response theory fails even for rather small external forces [7].

The paper is organized as follows. In section 2 the semiadiabatic theory is outlined and applied in section 3 to the stochastic resonance of an overdamped Brownian particle in a symmetric double well potential. section 4 provides a conclusion and an outlook.

2. Semiadiabatic Floquet theory

In the past hundred years Floquet theory has been applied to many different fields often related to stability problems of nonlinear oscillators [8]. The theoretical study of periodically driven quantum systems is based on Floquet theory [9] and only recently it has been utilized in the theory of periodic stochastic processes [5, 10] and of periodic quantum Markovian processes [11].

We begin this section with a short review of the Floquet theory of periodic Markov processes. In the next step we use the instantaneous eigenfunctions of the time dependent master operator introduced by Bunde and Gouyet [12] as a basis set and express the Floquet functions and corresponding Floquet equations in terms of this basis set. This representation was used by various authors to improve the adiabatic approximation of periodically driven quantum systems [13]. It allows us to identify the dominant slow terms and corresponding fast relaxing corrections and to build up a systematic perturbation theory based on this separation.

2.1. General properties of periodic Markov processes

We consider Markovian processes which are periodic in time. The state space of the process may be discrete or continuous. In the first case, $p(t)$ denotes the vector whose components are the probabilities to find the system at time t in its possible states and, correspondingly, the probability density at time t in case of a continuous state space [14]. For a discrete state space observables also are vectors, and functions for a continuous state space. In both cases, the expectation value $\langle f(t)\rangle$ of an observable f at time t is given by a scalar product:

$$\langle f(t)\rangle = (f, p(t))\,, \tag{1}$$

where the scalar product is defined as

$$(f, p) = \sum_i f_i p_i, \tag{2}$$

in the discrete case, and

$$(f, p) = \int \mathrm{d}x\, f(x) p(x), \tag{3}$$

in the continuous case where both the sum in (2) and the integral in (3) are extended over the whole state space.

For a Markov process, the time-evolution of $p(t)$ is given by a master equation, i.e. a linear evolution equation of first order in time:

$$\frac{\partial}{\partial t} p(t) = L(t) p(t), \tag{4}$$

where $L(t)$ denotes the master operator. Its particular form depends on the nature of the state space and the dynamics of the process considered. It is a matrix in case of a discrete state space and an integral, differential, or mixed operator for a continuous state space [15]. In any case, the conservation of total probability and of positivity are important properties of the operator $L(t)$. The periodicity of the process imposes a periodic time dependence on the master-operator:

$$L(t+T) = L(t). \tag{5}$$

Here T denotes the period of the process.

Since $L(t)$ is invariant under discrete time translations, according to Floquet theory [10], the solutions of (4) can be build up from exponential and periodic functions of time, $\exp\{\mu_i t\}$ and $v_i(t)$, respectively:

$$p(t) = \sum_i a_i \mathrm{e}^{\mu_i t} v_i(t). \tag{6}$$

The Floquet exponents μ_i and the periodic Floquet functions $v_i(t)$ are solutions of the Floquet equation:

$$\frac{\partial}{\partial t} v_i(t) + \mu_i v_i(t) = L(t) v_i(t), \tag{7}$$

$$v_i(t+T) = v_i(t). \tag{8}$$

Similarly, the time evolution of conditional expectations of observables is determined by the backward equation:

$$-\frac{\partial}{\partial t} f(t) = L^+(t) f(t), \tag{9}$$

where the backward operator $L^+(t)$ is the adjoint of $L(t)$ with respect to the above defined scalar product. Using the periodic solutions of the corresponding adjoint Floquet equation

$$-\frac{\partial}{\partial t} u_i(t) + \mu_i u_i(t) = L^+(t) u_i(t), \tag{10}$$

$$u_i(t+T) = u_i(t). \tag{11}$$

the conditional probability $P(x,t|y,s)$ of finding the system at time t at the state x, provided it was at the earlier time s at the state y, can be represented as:

$$P(x,t|y,s) = \sum_i \mathrm{e}^{\mu_i(t-s)} v_i(x,t) u_i(y,s) \quad \text{for } t \geq s. \tag{12}$$

Here we have used that the two sets of Floquet functions are biorthonormal as functions of space and time in the following sense:

$$\frac{1}{T}\int_0^T \mathrm{d}t (u_i(t), v_j(t)) = \delta_{i,j}. \tag{13}$$

In passing we note that this defines a scalar product between time periodic observables and time periodic probabilities. In quantum mechanics it corresponds to the scalar product of time periodic wave functions constituting the product Hilbert space introduced by Sambe [16]. In this space the Floquet equations appear as ordinary eigenvalue problems of the extended forward and backward operators $L(t) - \partial/\partial t$ and $L^+(t) + \partial/\partial t$, respectively. Further we assumed that both sets are complete in its respective vector or function spaces, i.e.

$$\sum_i v_i(x,t) u_i(y,t) = \delta(x-y). \tag{14}$$

With the Dirac δ-function, this holds for a continuous state space and with a Kronecker δ for a discrete space.

Finally we note that as a consequence of the conservation of total probability one of the Floquet exponents vanishes, $\mu_0 = 0$. The corresponding Floquet function solving the backward equation (10) is a constant function (or vector) $u_0(t) = 1$. If the dynamics of the system is confining, i.e. if the system cannot escape to infinity, a normalizable Floquet function $v_0(t)$ satisfies (7) with $\mu_0 = 0$. It is a periodic function of time with the driving period T. If there is only one such solution, i.e. if $\mu_0 = 0$ is not degenerate, it is nowhere negative and represents the asymptotic distribution that is approached from an arbitrary initial distribution [17]. Due to the conservation of positivity all nonzero Floquet exponents, μ_i, $i > 0$, have a negative real part.

2.2. The instantaneous basis

As a linear operator the master operator $L(t)$ has eigenvalues and eigenfunctions at each fixed time t. These *instantaneous* eigenvalues $\lambda_i(t)$ and eigenfunctions $\psi_i(t)$ are defined by

$$L(t)\psi_i(t) = \lambda_i(t)\psi_i(t). \tag{15}$$

Correspondingly, there are the instantaneous eigenfunctions $\varphi_i(t)$ of the backward operator:

$$L^+(t)\varphi_i(t) = \lambda_i(t)\varphi_i(t). \tag{16}$$

At a fixed time the instantaneous eigenfunctions of the master and the backward operator form a biorthogonal set with respect to the instantaneous scalar product (2) or (3) :

$$(\varphi_i(t), \psi_j(t)) = \delta_{i,j}. \tag{17}$$

Because of the periodicity of $L(t)$, the instantaneous eigenvalues are periodic functions of time:

$$\lambda_i(t+T) = \lambda_i(t), \tag{18}$$

and also the eigenfunctions can be chosen periodic:

$$\psi_i(t+T) = \psi_i(t), \tag{19}$$

$$\varphi_i(t+T) = \varphi_i(t). \tag{20}$$

This can always be achieved by means of an appropriate gauge transformation

$$\psi_i(x,t) \to \mathrm{e}^{\chi(t)}\psi_i(x,t),$$
$$\varphi_i(x,t) \to \mathrm{e}^{-\chi(t)}\varphi_i(x,t), \tag{21}$$

which leaves invariant the instantaneous scalar products (2) and (3) and also the Floquet scalar product (13). Admissible gauge functions $\chi(t)$ are independent of x but otherwise arbitrary.

As for the Floquet functions we also assume completeness of the instantaneous eigenfunctions:

$$\sum_i \psi_i(x,t)\varphi_i(y,t) = \delta(x-y). \tag{22}$$

For a confining dynamics connecting all states $\lambda_0(t) = 0$ is a nondegenerate eigenvalue of both the master and the backward operator. The corresponding eigenfunction of the backward operator is constant, $\varphi_0(t) = 1$ and the one of the master operator $\psi_0(t)$ is nowhere negative and normalized to unity:

$$(\varphi_0(t), \psi_0(t)) = \int \mathrm{d}x \psi_0(x,t) = 1. \tag{23}$$

Hence, the instantaneous eigenfunction $\psi_0(t)$ has the properties of a probability. However, since in general it does not solve the master equation it does not describe a possible time evolution of the considered system. Only in the fully adiabatic limit it coincides with the asymptotic periodic probability of the system [10].

