Markov processes driven by quasi-periodic deterministic forces

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Abstract. Quasi-periodically driven Markovian systems are rendered stationary by enlarging the state space. The eigenvalues and eigenvectors of the master operator of the enlarged process are considered for very slow driving forces in the adiabatic limit. This limit has to be performed always in the minimally enlarged state space and, for two driving frequencies in general leads to results that discontinuously depend on the frequency ratio.

1 Introduction

Stochastic systems that undergo a systematic time-dependent perturbation are frequently met in many fields of science. Beyond the classical experimental techniques of probing a system by a weak periodically varying force that are well described by linear response theory [1, 2], nonlinear effects in periodically driven systems have recently found great interest because of their important and often surprising properties. Stochastic resonance [3] and rocking ratchets [4] are prominent examples thereof. In many cases, however, a description of the time dependent driving force by a periodic function may only give a first qualitative approximation of the real process. For example, in the dynamics of the ice ages for which the mechanism of stochastic resonance was suggested for the first time [5], the earth's orbital parameters are varying with several periods that all earlier had been identified from climatological records [6] and later were included into a stochastic resonance model that was numerically investigated [7]. In recent years stochastic resonance has been studied for aperiodic, mostly broad band signals [8]. We only mention that acting with time dependent strong forces on quantum systems brings about countless effects many of which are under active investigation [9].

   In the present paper we consider a classical Markovian system which is driven by a force consisting of two periodic components with in general incommensurate periods. It is then still possible to give a spectral representation of the conditional probability of the process by means of an extension of the system's state space in an analogous way as for a periodic driving force [10]. The conditional probability then is expressed in terms of the eigenvalues and eigenfunctions of an extended master operator of the

considered process. These general properties are described in section 2. In section 3 we assume that the driving process is slow compared to all intrinsic time scales of the undriven process. We show that a WKB theory provides the framework in which the respective adiabatic limit can be systematically treated. However, it turns out that the ratio of the fundamental frequencies of the two components of the driving force plays an important role in that rational and irrational frequency ratios have to be treated differently. In view of the fact that the rational numbers sit densely within the irrational ones this presents a somewhat unpleasant situation if one is interested in the system's behavior as a function of the fundamental driving frequencies. In section 4 we show by means of an exactly solvable example that this aspect is not restricted to the range of validity of the adiabatic limit. The paper closes with a summary.

2 Stationary embedding

First we consider a family of time-homogeneous Markovian processes $x_q(t)$ that take values in the state space X and depend on two adjustable paramaters $q = (q_1, q_2)$. Each process of the family is governed by a master operator $\Lambda(q)$ which determines the time evolution of the conditional probability $P_q(x, t|x', s)$ to find the process at time t at x provided it was at the earlier time s at x' [11]:

$$\frac{\partial}{\partial t} P_q(x, t|x', s) = \Lambda(q) P_q(x, t|x', s), \tag{1}$$

where $\Lambda(q)$ acts on the x-dependence of the conditional probability. The master equation (1) has to be solved with the initial condition

$$P_q(x, s|x', s) = \delta(x - x'). \tag{2}$$

For a continuous state space X, $\delta(x - x')$ is a Dirac δ-function, and a Kronnecker δ for a discrete space X. Because the master operator is independent of time, the conditional probability is a function of the difference $t - s$ only and the process is homogeneous in time. If the dynamics is confining that means that the process will not typically wander through ever increasing regions of the state space, for large times, $x(t)$ will become a stationary process.

Eigenfunction expansions are a particularly useful tool for the investigation of time homogeneous Markovian processes [12]. We suppose that the left and right eigenvalue equations have solutions $\varphi_i(x, q)$ and $\psi_i(x, q)$, respectively,

$$\begin{aligned} \Lambda(q)\psi_i(x, q) &= \lambda_i(q)\psi_i(x, q), \\ \Lambda^+(q)\varphi_i(x, q) &= \lambda_i(q)\varphi_i(x, q), \end{aligned} \tag{3}$$

that form biorthogonal and complete sets of functions in the space of absolutely integrable functions on the state space X (or absolutely summable sequences in the case of a discrete X) and its dual space:

$$\begin{aligned} (\varphi_i(x, q), \psi_j(x, q)) &= \delta_{i,j}, \\ \sum_i \psi_i(x, q)\varphi_i(x', q) &= \delta(x - x'). \end{aligned} \tag{4}$$

Here the δ-function is the same as in Eq. (2) and the brackets denote the scalar product between pairs of elements of these spaces:

$$(\varphi, \psi) = \begin{cases} \int \mathrm{d}x \varphi(x)\psi(x) & \text{for } X \text{ continuous,} \\ \sum_k \varphi_k \psi_k & \text{for } X \text{ discrete.} \end{cases} \tag{5}$$

Because of the conservation of total probability, there always exists a constant left-eigenvector $\varphi_0(x) = 1$ to which the eigenvalue $\lambda_0(q) = 0$ belongs. If the dynamics is confining so that the process $x(t)$ becomes stationary for sufficiently long times there exists a stationary normalizable solution of the master equation which is the right-eigenvectors of $\Lambda(q)$ belonging to $\lambda_0(q) = 0$. For the sake of simplicity we assume that this eigenvalue is not degenerate. The other eigenvalues $\lambda_i(q)$ are typically also nondegenerate provided there are no particular symmetries of the system.

