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Statistics of entrance times

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Abstract

The statistical properties of the transitions of a discrete Markov process are investigated in terms of entrance times. A simple formula for their density is given and used to measure the synchronization of a process with a periodic driving force. For the McNamara–Wiesenfeld model of stochastic resonance we find parameter regions in which the transition frequency of the process is locked with the frequency of the external driving.

Keywords: Non-stationary Markov processes; Point processes; Entrance times; Rice frequency; Stochastic resonance

1. Introduction

Many processes that one meets in nature or in technical devices display sudden isolated events such as the firing of neurons, the change of magnetization in a magnetic storage device, the switching in a telegraphic signal, the beat of a heart or large spikes in an EEG signal. These events can often be described in either of two ways, namely, as those instants at which a continuous process crosses a critical threshold, or at which a discrete process enters a particular state. In general such times will constitute a random series also known as a point process [1]. These processes can be characterized by distribution functions, i.e., by the joint density with which n events occur at given times t_1, t_2, \dots, t_n . One can consider different types of distribution functions according to whether additional events may be found in the time intervals between neighboring times (t_i, t_{i+1}) , where $i = 1, \dots, n - 1$, or whether no events must occur between the specified times.

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Here we mainly will restrict ourselves to the simple situation of a Markovian process $x(t)$ in continuous time t that may visit only two states, $x = 1, 2$, for which we will consider the entrance time densities $W_j(t)$, $j = 1, 2$. They give the average number of transitions into the state j at time t per unit time. As a Markovian process $x(t)$ is characterized by the transition rates $\kappa_1(t)$ and $\kappa_2(t)$ specifying the transition probabilities from state 1 to 2 in infinitesimal time dt and vice versa, respectively:

$$\kappa_1(t) dt = \text{Prob}(x(t+dt) = 2 | x(t) = 1), \quad (1)$$

where $\text{Prob}(A|B)$ denotes the probability of A under the condition B . The other rate, $\kappa_2(t)$, is defined accordingly. Here we allow for an arbitrary time dependence of the rates. For the moment we leave this time dependence unspecified. There are many physical processes that can be described by this simple model such as stochastic resonance [2–5] in which case the rates vary periodically in time [6], or the closing and opening dynamics of ionic channels [7] for which the rates depend on the randomly switching conformation of the proteins building the channel. Between these extremes the rates may vary quasi-periodically [8] or in some aperiodic manner like a pulse or a ramp [9]. A two state model with both periodically and randomly modulated rates was investigated in Ref. [10].

The dynamics of a Markovian two state process is determined by the master equation

$$\dot{p}_1(t) = -\kappa_1(t)p_1(t) + \kappa_2(t)p_2(t), \quad \dot{p}_2(t) = \kappa_1(t)p_1(t) - \kappa_2(t)p_2(t) \quad (2)$$

for the probabilities $p_i(t)$ to find the process in state $i = 1, 2$ at time t . The dot denotes the time derivative. The solution of the master equation is readily found:

$$\begin{aligned} p_1(t) &= e^{-K(t)+K(t_0)} p_1(t_0) + \int_{t_0}^t ds e^{-K(t)+K(s)} \kappa_2(s), \\ p_2(t) &= e^{-K(t)+K(t_0)} p_2(t_0) + \int_{t_0}^t ds e^{-K(t)+K(s)} \kappa_1(s) = 1 - p_1(t), \end{aligned} \quad (3)$$

where

$$K(t) = \int_0^t d\tau (\kappa_1(\tau) + \kappa_2(\tau)). \quad (4)$$

We will consider the case when $K(t)$ diverges for $t \rightarrow \infty$ and, hence, $K(t)$ increases with time. Then the first terms of the right-hand sides of the above equations for $p_1(t)$ and $p_2(t)$ decrease with time and an asymptotic state is approached to which the initial conditions no longer contribute. Formally, the asymptotic probabilities $p_i^{as}(t)$, $i = 1, 2$ can be obtained by letting the process start in the infinite past

$$p_1^{as}(t) = \int_{-\infty}^t ds e^{-K(t)+K(s)} \kappa_2(s), \quad p_2^{as}(t) = \int_{-\infty}^t ds e^{-K(t)+K(s)} \kappa_1(s). \quad (5)$$

Due to the variability of the rates $\kappa_1(t)$ and $\kappa_2(t)$ these probabilities keep changing but are independent of the initial values from which they have been started.