2.3. The Floquet functions in the basis of the instantaneous eigenfunctions

The Floquet functions can be expanded in terms of the respective instantaneous eigenfunctions of $L(t)$ and $L^+(t)$:

$$v_k(x,t) = \sum_i c_{ki}(t)\psi_i(x,t), \tag{24}$$

and correspondingly

$$u_k(x,t) = \sum_i d_{ki}(t)\varphi_i(x,t). \tag{25}$$

From the orthogonality relations (13) and (17) a set of equations relating the coefficients $c_{kj}(t)$ and $d_{kj}(t)$ follows:

$$\frac{1}{T}\sum_i \int_0^T \mathrm{d}t c_{ji}(t) d_{ki}(t) = \delta_{jk}. \tag{26}$$

Using the Floquet equations (7), (10) together with (17) we find the following equations of motion for the coefficients $c_i(t)$ and $d_i(t)$:

$$\dot{c}_{kj}(t) \quad = (\lambda_j(t) - \mu_k)c_{kj}(t) - \sum_i c_{ki}(t)\left(\varphi_j(t), \dot{\psi}_i(t)\right), \tag{27}$$
$$-\dot{d}_{kj}(t) = (\lambda_j(t) - \mu_k)d_{kj}(t) - \sum_i d_{ki}(t)\left(\varphi_i(t), \dot{\psi}_j(t)\right). \tag{28}$$

As a consequence of the periodicity of both the Floquet functions and the instantaneous eigenfunctions, the coefficients also are periodic in time:

$$c_{ki}(t+T) = c_{ki}(t), \tag{29}$$
$$d_{ki}(t+T) = d_{ki}(t). \tag{30}$$

Some exact relations follow from the conservation of normalization. We assume that the Floquet exponent $\mu_0 = 0$ and the instantaneous eigenvalue $\lambda_0(t) = 0$ are not degenerate. Choosing the index $j = 0$ in (27) and using $\partial/\partial t\,(\varphi_0(t), \psi_i(t)) = \left(\varphi_0(t), \dot{\psi}_i(t)\right) = 0$ we find

$$\dot{c}_{k0}(t) = -\mu_k c_{k0}(t). \tag{31}$$

For $\mu_k \neq 0$ this equation only has the trivial periodic solution

$$c_{k0}(t) = 0, \tag{32}$$

and a constant solution for $\mu_0 = 0$ which can be chosen as unity:

$$c_{00}(t) = 1. \tag{33}$$

The remaining coefficients of the asymptotic probability $v_0(t)$ satisfy the following set of inhomogeneous equations

$$\dot{c}_{0j}(t) = \lambda_j(t)c_{0j}(t) - \sum_{i>0} c_{0i}(t)\left(\varphi_j(t), \dot{\psi}_i(t)\right) - \left(\varphi_j(t), \dot{\psi}_0(t)\right). \tag{34}$$

These equations have uniquely defined periodic solutions.

Since $u_0(t) = \varphi_0(t) = 1$ the coefficients $d_{0j}(t)$ are given by:

$$d_{0j}(t) = \delta_{0,j}. \tag{35}$$

All other coefficients result as periodic solutions of the following equations:

$$\dot{c}_{kj}(t) = (\lambda_j(t) - \mu_k)c_{kj}(t) - \sum_{i>0} c_{ki}(t)\left(\varphi_j(t), \dot{\psi}_i(t)\right) - \delta_{k,0}\left(\varphi_j(t), \dot{\psi}_0(t)\right) \quad \text{for } j > 0, \tag{36}$$

$$-\dot{d}_{kj}(t) = (\lambda_j(t) - \mu_k)d_{kj}(t) - \sum_{i>0} d_{ki}(t)\left(\varphi_i(t), \dot{\psi}_j(t)\right) \quad \text{for } k > 0. \tag{37}$$

Note that the index i of the sum in (36) and (37) assumes only positive values because of (35) and $\left(\varphi_0(t), \dot{\psi}_j(t)\right) = 0$.

In general, solving the exact equations (34), (27) and (28) is as difficult as solving the Floquet equations (7), (10).

2.4. The semiadiabatic limit

Motivated by the discussion of the relevant parameter regime for stochastic resonance we now assume that the absolute values of all non-zero instantaneous eigenvalues are much larger than the frequency $\Omega = 2\pi/T$ of the process, with the possible exception of a single instantaneous eigenvalue, say $\lambda_1(t)$:

$$|\lambda_j(t)| \gg \Omega \quad \text{for all} \quad j > 1. \tag{38}$$

A process satisfying this condition is in the *semiadiabatic limit*. The rates of change of the instantaneous eigenfunctions are of the order of Ω and, hence, small compared to the absolute values of all instantaneous eigenvalues $\lambda_i(t)$, $i > 1$. In general this implies that the matrix elements of the time derivatives also are small:

$$|\left(\varphi_i(t), \dot{\psi}_j(t)\right)| \ll \min_{k>1} |\lambda_k(t)|. \tag{39}$$

If the inequalities (38) are satisfied the inequalities (39) may only be violated at isolated times t at which an instantaneous eigenvalue $\lambda_i(t)$ is degenerate with a geometric multiplicity being smaller than the algebraic multiplicity. We will not treat this interesting case here.

Provided the strong form (39) of the semiadiabatic conditions holds, the coupling between equations (34) and (36) is weak and the coefficients $c_{ij}(t)$ can be obtained by means of perturbation theory. Similarly, the coefficients $d_{ij}(t)$ are only weakly coupled in (37) and can also be calculated in perturbation theory.

In the remainder of this section we outline the main steps of the perturbation theory and give approximate results for the coefficients and the Floquet exponents.

2.4.1. The coefficients $c_{0j}(t)$. First we consider (34) with $k = 0$ for $j = 1$ and $j > 1$ separately:

$$\dot{c}_{01}(t) = [\lambda_1(t) - \left(\varphi_1(t), \dot{\psi}_1(t)\right)]c_{01}(t) - \left(\varphi_1(t), \dot{\psi}_0(t)\right) - \sum_{i>1} c_{0i}(t) \left(\varphi_1(t), \dot{\psi}_i(t)\right), \tag{40}$$

$$\dot{c}_{0j}(t) = [\lambda_j(t) - \left(\varphi_1(t), \dot{\psi}_1(t)\right)]c_{0j}(t) - \left(\varphi_j(t), \dot{\psi}_0(t)\right) - \sum_{i>1, i\neq j} c_{0i}(t) \left(\varphi_j(t), \dot{\psi}_i(t)\right) \text{ for } j > 1. \tag{41}$$

According to the semiadiabatic assumption (38), $\lambda_j(t)$ with $j > 1$ is a fast rate. It dominates the decay of $c_{0j}(t)$ in eq. (41) towards its asymptotic, slowly varying form. Consequently, the derivative term $\dot{c}_{0j}(t)$ can be neglected yielding an algebraic equation for the coefficients:

$$c_{0j}(t) = \frac{\left(\varphi_j(t), \dot{\psi}_0(t)\right)}{\lambda_j(t) - \left(\varphi_j(t), \dot{\psi}_j(t)\right)} + \frac{\left(\varphi_j(t), \dot{\psi}_1(t)\right)}{\lambda_j(t) - \left(\varphi_j(t), \dot{\psi}_j(t)\right)} c_{01}(t) + \sum_{i>1, i\neq j} \frac{\left(\varphi_j(t), \dot{\psi}_i(t)\right)}{\lambda_j(t) - \left(\varphi_j(t), \dot{\psi}_j(t)\right)} c_{0i}(t) \quad \text{for} \quad j > 1. \tag{42}$$

The solution of this equation has two terms; one is linear in, and the other one independent of, $c_{01}(t)$:

$$c_{0j}(t) = H_{0j}(t) + H_{1j}(t)c_{01}(t), \tag{43}$$

where $H_{\sigma j}(t)$, $\sigma = 0, 1$ satisfy:

$$H_{\sigma j}(t) = \frac{\left(\varphi_j(t), \dot{\psi}_\sigma(t)\right)}{\lambda_j(t) - \left(\varphi_j(t), \dot{\psi}_j(t)\right)} + \sum_{i>1, i\neq j} \frac{\left(\varphi_j(t), \dot{\psi}_i(t)\right)}{\lambda_j(t) - \left(\varphi_j(t), \dot{\psi}_j(t)\right)} H_{\sigma i}(t). \tag{44}$$

An iterative solution of this equation with the inhomogeneity as starting value yields an expansion in terms of powers of the matrix elements $\left(\varphi_j(t), \dot{\psi}_i(t)\right)$. Including terms up to second order one obtains:

$$H_{\sigma j}(t) = \frac{\left(\varphi_j(t), \dot{\psi}_\sigma(t)\right)}{\lambda_j(t) - \left(\varphi_j(t), \dot{\psi}_j(t)\right)} + \sum_{i>1, i\neq j} \frac{\left(\varphi_j(t), \dot{\psi}_i(t)\right)}{\lambda_j(t) - \left(\varphi_j(t), \dot{\psi}_j(t)\right)} \times \frac{\left(\varphi_i(t), \dot{\psi}_\sigma(t)\right)}{\lambda_i(t) - \left(\varphi_i(t), \dot{\psi}_i(t)\right)} + \ldots \tag{45}$$

Putting (43) into (40) one finds an inhomogeneous differential equation for $c_{01}(t)$ of the form:

$$\dot{c}_{01}(t) = \nu_{01}(t)c_{01}(t) + h_{01}(t), \tag{46}$$

where

$$\nu_{01}(t) = \lambda_1(t) - \left(\varphi_1(t), \dot{\psi}_1(t)\right) - \sum_{i>1} H_{1i}(t) \left(\varphi_1(t), \dot{\psi}_i(t)\right), \tag{47}$$

and

$$h_{01}(t) = -\left(\varphi_1(t), \dot{\psi}_0(t)\right) - \sum_{i>1} H_{0i}(t) \left(\varphi_1(t), \dot{\psi}_i(t)\right). \tag{48}$$

The rate $\nu_{01}(t)$ may in general be of the same order of magnitude as the driving frequency Ω or even smaller. Therefore one must resort to the exact periodic solution of eq. (46):

$$c_{01}(t) = \frac{\int_0^T ds \exp\left\{\int_s^T ds' \nu_{01}(s')\right\} h_{01}(s)}{1 - \exp\left\{\int_0^T ds \nu_{01}(s)\right\}} \exp\left\{\int_0^t ds \nu_{01}(s)\right\} + \int_0^t ds \exp\left\{\int_s^t ds' \nu_{01}(s')\right\} h_{01}(s). \tag{49}$$

This is a unique solution provided $\nu_{01}(t) \neq 0$.