Under the above stated conditions the conditional probability $P(x, t|x', s)$ can be expanded in terms of the eigenvalues and eigenvectors resulting in the spectral representation, i.e.:

$$P_q(x, t|x', s) = \sum_i \mathrm{e}^{\lambda_i(q)(t-s)} \psi_i(x, q)\varphi_i(x', q). \tag{6}$$

We now consider a situation in which the parameters are changing in time. We assume that each parameter periodically depends on a phase that linearly grows in time:

$$\begin{aligned} q_i(\theta_1 + 2\pi) &= q_i(\theta), \\ \theta_i &= \Omega_i t. \end{aligned} \tag{7}$$

In general the frequencies Ω_1 and Ω_2 are different. The phases θ_1 and θ_2 can be considered as the coordinates on a torus $\mathcal{T}_2$. Moving according to Eq. (7) they cover the torus densely in time if the frequency ratio Ω_1/Ω_2 is irrational and otherwise run on a closed trajectory. In any case, the master operator becomes time dependent and the process $x(t)$ is no longer stationary. It is this particular type of non stationary processes that we want to investigate here.

The time evolution of the conditional probability $P(x, t|x', s)$ is described by Eq. (1) where the parameters of the master operator q now vary with time. An alternative description results if one enlarges the state space by incorporating the time-dependent parameters of the process as additional coordinates. Since the parameters are determined by their respective phases the new state space can be identified with $X \otimes \mathcal{T}_2$. The master operator of the extended process must also generate the time dependence of the former parameters. Because the phases move with constant velocity the respective part of the master operator is simply given by the Liouville operator:

$$L_\Omega = -\Omega \cdot \nabla_\theta \equiv -\Omega_1 \frac{\partial}{\partial \theta_1} - \Omega_2 \frac{\partial}{\partial \theta_2}. \tag{8}$$

The total master operator becomes:

$$L = \Lambda_\theta + L_\Omega, \tag{9}$$

where

$$\Lambda_\theta = \Lambda(q(\theta)). \tag{10}$$

The master operator L generates a Markov process in the extended state space $X \otimes \mathcal{T}_2$ which is homogeneous in time. The spectral representation of its conditional probability $P(x, \theta, t | x', \theta', s)$ can be constructed from the eigenvectors $v_i(x, \theta)$, $u_i(x, \theta)$ and eigenvalues μ_i of the extended master operator:

$$\begin{aligned} Lv_i(x, \theta) &= \mu_i v_i(x, \theta), \\ L^+ u_i(x, \theta) &= \mu_i u_i(x, \theta), \end{aligned} \tag{11}$$

where $L^+ = \Lambda_\theta^+ - L_\Omega$. As functions of θ, the eigenvectors $v_i(x, \theta)$ and $u_i(x, \theta)$ must be periodic. The right eigenvectors must be absolutely integrable on $X \otimes \mathcal{T}_2$ and the left eigenvectors must be elements of the space dual to that of absolutely integrable functions. This extension is analogous to the one for periodically driven quantum systems [13]. Because of the periodicity with respect to the phase θ the eigenvalues μ_i come in equivalence classes with equal real parts and imaginary parts differing by $\Omega_1 k_1 + \Omega_2 k_2$ with k_1, $k_2 \in \mathbb{Z}$. The corresponding eigenfunctions $v_i(x, \theta)$ are different within such a group by the factors $\exp\{i(k_1\theta_1 + k_2\theta_2)\}$. The same behavior is known for periodically driven processes [14].

The biorthogonality and completeness relations then read:

$$\begin{aligned} ((u_i(x, \theta), v_j(x, \theta))) &= \delta_{i,j}, \\ \sum_i v_i(x, \theta) u_i(x', \theta') &= \delta(x - x')\delta(\theta - \theta'). \end{aligned} \tag{12}$$

The double brackets denote the scalar product between the enclosed elements from the space of absolutely integrable functions on $X \otimes \mathcal{T}_2$ and from the dual space:

$$((\varphi(x, \theta), \psi(x, \theta))) = \frac{1}{4\pi^2} \int_{\mathcal{T}_2} \mathrm{d}^2\theta \, (\varphi(x, \theta), \psi(x, \theta)) \,. \tag{13}$$

Analogously to the spectral representation (6) of the conditional probability for the process with frozen parameters, one can express the conditional probability $P(x, \theta, t | x', \theta', s)$ in the extended state space in terms of the eigenvectors of the extended master operator. Because the phases change deterministically from the initial values $\theta_i(s) = \Omega_i s$ to $\theta_i(t) = \Omega_i t$ the extended conditional probability contains a δ-function in the phases. The conditional probability of the time-inhomogeneous process $x(t)$ can be expressed in terms of the eigenvalues and eigenfunctions of the extended process, see Appendix A:

$$P(x, t | x', s) = {\sum_i}^\star \mathrm{e}^{\mu_i(t-s)} v_i(x', \theta(t)) u_i(x', \theta(s)), \tag{14}$$

where the star at the sum indicates that it runs only over a single representative of each of the above defined equivalence classes of eigenvalues. For rational frequency ratios Ω_1/Ω_2 the corresponding eigenfunctions are conveniently normalized such that $(u_i(x, 0), v_1(x, 0)) = 1$.