In passing we note that the asymptotic probabilities satisfy a set of coupled integral equation. It can be obtained as the formal solutions of Eq. (2) if in the equation for $p_1(t)$ the probability $p_2(t)$ on the right-hand side is considered as a known function of

t and treated as an inhomogeneity and vice versa for the equation for $p_2(t)$. Finally, one again sends the initial time t_0 to $-\infty$. The resulting equations become

$$p_1^{as}(t) = \int_{-\infty}^t ds P_1(t|s) \kappa_2(s) p_2^{as}(s), \quad p_2^{as}(t) = \int_{-\infty}^t ds P_2(t|s) \kappa_1(s) p_1^{as}(s), \quad (6)$$

where $P_i(t|s)$ denotes the waiting time probability in the state i , i.e., the probability that the process stays in this state from time s until t without an interruption. It is given by [11]

$$P_i(t|s) = e^{-\int_s^t dt' \kappa_i(t')}. \quad (7)$$

On the other hand, specifying the initial values $p_i(t_0)$ in Eq. (4) as the probabilities for the certain and the impossible event, $\delta_{i,j}$, $j = 1, 2$, respectively, one obtains the conditional probabilities $p(i, t|j, t_0)$:

$$\begin{aligned} p(1, t|1, t_0) &= e^{-K(t)+K(t_0)} + \int_{t_0}^t ds e^{-K(t)+K(s)} \kappa_2(s), \\ p(1, t|2, t_0) &= \int_{t_0}^t ds e^{-K(t)+K(s)} \kappa_2(s), \\ p(2, t|1, t_0) &= \int_{t_0}^t ds e^{-K(t)+K(s)} \kappa_1(s), \\ p(2, t|2, t_0) &= e^{-K(t)+K(t_0)} + \int_{t_0}^t ds e^{-K(t)+K(s)} \kappa_1(s). \end{aligned} \quad (8)$$

In Section 2 we determine explicit expressions for the entrance time densities for a general Markovian two state process in terms of the probabilities $p_i(t)$ and the rates $\kappa_i(t)$ and show that they satisfy a coupled set of integral equations that by now exclusively have been used to determine these quantities [12,13]. We further indicate how multi-entrance time distribution functions can be formulated in terms of the single time probabilities (3), the conditional probabilities (8) and the rates $\kappa_i(t)$ and finally generalize the expression for the single entrance time density to Markovian processes with an arbitrary finite number of states.

In Section 3 we discuss the entrance times for a periodically driven two state process that may show the effect of stochastic resonance. For these processes we formulate a necessary criterion for the synchronization of the driving force and the driven process. It requires that on average just one entrance into each state takes place within a period of the driving force. This criterion can be understood on the basis of a conveniently defined generalized Rice phase as a frequency synchronization [14]. For the particular model of McNamara and Wiesenfeld [3] we determine the entrance time densities and the number of entrances per period of the driving force. We discuss their dependence on the frequency and amplitude of the driving force and on the noise strength. In particular, we identify parameter regions where frequency locking occurs according to the considered synchronization criterion.