2.4.2. The coefficients $c_{1j}(t)$ and the Floquet exponent μ_1. First we consider the equation for $c_{1j}(t)$ with $j > 1$:

$$\dot{c}_{1j}(t) = [\lambda_j(t) - \mu_1 - \left(\varphi_j(t), \dot{\psi}_j(t)\right)] c_{1j}(t) - \left(\varphi_j(t), \dot{\psi}_1(t)\right) c_{11}(t) - \sum_{i>1, i\neq j} c_{1i}(t) \left(\varphi_j(t), \dot{\psi}_i(t)\right) \text{ for } j > 1. \tag{50}$$

Since the rate $\lambda_j(t)$ with $j > 1$ is fast and dominates the other terms in the square bracket, the derivative $\dot{c}_{1j}(t)$ can be neglected to yield:

$$c_{1j}(t) = A_{j1}(\mu_1, t) c_{1j}(t) + \sum_{i>1, i\neq j} A_{ji}(\mu_1, t) c_{1i}(t), \tag{51}$$

where

$$A_{ji}(z, t) = \frac{\left(\varphi_j(t), \dot{\psi}_i(t)\right)}{\lambda_j(t) - z - \left(\varphi_j(t), \dot{\psi}_j(t)\right)}. \tag{52}$$

The solution of this equation in terms of $c_{11}(t)$ is:

$$c_{1j}(t) = R_{1j}(t) c_{11}(t), \tag{53}$$

where the factor $R_{ij}(t)$ obeys a set of inhomogeneous equations:

$$R_{1j}(t) = A_{j1}(\mu_1, t) + \sum_{i>1, i\neq j} A_{ji}(\mu_1, t) R_{1i}(t). \tag{54}$$

Inserting (53) into (36) with $k = j = 1$ yields a homogeneous differential equation for $c_{11}(t)$:

$$\dot{c}_{11}(t) = [\nu_{11}(t) - \mu_1] c_{11}(t), \tag{55}$$

where

$$\nu_{11}(t) = \lambda_1(t) - \left(\varphi_1(t), \dot{\psi}_1(t)\right) - \sum_{i>1} \left(\varphi_1(t), \dot{\psi}_i(t)\right) R_{1i}(t). \tag{56}$$

In order that (55) possess a periodic solution the time average of $\nu_{11}(t)$ must coincide with μ_1:

$$\mu_1 = \frac{1}{T}\int_0^T \mathrm{d}t\nu_{11}(t). \tag{57}$$

This is an implicit equation for μ_1 which still depends on the yet unknown factors $R_{1j}(t)$. These can be found iteratively from (54). The first iteration yields:

$$R_{1j}(t) = A_{j1}(\mu_1, t) + \sum_{i>1, i\neq j} A_{ji}(\mu_1, t)A_{i,1}(\mu_1, t). \tag{58}$$

In a first step, we neglect the sum in (56) when calculating the time average of $\nu_{11}(t)$. This yields a first approximation of μ_1 which can be inserted into (58). The resulting expression for $R_{1j}(t)$ yields with (56) an improved expression for $\nu_{11}(t)$ which can be put into (57) yielding a better value for μ_1. This procedure can be repeated iteratively. Including corrections of first order from the higher instantaneous states one finds for the first nonvanishing Floquet exponent:

$$\mu_1^{(1)} = \frac{1}{T}\int_0^T \mathrm{d}t\left\{\lambda_1(t) - \left(\varphi_1(t), \dot\psi_1(t)\right) - \sum_{i>1}\left(\varphi_1(t), \dot\psi_i(t)\right) A_{i1}(\mu_1^{(0)}, t)\right\}, \tag{59}$$

where the zeroth order approximation is given by:

$$\mu_1^{(0)} = \int_0^T ds[\lambda_1(s) + \left(\varphi_1(s), \dot\psi_1(s)\right)] \tag{60}$$

Since $|\mu_1^{(0)}|, \left(\varphi_j(t), \dot\psi_1(t)\right) \ll |\lambda_j(t)|$ one finds that $A_{j1}(\mu_1^{(0)}, t) = O(\Omega/\lambda_j(t))$. Therefore the the iterative solution of (51) and (57) is *a posteriori* justified. In particular, the sum term on the right hand side of (59) is a small correction in the semiadiabatic limit. A more detailed analysis of this term is given in Appendix B for the particular example discussed in section 3. For $c_{11}(t)$ one finally obtains

$$c_{11}(t) = \exp\left\{\int_0^t ds[\nu_{11}(s) - \mu_1]\right\} c_{11}(0). \tag{61}$$

Because of (57) this is indeed a periodic function of time with the period T.

2.4.3. The coefficients $c_{kj}(t)$ and Floquet exponents μ_k for $k > 1$. All other equations for the coefficients $c_{ij}(t)$ with $i > 1$ can be treated in an analogous way: the time derivatives of the nondiagonal coefficients with $i \neq j$ can be neglected leading to algebraic equations that can be solved in terms of the diagonal coefficients. Putting these expressions into the equations for the diagonal coefficients one finds decoupled homogeneous differential equations of first order. They possess periodic solutions only if the respective rate coefficients vanish when averaged over a period. This gives implicit equations for the corresponding Floquet exponents which can be solved iteratively, as explained above for μ_1.

We summarize the results. The nondiagonal coefficients $c_{kj}(t)$ with $k > 0$ are proportional to the diagonal ones:

$$c_{kj}(t) = R_{kj}(t)c_{kk}(t), \tag{62}$$

where the functions $R_{kj}(t)$ are the solutions of the following set of algebraic equations:

$$R_{kj}(t) = A_{jk}(\mu_k, t) + \sum_{i>0, i\neq k,j} A_{j,i}(\mu_k, t)R_{ki}(t). \tag{63}$$

With the inhomogeneity as starting value, one iteration of this equation gives:

$$R_{kj}(t) = A_{jk}(\mu_k, t) + \sum_{i>0, i\neq j} A_{ji}(\mu_k, t) A_{ik}(\mu_k, t) + ... \tag{64}$$

The diagonal coefficients are:

$$c_{kk}(t) = \exp\left\{\int_0^t ds[\nu_{kk}(s) - \mu_k]\right\} c_{kk}(0), \tag{65}$$

where

$$\nu_{kk}(t) = \lambda_k(t) - \left(\varphi_k(t), \dot{\psi}_k(t)\right) - \sum_{i>0, i\neq k} \left(\varphi_k(t), \dot{\psi}_i(t)\right) R_{ki}(t). \tag{66}$$

Requiring the periodicity of the diagonal elements one finds an implicit equation for each Floquet exponent μ_k:

$$\mu_k = \frac{1}{T}\int_0^T \mathrm{d}t\nu_{kk}(t). \tag{67}$$

Using (66) one obtains after one iteration of (64):

$$\mu_k = \frac{1}{T}\int_0^T \mathrm{d}t\left\{\lambda_k(t) - \left(\varphi_k(t), \dot{\psi}_k(t)\right) - \sum_{i>0, i\neq k} \left(\varphi_k(t), \dot{\psi}_i(t)\right) A_{ik}(\mu_k^{(1)}, t) + ...\right\}, \tag{68}$$

where

$$\mu_k^{(1)} = \frac{1}{T}\int_0^T \mathrm{d}t[\lambda_k(t) + \left(\varphi_k(t), \dot{\psi}_k(t)\right)] \tag{69}$$

is the Floquet exponent including corrections up to first order in Ω/λ_k.

2.4.4. The coefficients $d_{kj}(t)$ For the coefficients $d_{kj}(t)$ one can proceed in an analogous way. For all $k > 0$, $k \neq j$ except $k = 1$, $j = 0$ one finds:

$$d_{kj}(t) = Q_{kj}(t) d_{kk}(t), \tag{70}$$

where

$$Q_{kj}(t) = A_{kj}(\mu_k, t) + \sum_{i>0, i\neq j,k} Q_{ki}(t) A_{ij}(\mu_k, t), \tag{71}$$

with $A_{kj}(z, t)$ as defined in (52). The diagonal coefficients become:

$$d_{kk}(t) = \exp\left\{-\int_0^t ds[\nu_{kk}(s) - \mu_k]\right\} d_{kk}(0) \quad \text{for all } k > 0. \tag{72}$$

The rates ν_{kk} are defined in (66). For the coefficient $d_{10}(t)$ one finds:

$$\begin{aligned} d_{10}(t) = \frac{\mathrm{e}^{\mu_1 t}}{1 - \mathrm{e}^{\mu_1 T}} \int_0^T ds\mathrm{e}^{\mu_1(T-s)} d_{11}(s) \left[\left(\varphi_1(s), \dot{\psi}_0(s)\right) + \sum_{i>1} Q_{1i}(s) \left(\varphi_i, \dot{\psi}_0(s)\right)\right] \\ + \int_0^t ds\mathrm{e}^{\mu_1(t-s)} d_{11}(s) \left[\left(\varphi_1(s), \dot{\psi}_0(s)\right) + \sum_{i>1} Q_{1i}(s) \left(\varphi_i, \dot{\psi}_0(s)\right)\right]. \end{aligned} \tag{73}$$

Finally we note that without loss of generality, one can chose unity for all constants $d_{kk}(0)$:

$$d_{kk}(0) = 1. \tag{74}$$

The remaining constants $c_{kk}(0)$ then follow from (26):

$$c_{kk}(0) = \left\{ 1 + \frac{1}{T} \sum_{j>0, j\neq k} \int_0^T \mathrm{d}t R_{kj}(t) Q_{kj}(t) \right\}^{-1}. \tag{75}$$

This completes the semiadiabatic theory for the simplest possible case where all contributions coming from large instantaneous eigenvalues can be treated adiabatically. Closer inspection shows that this is only correct if the distances between all instantaneous eigenvalues but the first two are large compared to the driving frequency Ω. The cases of crossings of instantaneous eigenvalues with defect index zero and of avoided crossings, which are excluded in the present treatment, can be handled by standard degenerate perturbation theory. This question and the more severe problem of crossings with a nonzero defect index will be treated elsewhere.