3   The adiabatic limit

In the limit when the driving frequencies Ω_1 and Ω_2 are small compared to all intrinsic characteristic rates and frequencies of the undriven system, one expects that an adiabatic procedure exists that should allow one to determine the eigenvalues and eigenvectors of the extended problem. We here will show that in the presence of more than one driving frequency problems of technical nature arise that make this limit and, as we shall see later the stationary embedding in general, rather cumbersome.

In a straightforward approach to the adiabatic limit one expands the eigenvectors of the extended problem in terms of the frozen basis with phase dependent coefficients [15]. For these coefficients one then finds coupled partial differential equations of first order which can be decoupled if one assumes the completeness of the frozen eigenvectors for all values of the phases and makes use of the above mentioned slowness of the external driving. Here we take an alternative route and discuss the adiabatic limit in the framework of WKB theory.

3.1   Adiabatic limit and WKB theory

In the adiabatic limit the average driving frequency $\epsilon = \sqrt{(\Omega_1^2 + \Omega_2^2)/\nu^2}$ measured in units of an appropriate characteristic rate ν of the frozen system is a small dimensionless parameter that appears as a coefficient of the phase derivatives in the master operator L of the extended system:

$$L = \Lambda_\theta - \epsilon\omega \cdot \nabla_\theta. \tag{15}$$

The frequencies $\omega_\alpha = \Omega_\alpha\nu/\sqrt{\Omega_1^2 + \Omega_2^2}$, $\alpha = 1$, 2, are of the order of the fast rate ν. The presence of a small parameter in front of the phase derivatives is reminiscent of a Schrödinger equation in the semiclassical limit. The main difference is that in the present case first order derivatives carry the small factor and not a Laplacian as in the Schrödinger equation. Notwithstanding this fact, we make the following WKB ansatz for an eigenvector of the extended problem (11):

$$v(x, \theta, \epsilon) = A(x, \theta, \epsilon) \exp\left\{\frac{1}{\epsilon}S(\theta)\right\}, \tag{16}$$

where on the one hand we have dropped the here irrelevant index i, and, on the other hand made explicit the ϵ-dependence. The exponential dependence on $1/\epsilon$ together with the ϵ-independent "action" $S(\theta)$ is supposed to take into account the singular behavior caused by the small coefficient of the phase derivatives. The remaining ϵ-dependence contained in $A(x, \theta, \epsilon)$ is assumed to be analytic:

$$A(x, \theta, \epsilon) = A^0(x, \theta) + A^1(x, \theta)\epsilon + O(\epsilon^2). \tag{17}$$

Putting the ansatz as given by Eqs. (16), (17) into Eq. (11) and sorting for equal powers of ϵ we find:

$$\left[\Lambda_\theta - \omega \cdot (\nabla_\theta S(\theta)) - \mu^0\right] A^0(x, \theta) = 0, \tag{18}$$

$$\left[\Lambda_\theta - \omega \cdot (\nabla_\theta S(\theta)) - \mu^0\right] A^1(x, \theta) = \mu^1 A^0(x, \theta) + \omega \cdot \nabla_\theta A^0(x, \theta). \tag{19}$$

We also expanded the eigenvalue μ in powers of ϵ: $\mu = \mu_0 + \mu_1\epsilon + \ldots$ On the left hand sides of the above equations the phase derivatives only act on $S(\theta)$ but not on the functions outside of the square brackets. Hence, the solution of the first equation is proportional to a frozen eigenvector, say, $\psi_j(x,\theta)$:

$$A_j^0(x,\theta) = N_j(\theta)\psi_j(x,\theta), \tag{20}$$

where $N_j(\theta)$ is a phase dependent factor that remains undetermined from the first equation. For the corresponding action $S_j(\theta)$ and eigenvalue μ_j^0 we find the equation:

$$\lambda_j(\theta) - \omega \cdot \nabla_\theta S_j(\theta) = \mu_j^0. \tag{21}$$

We shall later come back to this equation from which one expects to find a periodic function: $S(\theta_1 + 2\pi, \theta_2) = S(\theta_1, \theta_2 + 2\pi) = S(\theta_1, \theta_2)$. If such a solution exists, the adiabatic approximation of the eigenvalue is readily found to be:

$$\mu_j^0 = \frac{1}{4\pi^2} \int_{T_2} d^2\theta \, \lambda_i(\theta). \tag{22}$$

Because the operator acting on the left hand side of Eq. (19) is singular, according to the Fredholm alternative [16], solutions for $A_j^1(x,\theta)$ exist only if the function on the right hand side is orthogonal to the kernel of the operator $\Lambda_\theta^+ + \omega \cdot \nabla_\theta - \mu_j^0$, i.e. to those functions that are mapped to zero by this operator. One shows by inspection that this kernel is spanned by the frozen left eigenvector $M(\theta)\varphi_j(x,\theta)$ where $M(\theta)$ is an arbitrary periodic function of the phase. Hence, it is necessary for Eq. (19) to possess a solution that the scalar product of $M(\theta)\varphi_j(x,\theta)$ and the function on the right hand side of Eq. (19) vanishes, i.e.:

$$\big(\!\big(M(\theta)\varphi_j(x,\theta), \big[\mu_j^1 N(\theta)\psi_j(x,\theta) + \omega \cdot \nabla_\theta(N(\theta)\psi_j(x,\theta))\big]\big)\!\big) = 0. \tag{23}$$