2. The entrance time density

For a two state process the entrance time density, say, for the state 1 at time t , $W_1(t)$, coincides with the number of entrances into state 1 within the infinitesimal time interval $[t, t + dt]$ divided by the length of the interval dt . This number is given by the probability to find the process in state 2 at time t and in state 1 at time $t + dt$. Hence, we have for the entrance time density:

$$W_1(t) = \text{Prob}(x(t + dt) = 1, x(t) = 2)/dt = \kappa_2(t)p_2(t), \quad (9)$$

where we used that the joint probability $\text{Prob}(x(t + dt) = 1, x(t) = 2)$ can be expressed as the product of the conditional probability $p(1, t + dt|2, t)$ and the probability $p_2(t)$. Because dt is infinitesimal, the conditional probability simplifies to $\kappa_2(t)dt$, finally leading to the last expression of the above two equations. Analogously, one obtains for the other state

$$W_2(t) = \kappa_1(t)p_1(t). \quad (10)$$

Expressions (9) and (10) for the entrance times represent the central result of this paper.

In previous works entrance times were determined in a rather cumbersome and incomplete way from a coupled set of integral equations that in the present case of a Markovian two state process take the form [12,13]:

$$W_1(t) = \int_{-\infty}^t ds \rho_2(t|s)W_2(s), \quad W_2(t) = \int_{-\infty}^t ds \rho_1(t|s)W_1(s), \quad (11)$$

where $\rho_i(t|s)$ is the first passage time density, i.e., the probability density for leaving the state i at time t after an uninterrupted sojourn in i since the previous time s . The first passage time density is given by the negative derivative of the waiting time probability in the state i , $P_i(t|s)$, see Eq. (7), with respect to the later time t [11]:

$$\rho_i(t|s) = \kappa_i(t)P_i(t|s). \quad (12)$$

The entrance time distributions in the asymptotic state, i.e., for $p_i(t) = p_i^{as}(t)$ as given by Eq. (5) indeed solve the integral equations (11). As a proof one puts the explicit expressions (9) and (10) using the asymptotic probabilities $W_1(t) = \kappa_2(t)p_2^{as}(t)$ and $W_2(t) = \kappa_1(t)p_1^{as}(t)$ into the integral equations, and recovers the integral equation (6) for the asymptotic probabilities $p_1^{as}(t)$ and $p_2^{as}(t)$.

We note though that even in this restricted case the integral equations are not equivalent to the explicit expressions for the asymptotic entrance time densities because the solutions of the linear homogeneous equations (11) only are determined up to a multiplicative factor. This factor is not fixed by normalization because $W_1(x)$ and $W_2(x)$ are densities of points and not probability densities [15]. The integral of an entrance density over all times generally diverges and the integral over a finite interval of times gives the average number of entrance points during this time.

On the other hand, some quantities as e.g. residence time distributions [16] depend on the entrance time densities in a way that they become independent of a multiplicative factor [12,13]. In such cases any solution of the integral equations (11) suffices.

With the entrance time densities only single time events are characterized. In order to specify the complete statistics a whole hierarchy of multi-event distribution functions can be introduced. This can be done in different ways, depending on whether one allows for the occurrence of further entrance events between two specified times or, alternatively, excludes their occurrence [1]. Here we discuss as exemplary cases the two-time distribution functions $Q_{i,j}(t,s)$ and $f_{i,j}(t,s)$ that give the densities of entrances into state i at time t and state j at time s , excluding or including, respectively, the occurrence of further entrances into the state i within the interval of specified times. Because the conditional and the unconditional distribution functions, $Q_{i,j}(t,s)$ and $f_{i,j}(t,s)$, respectively, are symmetric under a joint change of the states i and j and the times t and s we can restrict ourselves to the case $t > s$. Considering the conditional distribution function we have to distinguish whether the states i and j are different or not. For different states i and j we find from the probability that the process jumps at time s from i to j , stays there until time t at which it jumps back to i :

$$Q_{i,j}(t,s) = \kappa_j(t)P_j(t|s)\kappa_i(s)p_i(s) \quad \text{for } i \neq j, \quad (13)$$

where $P_j(t|s)$ is defined in Eq. (7). For equal states $i=j$ a jump must take place from i to the other state \bar{i} at some intermediate time $t' \in (s,t)$. One then finds

$$Q_{i,i}(t,s) = \kappa_{\bar{i}}(t) \int_s^t dt' P_{\bar{i}}(t|t')\kappa_i(t')P_i(t'|s)\kappa_{\bar{i}}(s)p_{\bar{i}}(s). \quad (14)$$