3. Application to stochastic resonance

As an application of the above developed theory we treat a bistable system that is coupled to a heat bath and driven by an external periodic force. The crucial assumption is that the driving period is long compared to the internal relaxation times of the system. In order to further simplify the discussion we assume that the periodic force alone is too weak to cause transitions between the two states, and that the noise is weak and therefore only rarely induces transitions.

3.1. The model

Here we consider the archetypical model of stochastic resonance of an overdamped Brownian particle in a symmetric double well. The particle moves according to a Langevin equation:

$$\dot{x}(t) = -\frac{\partial}{\partial x} U(x,t) + \xi(t), \tag{76}$$

where $x(t)$ denotes the particle's position at time t, and $\xi(t)$ is zero mean Gaussian white noise with thermal energy β^{-1} characterizing the noise strength:

$$\langle \xi(t) \rangle = 0, \qquad \langle \xi(t)\xi(s) \rangle = 2\beta^{-1}\delta(t-s). \tag{77}$$

The time dependent potential $U(x,t)$ consists of a symmetric double well potential and the potential of the periodic driving force with frequency Ω, see figure 1:

$$U(x,t) = \frac{1}{4}(x^2-1)^2 - Ax\cos(\Omega t). \tag{78}$$

The dimensionless amplitude A measures the strength of the driving force. We assume that A is so small that the potential always has two minima, i.e.:

$$A < \frac{2}{3\sqrt{3}}. \tag{79}$$

The local minima $x_\pm(t)$ and the local maximum $x_0(t)$ are solutions of the equation:

$$x^3 - x - A\cos(\Omega t) = 0. \tag{80}$$

The values of the local extrema of the potential are denoted by

$$U_\sigma(t) = U(x_\sigma(t), t), \qquad \sigma = +, -, 0, \tag{81}$$

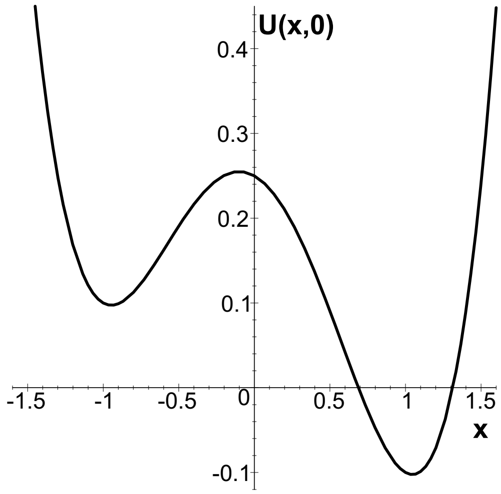

Figure 1. Potential $U(x,0)$ for $A = 0.1$.

and the respective curvatures by:

$$\omega_b^2(t) = -\frac{\partial^2}{\partial x^2}U(x_0(t),t),$$
$$\omega_\pm^2(t) = \frac{\partial^2}{\partial x^2}U(x_\pm(t),t). \tag{82}$$

The barrier heights as seen from the wells are given by:

$$V_\pm(t) = U_0(t) - U_\pm(t). \tag{83}$$

The master operator $L(t)$ for the process defined by the Langevin equation (76) is given by the following Fokker–Planck operator [14]:

$$L(t) = \frac{\partial}{\partial x}U'(x,t) + \beta^{-1}\frac{\partial^2}{\partial x^2}. \tag{84}$$

This operator is invariant under a joint inversion of space and a shift of time by half a period, $x \to -x$ and $t \to t + T/2$. Accordingly, both the Floquet functions and the instantaneous eigenfunctions have definite parities under this symmetry. Since the Fokker–Planck operator $L(t)$ is an even function of time, all instantaneous eigenvalues and eigenfunctions also are even functions of time.

3.2. The first instantaneous eigenvalues and eigenfunctions

The instantaneous groundstate $\psi_0(x,t)$ of the Fokker–Planck operator is given by the Boltzmann distribution:

$$\psi_0(x,t) = Z^{-1}(t)\exp\{-\beta U(x,t)\}, \tag{85}$$

where for low bath temperatures, the partition function is given by:

$$Z(t) = \sqrt{\frac{2\pi}{\beta}}\left\{\frac{\exp\{-\beta U_+(t)\}}{\omega_+(t)} + \frac{\exp\{-\beta U_-(t)\}}{\omega_-(t)}\right\}. \tag{86}$$

The Fokker–Planck operator $L(t)$ satisfies an instantaneous version of detailed balance with respect to this ground state [18]:

$$L(t)\psi_0(x,t) = \psi_0(x,t)L^+(t). \tag{87}$$

As a consequence the instantaneous eigenfunctions of the Fokker–Planck and the backward operator are simply related to each other:

$$\psi_i(x,t) = \varphi_i(x,t)\psi_0(x,t). \tag{88}$$

In the weak noise limit the first eigenfunction $\varphi_1(x,t)$ of the backward operator is well approximated by a linear combination of a constant function and an error function satisfying the homogeneous backward equation [19, 20]:

$$\varphi_1(x,t) = B(t) + \frac{C(t)}{\sqrt{2\pi}} \int_{-\infty}^{\omega_b(t)\sqrt{\beta}(x-x_0(t))} \mathrm{d}z \mathrm{e}^{-z^2/2}. \tag{89}$$

This function approaches the value $B(t)$ for $x \to -\infty$ and $B(t) + C(t)$ for $x \to \infty$. The functions $B(t)$ and $C(t)$ have to be such that the biorthonormality relations (17) are satisfied, i.e. that the scalar product of $\varphi_1(x,t)$ and $\psi_0(x,t)$ vanishes and that of $\varphi_1(x,t)$ and $\psi_1(x,t) = \varphi_1(x,t)\psi_0(x,t)$ is normalized to unity. In leading order in β one obtains from these requirements:

$$B(t) = \sqrt{(r(t)}, \tag{90}$$

$$C(t) = -\frac{1+r(t)}{\sqrt{r(t)}}, \tag{91}$$

where

$$r(t) = \frac{\omega_-(t)}{\omega_+(t)} \exp\left\{-\beta(U_+(t) - U_-(t))\right\}. \tag{92}$$

The corresponding instantaneous eigenvalue $\lambda_1(t)$ is given by the sum of the rates $k_\pm(t)$ which characterize the transitions from left to right and vice versa in the frozen potential $U(x,t)$:

$$\lambda_1(t) = -(k_+(t) + k_-(t)). \tag{93}$$

In the weak noise limit, the rates are given by the Kramers expressions [19, 20]:

$$k_\pm(t) = \frac{\omega_b(t)\omega_\pm(t)}{2\pi} \exp\left\{-\beta V_\pm(t)\right\}. \tag{94}$$

One can construct higher eigenfunctions in a WKB like way but we will not pursue this question further since we will not need them here. We only note that the higher instantaneous eigenvalues are of the order of the well frequencies $\omega_\pm(t)$ and integer multiples thereof.

3.3. The first Floquet functions

Based on the assumption that the thermal energy of the bath is much smaller than the minimal barrier one expects a large gap between the first two Floquet exponents and the remaining ones. Using the lowest approximation for the Floquet exponents, i.e. $\mu_k^{(0)} = \int_0^T \mathrm{d}t\lambda_k(t)/T$, one indeed finds an exponential separation of the first exponent from the rest: $|\mu_2|/|\mu_1| = O(\exp\{\beta V_{\min}\}$, where $V_{\min} = \min_{t\in(0,T)} V_+(t)$. Consequently, the long-time dynamics that characterizes the relevant aspects of stochastic resonance is governed by the first two Floquet exponents and the corresponding Floquet states, which will be determined in the following subsections.