Using the fact that this must hold for arbitrary functions $M(\theta)$ we can get rid of the integration over the phases contained in the double bracket scalar product. Further we make use of the normalization of the frozen eigenvectors and obtain the following equation for the unknown factor $N_j(\theta)$:

$$\omega \cdot \nabla_\theta N_j(\theta) + \big[\mu_j^1 + (\varphi_j(x,\theta), \omega \cdot \nabla_\theta \psi_j(x,\theta))\big] N_j(\theta) = 0. \tag{24}$$

For the logarithm of $N_j(\theta)$ we find an equation of the same type as for the action, see Eq. (21). If it has a periodic solution, the first correction of the eigenvalue becomes:

$$\mu_j^1 = - \big(\!\big(\varphi_j(x,\theta), \omega \cdot \nabla_\theta \psi_j(x,\theta)\big)\!\big). \tag{25}$$

We now come back to the discussion of Eq. (21) and its solutions, see also Appendix B. According to the Fredholm alternative, the existence of solutions is determined by the kernel of the operator $\omega \cdot \nabla_\theta$ and the inhomogeneity $\mu_j^0 - \lambda_j(\theta)$. This kernel always contains the constant function. Equation (22) guarantees that the inhomogeneity is orthogonal to the constant part of the kernel. If the frequency ratio $\omega_1/\omega_2 = \Omega_1/\Omega_2$ is irrational there are no other functions in the kernel than the

constants and the solution of Eq. (21) is unique up to an irrelevant constant factor that can be absorbed in the normalization of the eigenfunction $u_j(x, \theta)$. If, however, the frequency ratio is rational, say $\omega_1/\omega_2 = n_1/n_2$, the kernel of the operator $\omega \cdot \nabla_\theta$ contains further periodic solutions on which the inhomogeneity must be orthogonal in order that solutions exist. Consequently, there are solutions of the inhomogeneous equation (21) only if the Fourier coefficients

$$c_{j;k,l} = \frac{1}{4\pi^2} \int_{\mathcal{T}_2} \mathrm{d}^2\theta \, \lambda_j(\theta) \exp\{-\mathrm{i}(k\theta_1 + l\theta_2)\} \tag{26}$$

vanish for all $k = -n_2 r$, $l = n_1 r$ with $r \in \mathbb{Z}$. This trivially is the case for the frozen eigenvalue $\lambda_0(\theta) = 0$ for which one finds in the adiabatic limit $\mu_0^0 = \mu_0^1 = 0$. Accordingly, one obtains for the asymptotic state $v_0^{\mathrm{ad}}(x, \theta)$ in the adiabatic limit irrespectively of whether the frequency ratio is rational or not:

$$v_0^{\mathrm{ad}}(x, \theta) = \psi(x, \theta). \tag{27}$$

For a driven Ornstein Uhlenbeck process the frozen eigenvalues are independent of the phase [14]. Consequently, also in this case the adiabatic limit can be performed for all eigenvalues and eigenfunctions whether the frequency ratio is rational or not.

In the general case, however, the frozen eigenvalues will depend on the phases, in particular, some of their Fourier coefficients $c_{j;-n_2 r, n_1 r}$ will not vanish and consequently, the Eqs. (21) and (24) will not have solutions. For a more detailed discussion see Appendix B.

3.2 Adiabatic limit with a rational ratio of driving frequencies

If the driving frequencies are in a rational relation $\omega_1/\omega_2 = \Omega_1/\Omega_2 = n_1/n_2$ the driving force is strictly periodic with the period $T = 2\pi n_1/\Omega_1 = 2\pi n_2/\Omega_2$ and it is sufficient to consider a single additional phase variable ϕ to render the process time homogeneous. The extended process then takes values in the state space $X \otimes \mathcal{T}_1$ with the phase varying on the circle $\mathcal{T}_1$. The master operator of the extended process is again the sum of the frozen master operator $\Lambda_\phi = \Lambda_\theta$ where $\theta = (n_1\phi, n_2\phi)$, and the infinitesimal phase shift operator $L_\Omega = -\Omega\partial/\partial\phi$, where $\Omega = 2\pi/T$ is the common frequency. In the adiabatic limit the ratio of the common frequency to a typical rate ν of the undriven system, $\epsilon = \Omega/\nu$, is a small parameter in terms of which we make the analogous WKB ansatz (16) as in the irrational case:

$$v(x, \phi, \epsilon) = A(x, \phi, \epsilon) \exp\left\{\frac{1}{\epsilon} S(\phi)\right\}. \tag{28}$$

With the expansion of $A(x, \phi, \epsilon)$ in terms of powers of ϵ we find a hierarchy of equations analogous to the Eqs. (18), (19) which are always solvable because there are no other solutions of $\mathrm{d}f/\mathrm{d}\phi = 0$ than a constant.