The unconditional distribution function $f_{i,j}(t,s)$ which allows for an arbitrary number of transitions between the times s and t is given by

$$f_{i,j}(t,s) = \kappa_{\bar{i}}(t)p(\bar{i},t|j,s)\kappa_{\bar{j}}(s)p_{\bar{j}}(s), \quad (15)$$

where the conditional probability $p(i,t|j,s)$ is defined in Eq. (8). It now is straightforward to determine higher order entrance time distribution functions.

Finally we come back to single entrance time densities for an arbitrary n -state Markovian processes the dynamics of which is characterized by the transition rates $\kappa_{i,j}(t)$ from the state j to state i . The entrance density into state i is then given by

$$W_i(t) = \sum_{j \neq i} \kappa_{i,j}(t)p_j(t), \quad (16)$$

where the sum runs over all states except i and $p_j(t)$ is the probability to find the process at time t in the state j . Its time dependence follows from the master equation defined by the transition probabilities $\kappa_{i,j}(t)$ [15].

3. Periodic processes and stochastic resonance

We now come back to the family of Markovian two state processes and consider as a particular class processes with transition rates that are periodic functions of time, i.e.,:

$$\kappa_i(t+T) = \kappa_i(t) \quad \text{for } i = 1, 2 \text{ and for all } t, \quad (17)$$

where T denotes the period characterizing an external driving force. Note that then the asymptotic state given by Eq. (5) is a periodic function of time.

For a periodic force a phase can be defined in many ways, though, it always will increase by a factor of 2π after time has increased by the period T . Following the general idea outlined in Ref. [14], one can also introduce a phase for the process $x(t)$ that increases by 2π each time the process enters, say, the state 1. The generalized Rice frequency [14] that belongs to the such defined (random) phase is determined by the density of those events that lead to an increase of the phase by a factor of 2π . In the present case these events are given by the transitions into state 1, and consequently the frequency is given by the density of entrance points $W_1(t)$.

Hence, with this definition of a random phase the density of entrance points acquires the meaning of a generalized Rice frequency. This opens the possibility to characterize a possible frequency synchronization of the force and the corresponding driven process. As a synchronization condition [10,14] we require that the Rice frequency averaged over a period of the driving force must coincide with the frequency $2\pi/T$ of the driving force. It results if the average number of entrances into state 1, N_1 , during one period T is one

$$N_1 \equiv \int_0^T dt W_1(t) = 1. \quad (18)$$

This is a necessary but not a sufficient condition because only the time and ensemble averaged behavior of the process enters. In order to allow for deviations of the average behavior one has to consider higher correlations of the entrance times. The variance $\sigma_N^2(t)$ of the number $N(t)$ of entrances into state 1 from time 0 up to time t can be expressed in terms of the two-time correlation function [15]

$$\sigma_N^2(t) = \langle (N(t) - \langle N(t) \rangle)^2 \rangle = \langle N(t) \rangle + 2 \int_0^t ds \int_0^s ds' g_{1,1}(s, s'), \quad (19)$$

where the correlation function $g_{1,1}(s, s') = f_{1,1}(s, s') - W_1(s)W_1(s')$ exponentially decays as a function of the time difference $s - s'$. Hence, for large times the variance grows on average linearly in time, i.e., $\lim_{t \rightarrow \infty} \sigma_N^2(t)/t = D/(2\pi^2)$. Both, the number $N(t)$ and the Rice phase $\Phi(t) = 2\pi N(t)$ undergo a diffusional motion with the diffusion constants $D/(2\pi)^2$ and D , respectively. This aspect will further be discussed somewhere else; in the following we only consider the average behavior as it is characterized by the density of entrance points and its total number N_1 in a period.