3.3.1. The Floquet groundstate Using (24) we expand the Floquet groundstate $v_0(x,t)$ belonging to the Floquet exponent $\mu_0 = 0$ in terms of the instantaneous eigenfunctions:

$$v_0(x,t) = \sum_i c_{0i}(t)\psi_i(x,t). \tag{95}$$

According to (33), $c_{00}(t) = 1$. The coefficient $c_{01}(t)$ is given by (47)–(49). The remaining coefficients $c_{0j}(t)$ are smaller than the first two by roughly a factor of $\Omega/\lambda_2(t) \sim \Omega/\omega_\pm$, see (42), and can therefore be neglected in (95). Hence, we find for the Floquet groundstate a superposition of the first two instantaneous eigenstates:

$$v_0(x,t) = \psi_0(x,t) + c_{01}(t)\psi_1(x,t). \tag{96}$$

In the expressions for $\nu_{01}(t)$ and $h_{01}(t)$ which determine $c_{01}(t)$ the contributions from the higher instantaneous eigenfunctions are also smaller than those from the instantaneous eigenvalue $\lambda_1(t)$ and from the matrix elements involving the first two instantaneous eigenfunctions. Neglecting the contributions of the higher instantaneous eigenfunctions which are of the order $O(\Omega/\lambda_k(t))$, $k > 1$, we find for the functions $\nu_{01}(t)$ and $h_{01}(t)$:

$$\nu_{01}(t) = \lambda_1(t) - \left(\varphi_1(t), \dot{\psi}_1(t)\right), \tag{97}$$

$$h_{01}(t) = -\left(\varphi_1(t), \dot{\psi}_0(t)\right). \tag{98}$$

Now one has to determine the matrix elements $\left(\varphi_1(t), \dot{\psi}_0(t)\right)$ and $\left(\varphi_1(t), \dot{\psi}_1(t)\right)$. Up to exponentially small corrections they are given by:

$$\left(\varphi_1(t), \dot{\psi}_0(t)\right) = -\dot{B(t)} + B(t)\frac{\dot{C(t)}}{C(t)}, \tag{99}$$

and

$$\left(\varphi_1(t), \dot{\psi}_1(t)\right) = -\frac{\dot{C(t)}}{C(t)}. \tag{100}$$

Details of the derivation are explained in Appendix A.

Finally it remains to perform the various time integrals in (49). In general this has to be done numerically. As a result, we show in figure 2 the Floquet function $v_0(x,t)$ for three different noise strengths. The frequency and amplitude of the external driving force are kept fixed. For the largest noise we find the maximum of $v_0(x,t)$ moving in phase with the driving force. In this case the coefficient $c_{01}(t)$ is small and consequently $v_0(x,t)$ almost coincides with the instantaneous eigenfunction $\psi_0(x,t)$. The system then is in the fully adiabatic regime. With decreasing noise, the instantaneous eigenvalue $\lambda_1(t)$ rapidly decreases and consequently $c_{01}(t)$ increases and additionally gets out of phase with the driving force. Thus the instantaneous eigenfunction $\psi_1(x,t)$ increasingly contributes to the Floquet ground state $v_0(x,t)$ in a way that leads the Floquet ground state to staying behind the driving force. We note that changing the amplitude does not alter the qualitative behaviour while for obvious reasons an increase of the driving frequency Ω has a similar result as decreasing the noise.

We recall that the Floquet function of the backward operator corresponding to $\mu_0(t) = 0$ is constant, see the last paragraph of section 2.1:

$$u_0(x,t) = 1. \tag{101}$$

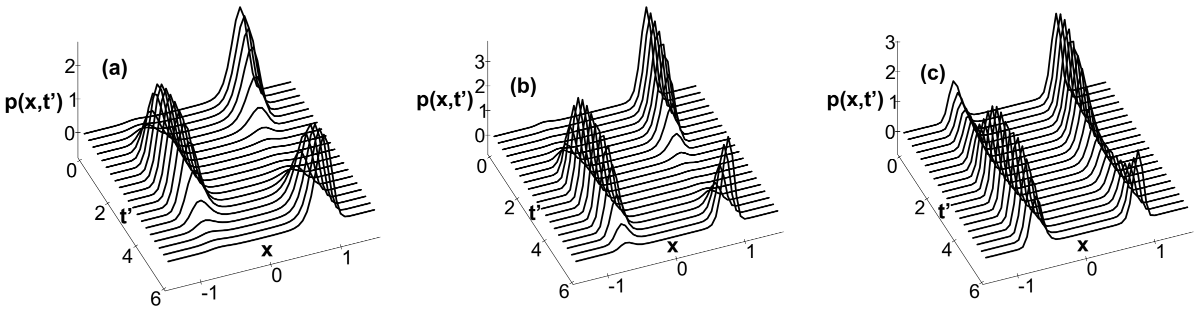

Figure 2. Floquet ground state $v_0(x,t)$ which equals the asymptotic probability density $p(x,t)$, see (113) for $A = 0.1$, $\Omega = 10^{-4}$ and $\beta = 20$ (a), $\beta = 40$ (b), and $\beta = 50$ (c).

3.3.2. The first excited Floquet exponent and the corresponding Floquet functions. The Floquet exponent μ_1 given by (60) can be further simplified. The time-integral of the matrix element $\left(\varphi_1(t), \dot{\psi}_1(t)\right)$ over a full period vanishes because the instantaneous eigenfunctions are even under time reversal, see section 3.1. Consequently, we find:

$$\mu_1^{(0)} = \frac{1}{T}\int_0^T \mathrm{d}t\lambda_1(t), \tag{102}$$

where $\lambda_1(t)$ is given by (93) and (94). The relative error of $\mu_1^{(0)}$ is of the order of $O(\Omega/\omega)$, see Appendix B. The integral can approximately be done in the limiting cases of a small amplitude A of the driving force, and weak noise. In the small amplitude limit one obtains

$$\mu_1 = -\frac{\sqrt{2}}{\pi}\mathrm{e}^{-\beta/4}\left\{1 + \frac{1}{4}(A\beta)^2(1 - \frac{3}{\beta} - \frac{69}{16\beta^2}) + O((A\beta)^4)\right\}. \tag{103}$$

Here βA is the relevant small quantity. On the other hand, in the weak noise limit, i.e. for $\beta \to \infty$, one finds:

$$\mu_1 = -\frac{\omega_b(0)\omega_-(0)}{\pi\sqrt{2\pi(x_0 - x_-(0))A\beta}}\mathrm{e}^{-\beta V_-(0)}. \tag{104}$$

So in both limits an Arrhenius type behaviour results, see figure 3.

The crossover between the asymptotic limits is rather gradual. In particular, for small values of A it is almost invisible since the 'activation energy' $V_-(0)$ differs only very little from the static value $1/4$.

For the first excited Floquet functions we find:

$$v_1(x,t) = c_{11}(t)\psi_1(x,t), \tag{105}$$
$$u_1(x,t) = d_{10}(t) + d_{11}(t)\varphi_1(x,t), \tag{106}$$

where we have neglected contributions from the higher instantaneous eigenfunctions which are smaller by a factor of $\Omega/\omega_\pm$.

Within the same approximation, the rates $\nu_{01}(t)$ and $\nu_{11}(t)$ coincide. The coefficient $c_{11}(t)$ then becomes:

$$c_{11}(t) = \exp\left\{\int_0^t \mathrm{d}s[\nu_{01}(s) - \mu_1]\right\}. \tag{107}$$

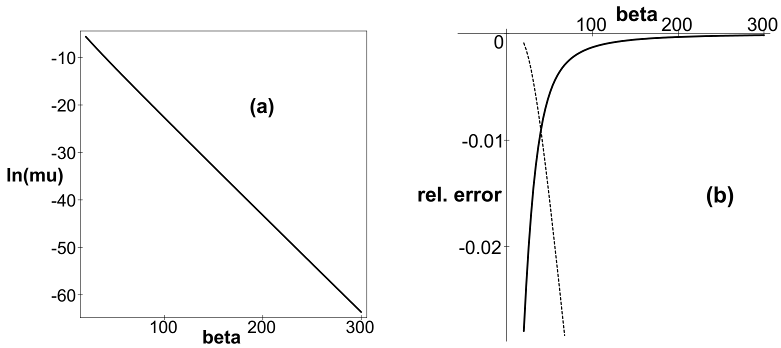

Figure 3. (a) Logarithm of the negative Floquet exponent $-\mu_1$ as a function of the inverse noise strength β for $A = 0.05$. Deviations from a linear Arrhenius plot are hardly detectable. In panel (b) the relative deviation of the logarithm of the negative Floquet exponent between the weak noise approximation (104) and the full result (102) is shown as solid line. The broken line shows the respective deviation of the small amplitude approximation (103). For the present choice of parameters, the weak noise approximation gives a reliable approximation throughout the whole range of noise for $\beta \geq 20$.