The lowest order term of the amplitude $A^0(x, \phi) = A(x, \phi, 0)$ is again proportional to an eigenvector $\psi_j(x, \phi)$ of the frozen master operator Λ_ϕ:

$$A_j^0(x, \phi) = N_j(\phi)\bar{\psi}_j(x, \phi), \tag{29}$$

where

$$N_j(\phi) = \exp\left\{ \int_\phi^{2\pi} \mathrm{d}\phi' \left(\bar{\varphi}_j(x,\phi'), \frac{\partial}{\partial\phi'} \bar{\psi}_j(x,\phi') \right) \right\},$$ (30)

and

$$\bar{\psi}_j(x,\phi) = \psi_j(x, n_1\phi, n_2\phi),$$
$$\bar{\varphi}_j(x,\phi) = \varphi_j(x, n_1\phi, n_2\phi).$$ (31)

For the eigenvalues one obtains in lowest order:

$$\mu_j^0 = \frac{1}{T} \int_0^T \mathrm{d}t \, \lambda_j(\Omega_1 t, \Omega_2 t) = \sum_{\substack{k,l \\ n_1 k + n_2 l = 0}} c_{j;k,l},$$ (32)

where $c_{j;k,l}$ are the Fourier coefficients of the frozen eigenvalue $\lambda_j(\theta_1, \theta_2)$, see Eq. (26). Both expressions also hold in the irrational case where in the integral the limit $T \to \infty$ has to be taken; the sum then collapses to the single term with $k = l = 0$ and one recovers the expression (22). In the adiabatic limit, the eigenvalues μ_j show a rather irregular behavior as a function of the frequency ratio Ω_1/Ω_2 jumping from $c_{j;0,0}$ at a typical irrational frequency ratio to a value differing by a partial sum of Fourier coefficients at a nearby rational ratio.

The action is readily expressed in terms of $\bar{\lambda}_j(\phi) = \lambda_j(n_1\phi, n_2\phi)$:

$$S_j(\phi) = \frac{1}{\nu} \int_\phi^{2\pi} \mathrm{d}\phi' \bar{\lambda}_j(\phi').$$ (33)

The results found by means of the WKB ansatz completely agree with the eigenvectors and eigenvalues in the adiabatic limit based on an expansion in terms of the corresponding frozen quantities [15].

We note that the exponent of the factor $N_j(\phi)$ is the first order correction of the action $S_j(\phi)$. The adiabatic limit represents a valid approximation if this correction which is determined by $\left(\bar{\varphi}_j(x,\phi), \Omega \partial \bar{\psi}_j(x,\phi)/\partial\phi \right)$, or, in case of an irrational frequency ratio, by $(\varphi_j(x,\theta), \Omega \cdot \nabla_\theta \psi_j(x,\theta))$, is small compared to the action. There are two fundamentally different possible reasons why this condition may be spoiled and why the adiabatic approximation may fail. A defective degeneracy at an isolated phase leads to a divergence of $(\varphi_j(x,\theta), \Omega \cdot \nabla_\theta \psi_j(x,\theta))$ for any finite Ω. Within the WKB formulation such a particular phase corresponds to a turning point. Details will be discussed elsewhere. The other reason for a breakdown of the adiabatic limit is the presence of a single slow mode in the frozen system corresponding to a single small but nonvanishing frozen eigenvalue. This corresponds to the semiadiabatic limit for a periodically driven process [15]. An extension of the respective theory will be developed elsewhere.

Before closing this section we shortly present the results for the left eigenvectors in the adiabatic limit.

3.3 The left eigenvectors

By means of the same kind of arguments as for the right eigenvectors $v_j(x, \theta)$ one finds that in the adiabatic limit the left eigenvectors are proportional to the frozen eigenvectors with a phase dependent factor which is the inverse of the one appearing in the right eigenvectors:

$$u_j(x, \theta) = \frac{1}{N(\theta)} \exp\left\{-\frac{1}{\epsilon} S(\theta)\right\} \varphi_j(x, \theta). \tag{34}$$

Again, for rational frequency ratios one has to resort to the periodic case.

4 Simple example

We consider a Markovian two-state process with transition probabilities that are quasiperiodic functions of time, an example that can be treated exactly. The dynamics of this model is determined by the master operator:

$$\Lambda_\theta = \begin{pmatrix} -\nu(\theta) & \gamma(\theta) \\ \nu(\theta) & -\gamma(\theta) \end{pmatrix}, \tag{35}$$

where $\nu(\theta)$ denotes the rate from state 1 to state 2 and $\gamma(\theta)$ the reverse one, both depending on the momentary phases θ. These vary in time as $\theta = (\Omega_1 t, \Omega_2 t)$. The right eigenvalue equation (11) then becomes:

$$\begin{aligned} -\nu(\theta)v_1(\theta) + \gamma(\theta)v_2(\theta) - \Omega \cdot \nabla_\theta v_1(\theta) &= \mu v_1(\theta) \\ \nu(\theta)v_1(\theta) - \gamma(\theta)v_2(\theta) - \Omega \cdot \nabla_\theta v_2(\theta) &= \mu v_2(\theta). \end{aligned} \tag{36}$$