The magnitudes of the rates $\kappa_i(t)$ typically are controlled by a noise strength. For vanishing noise the rates also vanish and monotonically increase with the noise strength. This leads to the growth of the number of entrances per period and eventually the synchronization condition (18) will be met. In the following we discuss the dependence of the entrance time density on the noise strength and on other relevant parameters in more detail.

First we consider a weakly driven process for which the rates can be linearized in the external force $f(t)$ which vanishes when averaged over a period T

$$\kappa_i(t) = \kappa_i^{(0)} + O(f(t)). \quad (20)$$

Consequently, we find for the asymptotic probabilities

$$p_1^{as}(t) = \frac{\kappa_2^{(0)}}{\kappa_1^{(0)} + \kappa_2^{(0)}} + O(f(t)), \quad p_2^{as}(t) = \frac{\kappa_1^{(0)}}{\kappa_1^{(0)} + \kappa_2^{(0)}} + O(f(t)). \quad (21)$$

Using these expressions in Eq. (18) one obtains as synchronization condition:

$$T = \frac{1}{\kappa_1^{(0)}} + \frac{1}{\kappa_2^{(0)}}, \quad (22)$$

where the corrections are of second order in the driving force $f(t)$ because the first order correction vanishes upon the average over a period. In the symmetric case, for $\kappa_1^{(0)} = \kappa_2^{(0)} = \kappa$, this relation simplifies to the well-known time scale matching condition for stochastic resonance [4]

$$\kappa T = 2. \quad (23)$$

As a typical example of a periodically driven process we now consider the McNamara–Wiesenfeld model of stochastic resonance which is defined by the rates [3]

$$\kappa_1(t) = \nu e^{-Q(1+a \cos \Omega t)}, \quad \kappa_2(t) = \nu e^{-Q(1-a \cos \Omega t)}, \quad (24)$$

where ν is an attempt frequency that, for the sake of simplicity, is assumed to be equal for both rates and independent of time. They are of the general form of a Kramers rate [17], however with time-dependent Arrhenius factors. The strength and frequency of the driving force are denoted by a and $\Omega = 2\pi/T$, respectively. The barrier height Q which is measured in units of the noise strength refers to the symmetric case in the absence of a driving force. In the following time will be measured in units of the inverse attempt frequency, accordingly Ω is measured in units of ν . Formally, this amounts to put $\nu = 1$ in the rate expressions (24).

Fig. 1 shows the resulting entrance time density $W_1(t)$ in the (periodic) asymptotic state as a function of time for different parameter values. For a sufficiently large value of the noise, i.e., for relatively small values of Q for which the rates are much larger than the driving frequency one observes two almost equally high peaks at the times at which the force vanishes, see also panel (a) of Fig. 2. In this case the process reaches the equilibrium relative to the instantaneous values of the rates before the rates have changed much. Hence, an adiabatic approximation can be performed for which the entrance time densities become

$$W_1^{ad}(t) = W_2^{ad}(t) = \frac{\kappa_1(t)\kappa_2(t)}{\kappa_1(t) + \kappa_2(t)}. \quad (25)$$

If the barrier height becomes larger, or, equivalently, the noise gets weaker, the rates and consequently the density of entrance points decrease. This happens preferentially for the first peak close to $T/4$ whereas the other peak which is close to $3T/4$ for strong noise, moves to later times and gains weight relatively to the first peak when the noise becomes weaker, see panel (a) of Fig. 1.