Accordingly, the coefficients $d_{11}(t)$ and $d_{10}(t)$ simplify to read:

$$d_{11}(t) = \exp\left\{-\int_0^t \mathrm{d}s[\nu_{01}(t) - \mu_1]\right\}, \tag{108}$$

$$\begin{aligned} d_{10}(t) = \frac{\mathrm{e}^{\mu_1 t}}{\mathrm{e}^{-\mu_1 T} - 1}\Bigg\{ &\int_0^T \mathrm{d}s \exp\left\{-\int_0^s \mathrm{d}s'\nu_{01}(s')\right\}\left(\varphi_1(s), \dot{\psi}_0(s)\right) \\ &+ \int_0^t \mathrm{d}s \exp\left\{-\int_0^s \mathrm{d}s'\nu_{01}(s')\right\}\left(\varphi_1(s), \dot{\psi}_0(s)\right)\Bigg\}. \end{aligned} \tag{109}$$

3.3.3. The conditional probability On the timescale of the period of the driving force the contributions of the higher eigenvalues have died out and only the two lowest states contribute to the conditional probability (12):

$$P(x, t|y, s) = v_0(x, t)u_0(y, s) + \mathrm{e}^{\mu_1(t-s)}v_1(x, t)u_1(y, s). \tag{110}$$

Within the quasi-adiabatic approximation it can be expressed in terms of the instantaneous eigenfunction:

$$\begin{aligned} P(x, t|y, s) = \psi_0(x, t)\Big\{ &1 + \left[c_{01}(t) + \mathrm{e}^{\mu_1(t-s)}c_{11}(t)d_{10}(s)\right]\varphi_1(x, t) \\ &+ \mathrm{e}^{\mu_1(t-s)}\varphi_1(x, t)\varphi_1(y, s)\Big\}. \end{aligned} \tag{111}$$

Conditioning at $s = 0$ and choosing t just a period later we find for the stroboscopic transition probability $P(x, T|y, 0)$:

$$P(x, T|y, 0) = \psi_0(x, 0)\Big\{1 + \left[c_{01}(0) + \mathrm{e}^{\mu_1 T}d_{10}(0)\right]\varphi_1(x, 0)$$

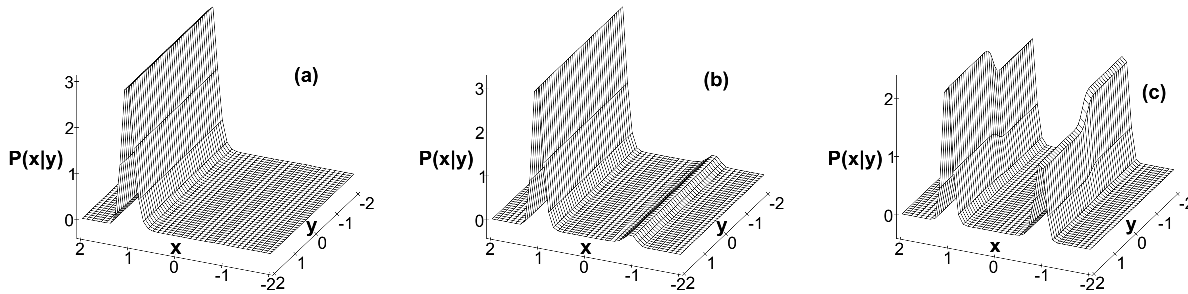

Figure 4. Stroboscopic probability $P(x,T|y)$ for $\Omega = 10^{-4}$. For $\beta = 30$ displayed in panel (a) the transitions between the two wells are fast enough such that the asymptotic state is reached within a period of the external driving force. In this case and also for $\beta = 40$ displayed in panel (b) the probability after a period therefore is independent of the initial state y at time $s = 0$. For $\beta = 40$ a small peak at the higher potential minimum at $x = -1$ indicates a slight mismatch between the process and the driving force. For $\beta = 50$ shown in panel (c), the asymptotic state is not reached after one period and consequently the stroboscopic probability shows a clear dependence on the initial state y.

$$+ \mathrm{e}^{\mu_1 T}\varphi_1(x,0)\varphi_1(y,0)\Big\}. \tag{112}$$

Figure 4 shows this stroboscopic probability for three different values of the noise strengths and fixed parameters of the driving force. Only for rather small noise strength the probability of finding the particle after a period at the position x depends on the the starting point y. Decreasing the amplitude of the driving force again weakens the bias between the two states and introduces a growing probability to find the system in the less favourable well.

If the initial time lies in the remote past, $s \to -\infty$, the conditional probability becomes independent of the condition y and yields a uniquely defined asymptotic probability density $p(x,t)$ which coincides with the Floquet function $v_0(x,t)$:

$$p(x,t) = \lim_{s\to-\infty} P(x,t|y,s) = v_0(x,t). \tag{113}$$

Hence, figure 2 shows the asymptotic probability density for different values of the noise strength and fixed driving force parameters. One finds that for sufficiently large noise the population is in phase with the driving force and predominantly stays within the more favourite well. With decreasing noise, the population increasingly lags behind the driving force and the probability to find the particle in the less favourite well raises until it only weakly depends on time showing some minor time-dependent modulation.

3.3.4. The first moments and the probability current The asymptotic probability density can be characterized by the moments of x:

$$\langle x(t)^n\rangle_{as} = \int_{-\infty}^{\infty} \mathrm{d}x x^n v_0(x,t), \quad n = 1,2,... \tag{114}$$

The first two moments are shown in figure 5 for various parameter values. One again sees how they get out of phase with the driving force if the noise becomes weaker. The transition of the

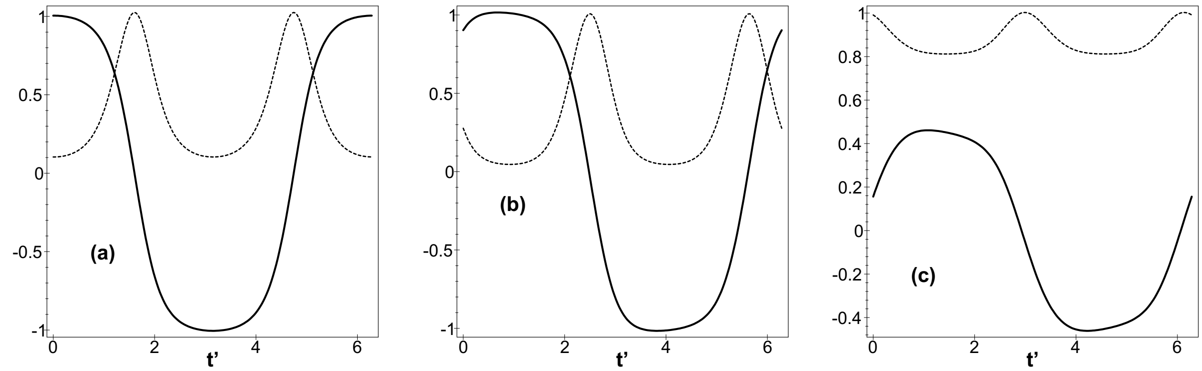

Figure 5. Average position $\langle x(t)\rangle$ (solid line) and its variance $\langle x^2(t)\rangle - \langle x(t)\rangle^2$ (dotted line) as a function of the scaled time $t' = \Omega t$ for $A = 0.1$, $\Omega = 10^{-4}$ for different noise strength $\beta = 20$ (a), $\beta = 40$ (b), and $\beta = 50$ (c). Note that with decreasing noise the average position stays behind the external forcing. The variance goes through a maximum when the average position passes through zero. With decreasing noise the minima of the variance first slightly decrease and then strongly increase.

mean value of the position from one well to the other is always accompanied by an increase of the second moment since then the probability of finding the particle in either well is equally high [21].

The probability current is yet another quantity that describes the transitions between the wells. It is given by

$$J(x,t) = -\frac{\partial U(x,t)}{\partial x}p(x,t) - \beta^{-1}\frac{\partial p(x,t)}{\partial x}. \tag{115}$$

For the asymptotic probability density at the location $x_0(t)$ of the instantaneous potential maximum the probability current becomes:

$$J(t) = J(x_0(t),t) = \frac{k_-(t)}{\sqrt{r(t)}}c_{01}(t). \tag{116}$$

In figure 6 the probability current $J(t)$ is shown for different parameter values. With decreasing noise the extrema of $J(t)$ are shifting to later times within a period. This is in complete agreement with the increasing lag of the first two moments relative to the driving frequency as the noise decreases. More remarkable is the behaviour of the maximal value of the probability current which displays a nonmonotonic behaviour as a function of the noise strength. It is largest in the region where the spectral amplification η explained below shows a maximum.

The spectral amplification η is defined as the integrated power of the peaks of the time averaged spectral density at the frequencies $\pm\Omega$ measured in units of the power of the driving field A^2 [5]. It can be expressed in terms of the first Fourier coefficient of the average position $\langle x(t)\rangle = \sum_{n=\infty}^{\infty} M_n \mathrm{e}^{\mathrm{i}n\Omega t}$, [5]:

$$\eta = \left(\frac{2|M_1|}{A}\right)^2. \tag{117}$$

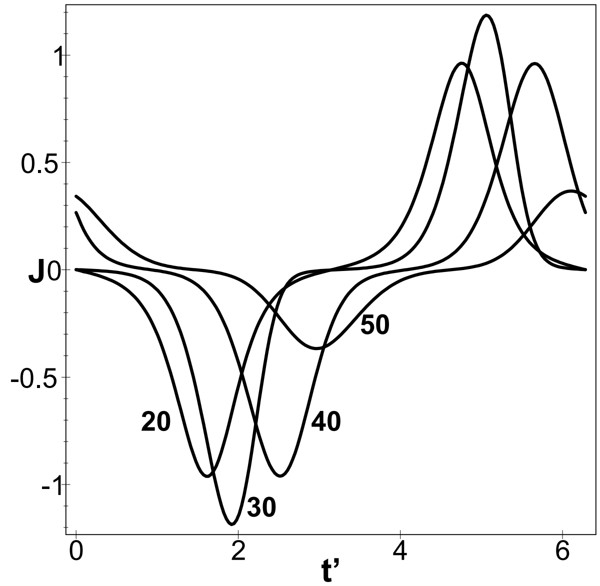

Figure 6. Probability current at the position $x_0(t)$ for $A = 0.1$, $\Omega = 10^{-4}$ and different noise strengths which are indicated by the number next to each curve. With decreasing noise the maximum of the probability current first increases and then decreases. The location of the extrema coincide with the zeros of the average position.