With a slight abuse of notation here the indices indicate the states 1 and 2. For simplicity, the index numbering the eigenvalues and corresponding eigenvectors is omitted. Adding the two equations yields for the sum:

$$-\Omega \cdot \nabla_\theta(v_1(\theta) + v_2(\theta)) = \mu(v_1(\theta) + v_2(\theta)). \tag{37}$$

As discussed above, a nontrivial periodic solution requires that the real part of μ vanishes. This leads to the equivalence class of eigenvalues belonging to $\mu = 0$:

$$\mu = \mathrm{i}(k_1\Omega_1 + k_2\Omega_2) \qquad k_1, \ k_2 \in \mathbb{Z}. \tag{38}$$

For an irrational frequency ratio and for $\mu = 0$ the sum of the two components is constant. A second equivalence class follows from the trivial solution of Eq. (37) for which $v_1(\theta) + v_2(\theta)$ vanishes. Then one finds from Eq. (36):

$$\left(\nu(\theta) + \gamma(\theta)\right)v_1(\theta) + \Omega \cdot \nabla_\theta v_1(\theta) = \mu v_1(\theta). \tag{39}$$

For the logarithm of $v_1(\theta)$ an inhomogeneous equation is obtained which has the same structure as Eq. (21) for the action. The inhomogeneity is given by $\mu - \nu(\theta) - \gamma(\theta)$

and, hence, a solution exists if the frequency ratio is irrational. Then the eigenvalue μ takes the form:

$$\mu = \frac{1}{4\pi^2} \int_{T_2} \mathrm{d}^2\theta (\nu(\theta) + \gamma(\theta)), \tag{40}$$

see also Eq. (22), and the component $v_1(\theta)$ becomes, see Appendix B:

$$v_1(\theta) = \exp\left\{ \int_0^{\Omega_1\theta_1 + \Omega_2\theta_2} \mathrm{d}x \right.$$
$$\left. h\left(\frac{\Omega_1}{\Omega_1^2 + \Omega_2^2} x + \Omega_2^2\theta_1 - \Omega_1\Omega_2\theta_2, \frac{\Omega_2}{\Omega_1^2 + \Omega_2^2} x - \Omega_1\Omega_2\theta_1 + \Omega_1^2\theta_2 \right) \right\}, \tag{41}$$

where

$$h(\theta) = \nu(\theta) + \gamma(\theta) - \mu. \tag{42}$$

There also exist solutions if the frequency ratio is rational, $\Omega_1/\Omega_2 = n_1/n_2$, and all Fourier coefficients of $\mu - \nu(\theta) - \gamma(\theta)$ with indices $k, l \in \mathbb{Z}$ satisfying $n_1 k + n_2 l = 0$ vanish. In the typical rational frequency case these coefficients will not do so and in order to achieve a stationary description of the process one has to resort to the lower dimensional extension of the state space with the single phase $\phi = n_1\theta_1 + n_2\theta_2$. This is in complete agreement with our previous findings in the adiabatic limit.

5   Conclusions

In this paper we gave the spectral representation of the conditional probability of a Markov process that is driven by a quasi-periodic force by means of an extension of the state space of the system. This generalizes the Floquet representation for a periodically process [10, 14]. In most cases, however, the spectral representation only provides a formal expression containing the usually unknown eigenvalues and eigenvectors of the Markovian master operator of the extended process.

For slowly driven processes the adiabatic limit can be formulated by means of a WKB ansatz. Unfortunately, it turns out that in most cases one must perform this limit in the smallest possible enlarged state space. This, however, depends on the frequency ratio in the case of two fundamental driving frequencies and, more generally, on the dimensionality of the subspace that the phases cover as time goes to infinity. As a consequence, the eigenvalues and eigenfunctions can show a rather weird behavior as functions of the frequency ratio Ω_1/Ω_2.

The discussion of the exactly solvable Markovian two state model shows that this unexpected feature is not an artifact of the adiabatic approximation. The relaxational eigenvalues and the corresponding eigenvectors discontinuously depend on the frequency ratio also outside the adiabatic regime.

Whether the singularities of the eigenvalues and eigenfunctions do have an influence on the time evolution of the conditional probability is in general not clear. In the case of a system with a finite state space the original master equation represents a finite

set of linear coupled differential equations the solutions of which depend continuously on the systems parameters as long as finite times are considered. For a finite state space the discontinuities of the eigenvalues and eigenfunctions must compensate each other in the spectral representation of the conditional probability. Further studies are necessary how this compensation comes about and whether it also works in the limit of infinitely many states.

As in quantum mechanics the adiabatic limit does not hold if branches of frozen eigenvalues cross each other. Crossings of the frozen eigenvalue as functions of the phases present already for periodically driven processes new aspects compared to a periodically driven quantum system because the frozen eigenvalues may be complex. In particular, the points where two real frozen eigenvalues merge and a pair of complex frozen eigenvalues are born are quite different from crossings of real eigenvalues in a quantum system. In the stochastic case these crossings are structurally stable, i.e. they generally do not disappear under the action of a small perturbation. Within the WKB approach to the adiabatic limit these points correspond to turning points. We will pursue this problem in a future publication.