If the noise strength is further decreased one finally enters a regime in which the maximal rates are much smaller than the frequency of the driving force. Then the asymptotic probabilities only feel the average driving force and hence become

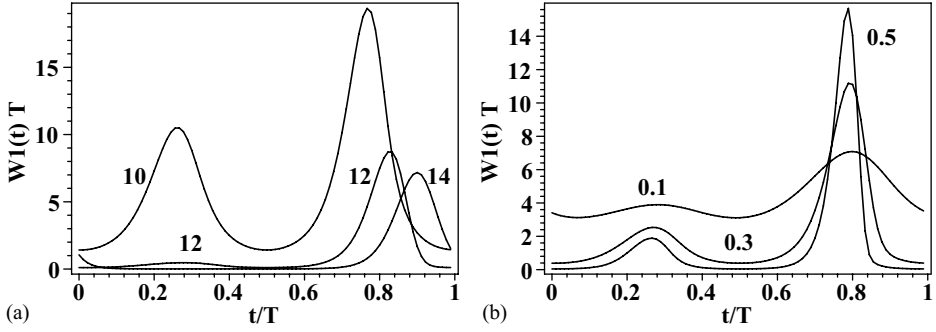


Fig. 1. For the McNamara Wiesenfeld model the density of entrance times is shown for the frequency $\Omega = 10^{-5}$ as a function of time. In panel (a) the driving strength has the fixed value $a = 0.3$ and the noise strength takes the values $Q = 10, 12, 14$. In panel (b) the respective curves are displayed for the noise strength $Q = 11$ and different values of the driving strength, $a = 0.1, 0.3, 0.5$. The entrance time density is scaled with the length of the period $T = 2\pi/\Omega$ such that the integral over the scaled time t/T gives the correct total number of entrances. With decreasing noise the first peak disappears and the second peak shifts to later times within one period, see panel (a); with increasing driving strengths the locations of the peaks hardly change, while the first peak disappears and the second peak increases, see panel (b).

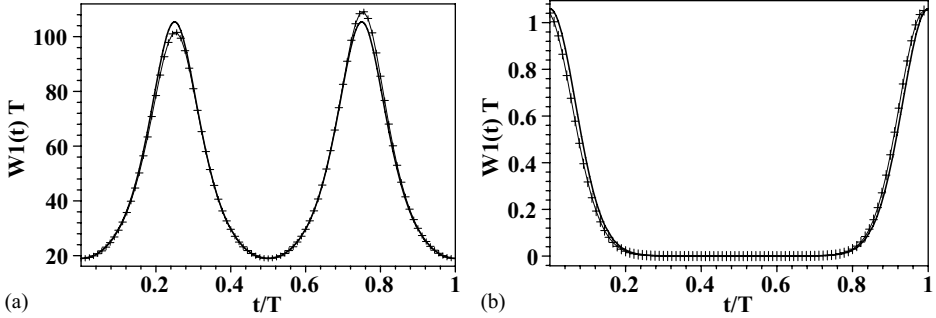


Fig. 2. Density of entrance times as a function of time for the driving strength $a = 0.3$, the driving frequency $\Omega = 10^{-5}$ and a large value of the noise strength $Q = 8$ is shown in panel (a) and for the small noise $Q = 18$ and the same other parameters in panel (b). The thin lines with crosses represent the respective exact expressions (9). In panel (a) the thick line shows the entrance time density in the adiabatic limit according to Eq. (25) and in panel (b) the weak noise approximation as given by Eq. (26).

approximately time independent and equal. Consequently one finds for the entrance time densities in the limit of weak noise:

$$W_1^{wn}(t) = \frac{1}{2} \kappa_2(t), \quad W_2^{wn}(t) = \frac{1}{2} \kappa_1(t), \quad (26)$$

where the superscripts indicate the limit of weak noise.

The dependence of the entrance time density on the period T of the driving force is qualitatively similar to that on the noise strength and therefore is not shown here. With increasing length of the period the first maximum of the entrance time density