Figure 7 shows the spectral amplification as a function of the noise strength β^{-1} at a fixed amplitude A for different frequencies Ω. The fully adiabatic limit provides an upper bound on η which is approached for sufficiently large noise. At small noise the spectral amplification rapidly vanishes. In the turnover region between these regimes one finds the maximal spectral amplification. This maximum increases and moves towards smaller noise intensities when the frequency Ω decreases. According to the simple time-scale matching condition mentioned in the introduction, see also [5], one expects to find the maximal spectral amplification when the period coincides with twice the average waiting time $T(\beta)$ in a well of the symmetric potential in the absence of a periodic force. For the potential considered here $T(\beta) = \sqrt{2}\pi \exp\{\beta/4\}$. In figure 8 we show the spectral amplification as a function of $x = -\ln(T(\beta)\Omega/\pi) = -\ln(\sqrt{2}\Omega) - \beta/4$. According to the time-scale matching condition the maximum of $\eta(x)$ is expected to lie at $x = 0$ independent of the value of the other parameters Ω and A. As can be seen in figure 8 deviations from this expected behaviour may occur. Therefore the time-scale matching condition can only be considered as a rule of thumb that qualitatively indicates where the optimal stochastic resonance conditions can be expected.

The dependence of the spectral amplification on the amplitude is shown in figure 9 as it approaches the fully adiabatic value with decreasing frequency Ω. For sufficiently large frequencies the spectral amplification has a maximum at a finite amplitude A [5]. With decreasing frequency, the maximum moves to $A = 0$ and a monotonic behaviour results.

Figure 10 shows the spectral amplification as a function of β^{-1} at a fixed frequency for different values of the amplitude A. With decreasing values of A the amplification approaches

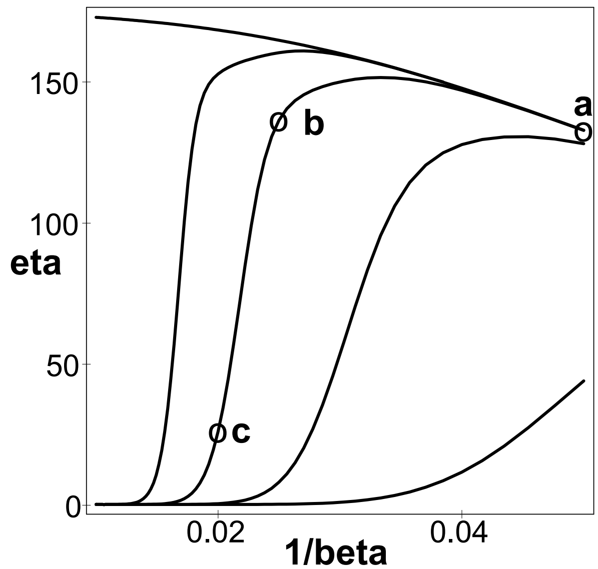

Figure 7. Spectral amplification η as a function of the noise strength β^{-1} for $A = 0.1$ and different values of the driving frequency $\Omega = 10^{-n}$. The uppermost curve represents the adiabatic limit corresponding to $n = \infty$. The following curves correspond to $n = 5$, $n = 4$, $n = 3$, and $n = 2$ from top to bottom. With increasing frequency the maximum of the spectral amplification decreases and moves towards larger noise strengths. For $\Omega = 0.01$ the maximal amplification will occur at a noise strength for which the present weak noise theory does not hold. The circles on the curve for $\Omega = 10^{-4}$ indicate the spectral amplification for the parameter values for which the asymptotic probabilities are shown in figure 2 and for the average positions in figure 5. The letters a, b, and c indicate the corresponding parts of these figures.

its limiting form as predicted by linear response theory in the limit of weak noise [7, 22]:

$$\eta = \frac{4\beta^2 k_0^2(b)}{4k_0^2(\beta) + \Omega^2}. \tag{118}$$

4. Summary

A theory for periodically driven Markovian processes in the semiadiabatic limit has been developed. It is based on the expansion of the Floquet functions of the master operator in terms of the instantaneous eigenfunctions of the same operator. By definition, in the semiadiabatic limit all but two of the instantaneous eigenvalues of the master operator have a much larger absolute value compared to the external driving frequency. This assumption makes possible the adiabatic elimination of most of the expansion coefficients. The ensuing algebraic equations can be solved in a perturbative way. In the present treatment possible complications have been ignored which might result from degenerate, or quasi degenerate instantaneous eigenvalues. They will be treated in a subsequent publication.

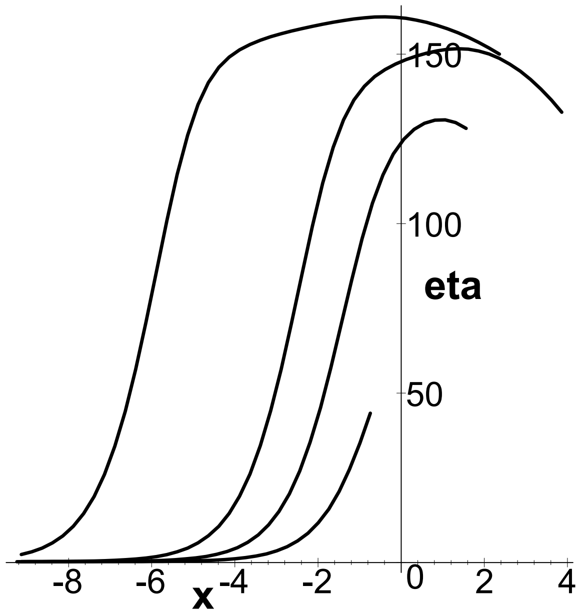

Figure 8. Spectral amplification as a function of the scaled noise strength $x = -\ln(\sqrt(2)\Omega) - \beta/4$ for $A = 0.1$ and different driving frequencies $\Omega = 10^{-5}$, $\Omega = 10^{-4}$, $\Omega = 10^{-3}$, and $\Omega = 10^{-2}$, from top to bottom. Only for the largest frequency the maximum is approximately located at $x = 0$ where it should be according to the $T(\beta) = \sqrt(2)\Omega \exp\{\beta/4\}$. The maxima for larger frequencies lie at substantially larger noise strengths.

A central quantity of the Floquet theory are the Floquet exponents μ. Any inverse Floquet multiplicator $\mathrm{e}^{-\mu t}$ can be regarded as a gauge factor that renders the time evolution of a Floquet state periodic. Though in general not being purely imaginary the Floquet exponents play the role of the phases in quantum mechanics. Following the analysis of [23] and [24] we can split each Floquet exponent into a dynamic phase $\langle\langle u(t), L(t)v(t)\rangle\rangle$ and a geometric, or Berry phase $\langle\langle u(t), \frac{\partial}{\partial t}v(t)\rangle\rangle$:

$$\mu = \langle\langle u(t), L(t)v(t)\rangle\rangle - \langle\langle u(t), \frac{\partial}{\partial t}v(t)\rangle\rangle, \tag{119}$$

where the double brackets denote the Floquet scalar product $\langle\langle f(t), g(t)\rangle\rangle = \int_0^T \frac{\mathrm{d}t}{T}\,(f(t), g(t))$ which appeared in (13). The functions $u(t)$ and $v(t)$ are the left and right Floquet functions, respectively, corresponding to the Floquet exponent μ.

The proposed semiadiabatic theory has been applied to a simple model of stochastic resonance consisting of a Brownian particle in a symmetric double well potential driven by a slow time dependent periodic force. For the sake of simplicity we made the following additional assumptions: In the absence of noise the external force is too weak to drive the particle from one well into the other. The strength of the noise is much smaller than the smallest barrier height of the time dependent potential, describing both the static double well and the contribution from the time dependent periodic force. Further, we have only considered the long time dynamics. Under these conditions it is sufficient to consider the first two Floquet functions and to approximate

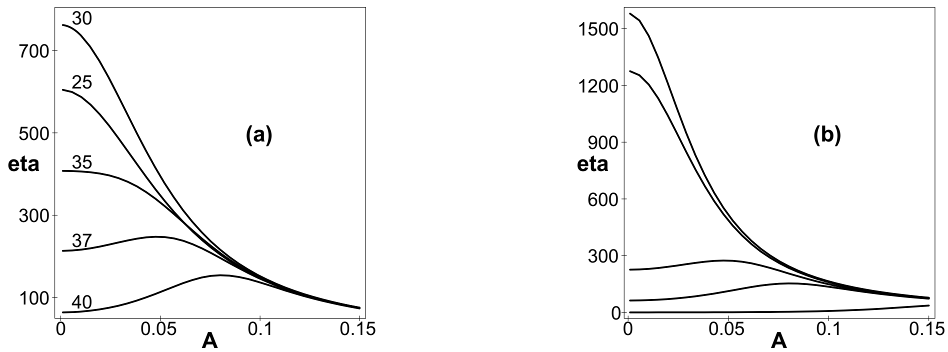

Figure 9. Spectral amplification η as a function of A. Panel (a) shows η for $\Omega = 10^{-4}$ and different values of β which are indicated next to the corresponding curve. Note the nonmonotonic behaviour displaying the stochastic resonance as β varies. Only for small noise the maximal amplification is found at a finite amplitude. Panel (b) shows η for $\beta = 40$ and different driving frequencies $\Omega = 10^{-3}$, 10^{-4}, 5×10^{-5}, 10^{-5}, and η in the fully adiabatic limit (from bottom to top). Upon decreasing the frequency the maximum of the spectral amplification moves from finite values of A to $A = 0$.