The adiabatic elimination fails also if there is a slow relaxational mode in the frozen systems as e.g. in bistable systems [17]. This case can be treated in the semiadiabatic limit [15]. The most relevant parameter regime of stochastic resonance with a quasi-periodic imput signal falls into this limit and will be discussed elsewhere.

A   The spectral representation of a Markov process with quasi-periodic external forcing

We first write the conditional probability of the time-inhomogeneous Markov process $x(t)$ which is externally driven by two periodically varying parameters as given by Eq. (7), in terms of the eigenvectors of the extended master operator:

$$P(x,t|x',s) = \sum_i e^{\mu_i(t-s)} c_i v_i(x, \theta(t)), \tag{43}$$

where the phases are taken at their deterministic physical values at time t, $\theta_{1,2}(t) = \Omega_{1,2}t$, and the constants c_i depend on the condition x' at time s. The sum only runs over the equivalence classes of the eigenvalues, i.e. disregards all integer multiples of $i\Omega_1$ and $i\Omega_2$, because within each class the respective eigenfunctions differ only by phase-dependent factors and therefore are not independent of each other if considered as functions of the physical state x alone. By inspection one finds that this ansatz fulfills the forward equation

$$\frac{\partial}{\partial t} P(x,t|x',s) = \Lambda_{\theta(t)} P(x,t|x',s). \tag{44}$$

The yet undetermined constants follow from the initial condition

$$P(x,s|x',s) = \delta(x - x'). \tag{45}$$

Their determination is straightforward if one makes use of the fact that the left and right eigenvectors of the extended master operator are biorthogonal as functions on the physical state space X:

$$(u_i(x,\theta), v_j(x,\theta)) = \delta_{i,j} f_j \left(\theta_2 - \frac{\Omega_2}{\Omega_1} \theta_1 \right), \tag{46}$$

where the functions $f_j(\theta_2 - (\Omega_2/\Omega_1)\theta_1)$ are periodic with respect to both phases θ_1 and θ_2 and unity at zero, $f(0) = 1$. As a consequence, one has $f_j(x) = 1$ for irrational frequency ratios. Multiplying both sides of Eq. (45) with $u_i(x, \theta(s))$ and integrating over the physical state space X one finds with Eq. (43) for the coefficients c_i the expression:

$$c_i = u_i(x', \theta(s)). \tag{47}$$

Together with Eq. (43) this gives the required result (14).

So it remains to show the validity of Eq. (46). For this purpose we multiply the equation for the right eigenfunction $v_i(x, \theta)$ with the left eigenfunction $u_j(x, \theta)$ and integrate over the physical state space X. Vice versa, we proceed with the equation for the left eigenfunction $u_j(x, \theta)$ which we multiply by $v_i(x,, \theta)$ and then integrate over X. The resulting equations are subtracted from each other so that the contribution $(u_j(x, \theta), \Lambda_\theta v_i(x, \theta))$ that is contained in both equations cancels. The result is a partial differential equation for the scalar products $N_{j,i}(\theta) = (u_j(x, \theta), v_i(x, \theta))$:

$$\Omega \cdot \nabla_\theta N_{j,i}(\theta) = (\mu_j - \mu_i) N_{j,i}(\theta). \tag{48}$$

For $\mu_i \neq \mu_j$ only the trivial solution $N_{i,j} = 0$ is periodic. For $\mu_i = \mu_j$ the general solution is a function of the single variable $\theta_2 - (\Omega_2/\Omega_1)\theta_1$ which is zero for the "physical" phases $\theta_1 = \Omega_1 t$ and $\theta_2 = \Omega_2 t$. The eigenfunctions can always be normalized so that $f(0) = 1$.

B   Periodic solutions of the equation $\boldsymbol{\omega \cdot \nabla_\theta S(\theta) = \lambda(\theta) - \mu^0}$

For the sake of simplicity we choose the time unit such that the characteristic rate ν of the frozen system is unity, hence, $\omega_1^2 + \omega_2^2 = 1$, and introduce as new coordinates on the torus $\mathcal{T}_2$ linear combinations of the phases θ_1 and θ_2:

$$\begin{aligned} x &= \omega_1 \theta_1 + \omega_2 \theta_2, \\ y &= -\omega_2 \theta_1 + \omega_1 \theta_2. \end{aligned} \tag{49}$$

In these coordinates the partial differential equation (21) simplifies to an ordinary one:

$$\frac{\partial}{\partial x} S(\omega_1 x - \omega_2 y, \omega_2 x + \omega_1 y) = \lambda(\omega_1 x - \omega_2 y, \omega_2 x + \omega_1 y) - \mu^0. \tag{50}$$

This equation is readily integrated to yield:

$$\begin{aligned} S(\theta_1, \theta_2) = \int_0^{\omega_1 \theta_1 + \omega_2 \theta_2} \mathrm{d}x \left[\lambda(\omega_1 x + \omega_2^2 \theta_1 - \omega_1 \omega_2 \theta_2, \omega_2 x - \omega_1 \omega_2 \theta_1 + \omega_1^2 \theta_2) - \mu^0 \right] \\ + s(-\omega_2 \theta_1 + \omega_1 \theta_2), \end{aligned} \tag{51}$$

where $s(-\omega_2\theta_1 + \omega_1\theta_2)$ is an integration constant that may still depend on $y = -\omega_2\theta_1 + \omega_1\theta_2$. It is relevant up to an arbitrary additive constant that can be absorbed into the normalization of the corresponding eigenvector $v(x, \theta, \epsilon)$. Because the frozen eigenvalues $\lambda(\theta)$ are periodic functions of the phases they can be represented as Fourier series:

$$\lambda(\theta_1, \theta_2) = \sum_{k,l} c_{k,l} e^{i(k\theta_1 + l\theta_2)}. \tag{52}$$