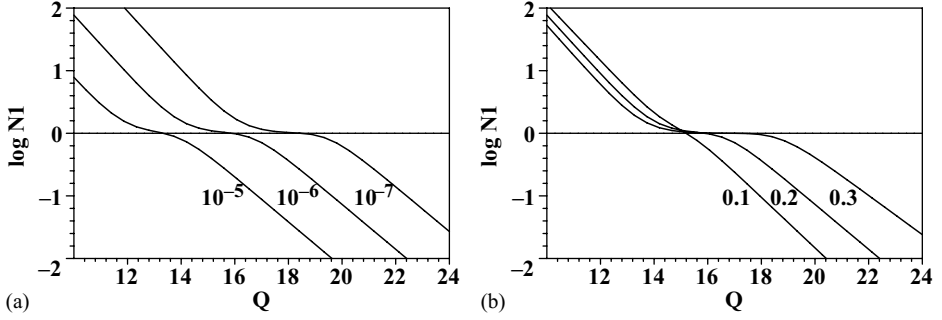


Fig. 3. The logarithm to base 10 of the total number of entrances within a period is shown as a function of the barrier height Q for different values of the frequency $\Omega = 10^{-5}, 10^{-6}, 10^{-7}$ at the fixed value of the driving strength $a = 0.2$ in panel (a) and for the fixed frequency $\Omega = 10^{-6}$ at different values of the driving strength $a = 0.1, 0.2, 0.3$ in panel (b). The positions of the locking intervals shift to larger values of the barrier height while their widths increase only little with decreasing period, see panel (a), whereas with increasing strength of the driving force their widths grow and their positions move to larger barrier heights, see panel (b).

decreases relatively to the second one which at the same time moves towards the end of the period. If the strength a of the driving force is increased the first peak of the entrance time density shrinks and the second one increases while the positions of the two peaks hardly change, see panel (b) of Fig. 1.

We now come to the discussion of the average number N_1 of entrances into state 1 within one period, as given by Eq. (18). As already mentioned, it monotonically decreases as a function of Q . Remarkably it shows a more or less pronounced plateau around the value of Q where it assumes the value 1, see Fig. 3. This behavior indicates that the transitions may synchronize with the external force not only for an isolated value of the noise strength but for a finite range of noise strengths, i.e., that the average frequency of transitions may be locked with the frequency of the driving force. Similar behavior is well known from periodically driven non-linear oscillators [18]. It also was found in the case of a periodically driven and weakly, or moderately, damped bistable Brownian oscillator for a driving strength slightly below the threshold above which the barrier vanishes [14]. Here we find an increase of the locking regions with increasing driving strength a which, however, always stays well below the threshold at $a = 1$, see panel (b) of Fig. 3. With decreasing frequency the locking regions shift to smaller noise strengths and grow slightly.

We observe a pronounced locking behavior if the driving force is sufficiently slow and its strength not too small. Then a time window exists within each period of the driving force during which a transition from one state to the other will occur almost certainly but a transition almost never occurs in the opposite direction within this window. If this time window is sufficiently short compared to the period T a small decrease of the noise strength will not change this behavior but only shift the window to a somewhat later time and slightly increase its length. In this way, with quite high precision, one transition per period will result in each direction in a whole range of noise strengths.

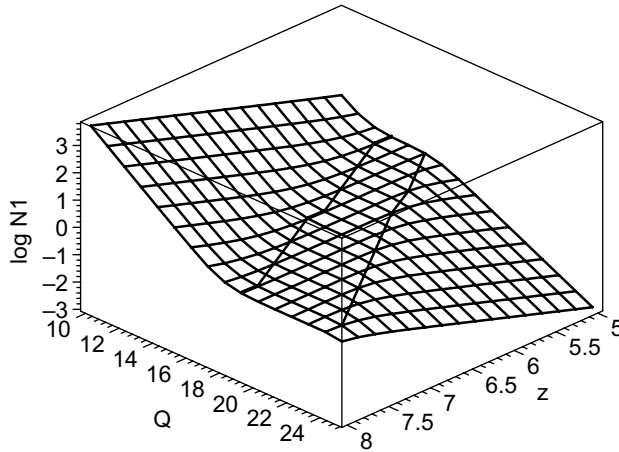


Fig. 4. The logarithm to base 10 of the average total number of entrances within a period as a function of the noise strength and the negative logarithm of the driving frequency, $z = -\log_{10} \Omega$ at the driving force $a = 0.2$ shows a pronounced plateau where the frequency of the process is locked with that of the driving. For the sake of better visibility two lines are shown that confine the plateau. Beyond these lines the Rice frequency deviates more than 5% from the driving frequency. The plateau widens for smaller noise strengths and driving frequencies.