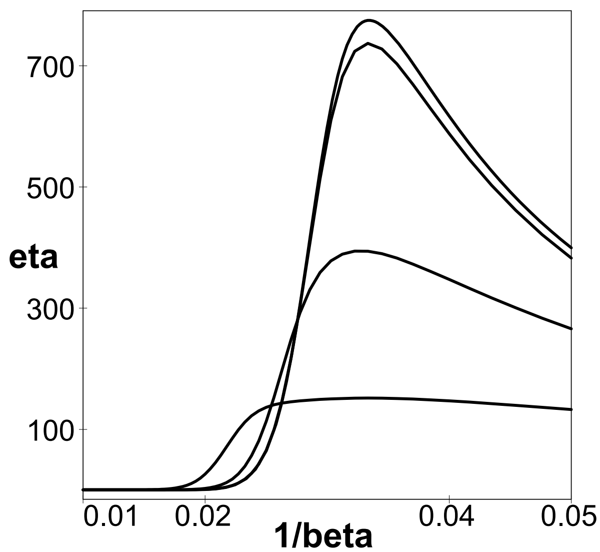

Figure 10. Spectral amplification η as a function of β^{-1} for $\Omega = 10^{-4}$ and different values of the amplitude $A = 0.1$, 0.05, 0.01 and η according to linear response theory, see (118). The maxima of η increase with decreasing amplitude. Linear response theory gives the curve with the largest maximum.

these functions by linear combinations of the first two instantaneous eigenfunctions. The latter are given by the well known weak noise expressions first derived by Kramers [19, 20]. Within this restricted parameter regime we have tested our theory in the limiting cases of linear response theory and in the fully adiabatic limit. In both cases quantitative agreement is observed.

All the above restrictions but the slowness of the external driving force can be released and the influence of large noise, large driving amplitudes and also corrections coming from the fast dynamics may be quantitatively studied within the semiadiabatic regime. Other interesting generalizations of the present model that can be approached by the presented semiadiabatic theory are the influence of a nonsymmetric static potential, nonsinusoidal periodic driving forces, in particular driving by two frequencies. Also the influence of inertia on the escape from a metastable state can be studied in presence of periodic driving.

The semiadiabatic theory can further be employed for analyzing various different natural and technical processes.

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Appendix A. The matrix elements $(\varphi_1(t), \dot{\psi}_0(t))$ and $(\varphi_1(t), \dot{\psi}_1(t))$

The time derivative of the first Floquet function $\varphi_1(x,t)$ follows from (89):

$$\dot{\varphi}_1(x,t) = \dot{B}(t) - \frac{\dot{C}(t)}{C(t)}B(t) + \frac{\dot{C}(t)}{C(t)}\varphi_1(x,t)$$
$$+\sqrt{\frac{\beta}{2\pi}}C(t)\left[\dot{\omega}_b(t)\left(x - x_0(t)\right) - \omega_b(t)\dot{x}_0(t)\right]\exp\left\{-\frac{\beta}{2}\omega_b^2(t)\left(x - x_0(t)\right)^2\right\} \quad \text{(A.1)}$$

According to (17) the scalar product with $\psi_0(x,t)$ yields the negative of $\left(\varphi_1(t), \dot{\psi}_0(t)\right)$, and, hence:

$$\left(\varphi_1(t), \dot{\psi}_0(t)\right) = -\dot{B}(t) + \frac{\dot{C}(t)}{C(t)}B(t) + g(t), \quad \text{(A.2)}$$

where

$$g(t) = g_0(t)\sqrt{r(t)}\mathrm{e}^{-\beta V_+(t)}\int_{-\infty}^{\infty}\mathrm{d}x\mathrm{e}^{-\beta v(x,t)}, \quad \text{(A.3)}$$

and

$$g_0(t) = \frac{\beta}{2\pi}\omega_+(t)\left[\dot{\omega}_b(t)\langle x(t)\rangle_v - x_0(t)\dot{\omega}_b(t) + \omega_b(t)\dot{x}_0(t)\right]. \quad \text{(A.4)}$$

Here, $v(x,t)$ defines the nonparabolic part of the potential $U(x,t)$, i.e., $U(x,t) = U_0(t) - \omega_b^2(t)/2(x - x_0(t))^2 + v(x - x_0(t), t)$. It reads:

$$v(x,t) = \frac{1}{4}x^4 + x_0(t)x^3. \quad \text{(A.5)}$$

A mean value $\langle\cdot\rangle_v$ with index v is taken with respect to the normalized probability density $\rho_v(x,t) = \exp\{-\beta v(x,t)\} / \int_{-\infty}^{\infty}\mathrm{d}x\exp\{-\beta v(x,t)\}$:

$$\langle f(t)\rangle = \frac{\int_{-\infty}^{\infty}\mathrm{d}x f(x)\exp\{-\beta v(x,t)\}}{\int_{-\infty}^{\infty}\mathrm{d}x\exp\{-\beta v(x,t)\}}. \quad \text{(A.6)}$$

The normalization integral $\int_{-\infty}^{\infty}\mathrm{d}x\exp\{-\beta v(x,t)\}$ is of the order of unity for amplitudes $A \leq 0.2$ and inverse noise strength $\beta \leq 100$. For extremely weak noise, with $A^3\beta \gg 1$ this integral oscillates within one period of the driving force between its minimal value $\Gamma(1/4)(\beta/4)^{1/4}/4$, where $\Gamma(z)$ is the gamma function, and its maximal value $\sqrt{2\pi/\beta}\exp\left\{(27x_0^m)^4\,\beta/4\right\}/(3x_0^m)$ where x_0^m denotes the maximal value of $x_0(t)$ which is approximately given by $x_0^m \approx |A|$. In any case, both products $\exp\{-\beta V_\pm\}\int_{-\infty}^{\infty}\mathrm{d}x\exp\{-\beta v(x,t)\}$ are exponentially small for large β.

The normalized probability density $\rho_v(x,t)$ has a maximum at $x^*(t) = -3x_0(t)$ and for $x_0(t) \neq 0$ a shoulder at $x = 0$ which disappears for very weak noise. Consequently, the average $\langle x(t)\rangle_v$ always is of the order of $x_0(t)$.

The first contributions on the right-hand side of (A.2) can be written as

$$\frac{\dot{C}(t)}{C(t)}B(t) - \dot{B}(t) = \alpha(t)\frac{\sqrt{r(t)}}{1+r(t)}, \tag{A.7}$$

where

$$\alpha(t) = \frac{\dot{\omega}_-(t)}{\omega_-(t)} - \frac{\dot{\omega}_+(t)}{\omega_+(t)} - \beta\left(\dot{U}_+(t) - \dot{U}_-(t)\right). \tag{A.8}$$

Here, $\alpha(t)$ is of the same order of magnitude as the function $g_0(t)$. If we combine these results we find that the contribution from $g(t)$ to the scalar product $\langle\varphi_1(t)\dot{\psi}_0(t)\rangle$ is smaller than the first two terms of the right-hand side of (A.2) by the exponential factor $(1+r(t))\exp\{-\beta V_+(t)\}\int_{-\infty}^{\infty}\mathrm{d}x\exp\{-\beta v(x,t)\}$ and, hence, can safely be neglected. A similar analysis shows that (100) is a valid approximation for the matrix element $\langle\varphi_1(t)\dot{\psi}_1(t)\rangle$ up to exponentially small corrections.

Appendix B. The error of $\mu_1^{(0)}$

According to (59) the leading error term of $\mu_1^{(0)}$ is given by the time average of $\sum_{i>1}\left(\varphi_1(t),\dot{\psi}_i(t)\right)A_{i1}(\mu_1^{(0)},t)$. Using (52) one finds that the first term $\left(\varphi_1(t),\dot{\psi}_i(t)\right) = -\left(\dot{\varphi}_1(t),\psi_i(t)\right)$ is exponentially small in the time dependent barrier heights $\beta V_\pm(t)$ and, hence, of the order of $\lambda_1(t)$. The other matrix elements $\left(\varphi_i,\dot{\psi}_1(t)\right)$ can be be analysed in WKB approximation [25]. For the doublet of Floquet functions, $\varphi_i(x,t)$, $i = 2,3$, each of which in the well regions is given by a Hermite polynomial of degree one, the matrix element $\left(\varphi_1(t),\dot{\psi}_i(t)\right)$ is of the order Ω while the other Floquet functions lead to exponentially small matrix elements. This gives the following estimate of the relative error of $\mu_1^{(0)}$:

$$\mu_1 = \mu_1^{(0)}(1 + O(\Omega/\omega)), \tag{B.1}$$

where we took the well frequency ω of the potential in absence of a driving force as a bound for the eigenvalues $\lambda_i(t)$ for $i = 2,3$ that gives the dominating contribution to the sum $\sum_{i>1}\left(\varphi_1(t),\dot{\psi}_i(t)\right)A_{i1}(\mu_1^{(0)},t)$.

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