Putting this into the expression (51) and exchanging integration and summation one obtains:

$$S(\theta_1, \theta_2) = \sum_{k,l \neq 0} c_{k,l} e^{i(k\omega_2 - l\omega_1)(\omega_2\theta_1 - \omega_1\theta_2)} \int_0^{\omega_1\theta_1 + \omega_2\theta_2} \mathrm{d}x\, e^{i(k\omega_1 + l\omega_2)x}$$
$$+ s(-\omega_2\theta_1 + \omega_1\theta_2). \tag{53}$$

As noted above, the constant μ^0 equals the average of $\lambda(\theta)$ over all phases and hence compensates the term with $k = l = 0$. For the further discussion of the existence of periodic solutions we have to distinguish between rational and irrational frequency ratios ω_1/ω_2. In case of a rational frequency ratio there exists a pair of integers n_1 and n_2 such that

$$\omega_1 = \frac{n_1}{\sqrt{n_1^2 + n_2^2}}, \quad \omega_2 = \frac{n_2}{\sqrt{n_1^2 + n_2^2}}. \tag{54}$$

If there is a nonvanishing coefficient $c_{k,l}$ with $k, l \in \mathbb{Z}$ satisfying $n_1 k + n_2 l = 0$ the corresponding integral in Eq. (53) is proportional to $n_1\theta_1 + n_2\theta_2$. In this case, no function $s(-\omega_2\theta_1 + \omega_1\theta_2)$ exists that would render (θ_1, θ_2) periodic. Consequently, the Eq. (21) then does not have a solution.

On the other hand, if the frequency ratio ω_1/ω_2 is irrational, the sum on the right hand side of Eq. (53) is a periodic function of the phases. Then the integration "constant" $s(-\omega_2\theta_1 + \omega_1\theta_2)$ must also be periodic in both phases, i.e.:

$$s(y - 2\pi\omega_2) = s(y + 2\pi\omega_1) = s(y). \tag{55}$$

The only solution of this equation for an irrational ratio ω_1/ω_2 is a constant that can be put to zero without loss of generality.

References

[1] R. Kubo, Rep. Prog. Phys. **29** (1966) 255
[2] P. Hänggi and H. Thomas, Phys. Rep. **88** (1982) 207
[3] For a recent, comprehensive review see L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70** (1998) 223
[4] For recent reviews see R. D. Astumian, Science **267** (1997) 917; F. Jülicher, A. Ajdari, and J. Prost, Rev. Mod. Phys. **69** (1997) 1269
[5] R. Benzi, A. Sutera, G. Parisi, and A. Vulpiani, J. Phys. A **14** (1981) L453

[6] M. Milankovitch, *Kanon der Erdbestrahlung und seine Anwendung auf das Eiszeiten-problem*, Royal Serbian Sciences, Spec. Pub. 132, section of Mathematical and Natural Sciences, vol. 33, Belgrade, 633pp., 1941 [*Canon of Insolation and Ice Age Problem*, English Translation by Israel Program for Scientific Translation and published for the U.S. Department of Commerce and the National Science Foundation, Washington 1969]

[7] G. Matteucci, Climate Dynamics **3** (1989) 179; **6** (1991) 67

[8] P. Jung and P. Hänggi, Phys. Rev. A **44** (1991) 8032; G. Hu, D. C. Gong, X. D. Weng, C. Y. Yang, G. R. Quing, and L. Rong, Phys. Rev. A **46** (1992) 3250; A. Neimann and L. Schimansky-Geier, Phys. Rev. Lett. **72** (1994) 2988; J. J. Collins, C. C. Chow and T. T. Imhoff, Phys. Rev. E **52** (1995) R3321

[9] For recent reviews see S.-I. Chu, Adv. Chem. Phys. **73** (1989) 739; M. Grifoni and P. Hänggi, Phys. Rep. **304** (1998) 229

[10] P. Jung, Phys. Rep. **234** (1993) 175

[11] N. G. van Kampen, *Stochastic Processes in Physics and Chemistry*, North Holland, Amsterdam 1992

[12] H. Risken, *The Fokker-Planck Equation*, Springer, Berlin 1989

[13] H. Sambe, Phys. Rev. A **7** (1973) 2203

[14] P. Jung and P. Hänggi, Phys. Rev. A **41** (1990) 2977

[15] P. Talkner, New J. Phys. **1** (1999) 4 (`http://www.njp.org/`)

[16] T. Kato, *Perturbation Theory for Linear Operators*, Springer, Berlin 1984

[17] P. Hänggi, P. Talkner, and M. Borkovec, Rev. Mod. Phys. **62** (1990) 251