The dependence of N_1 as a function of the frequency is similar to its noise dependence: with increasing frequency the number of entrances per period decreases. It shows locking within a frequency interval which becomes longer for weaker noise and for larger driving periods, see Fig. 4.

4. Conclusions

A simple expression for the entrance time density of general discrete Markov processes with time dependent rates has been given. For a two state process distribution functions of two entrance times and corresponding conditional distribution functions have been obtained. The generalization to Markov processes with an arbitrary number of states and to many time distribution functions describing the complete statistics of the point process of entrance times has been indicated and can be performed easily.

For a periodically driven two state process the entrance time density has been used to study the synchronization of the driving force and the driven process. The clue here is that the average number of transitions into either of the two states within a period determines an average frequency of the process. This phase is the discrete counterpart of the Rice phase of a smooth processes. A comparison of stochastic resonance and synchronization shows that the two effects are closely related but yet not identical. Employing the spectral amplification [5] as a quantitative measure for stochastic resonance we generally find the maximal amplification at a noise level at which more than one entrance takes place into either state within a period of the driving force,

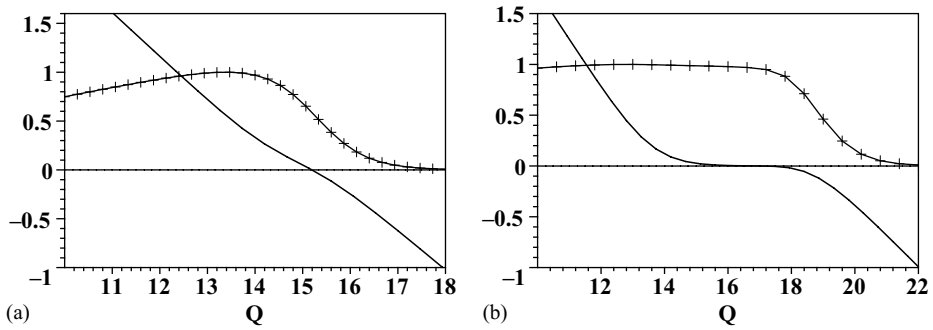


Fig. 5. The logarithm to base 10 of the total number of entrances within a period (thick line) and the spectral amplification normalized to its maximal value (thin line with crosses) are shown as functions of the noise strength for the driving frequency $\Omega = 10^{-6}$ in panel (a) for the driving force $a = 0.1$ and in panel (b) for $a = 0.3$.

i.e., the synchronization occurs for a smaller noise strength than stochastic resonance, see Fig. 5. A large locking regime is accompanied by a very extended maximum of the spectral amplification. The locations of the amplification plateau and of the locking regime are only slightly shifted relative to each other, the former being extending to larger noise strengths than the latter. A major difference of the two notions is that the synchronization leads to a *bona fide* resonance, i.e., it not only shows up as a function of the noise strength but also as a function of frequency, see Fig. 4.

Another measure of stochastic resonance that was proposed in literature and that also yields a *bona fide* resonance is based on the residence time statistics. It is given by the area under the peak of the residence time distribution at half the period [16]. According to this criterion this area is maximal at stochastic resonance. In general, this also happens at a slightly different noise strength than the one where the spectral amplification has its maximum. This approach was criticized by Choi et al. [13]. One of their arguments refers to the fact that the contributions to the area caused by spontaneous transitions and by the driving force are difficult to separate, in particular, in the region of stochastic resonance. In contrast, the number of entrance times on which the present criterion is based provides a direct measure that does not need further processing. It therefore appears as an ideal tool to investigate and characterize the response of a bistable system to a periodic driving force both in theory and experiment.

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