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Lei Zhao

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Quasi-periodic Almost-collision Orbits in the Spatial Three-body Problem

LEI ZHAO
Rijksuniversiteit Groningen

Dedicated to Alain Chenciner for his 70th birthday

Abstract

In a system of particles, quasi-periodic almost-collision orbits are collisionless orbits along which two bodies become arbitrarily close to each other – the lower limit of their distance is zero but the upper limit is strictly positive – and are quasi-periodic in a regularized system up to a change of time. Their existence was shown in the restricted planar circular three-body problem by A. Chenciner and J. Llibre, and in the planar three-body problem by J. Féjoz. In the spatial three-body problem, the existence of a set of positive measure of such orbits was predicted by C. Marchal. In this article, we present a proof of this fact.

1 Introduction

1.1 Quasi-periodic almost-collision orbits

In Chazy's classification of final motions of the three-body problem (see [2, p. 83]), possible final velocities were not specified for two particular kinds of possible motions: bounded motions, *i.e.* those motions along which the mutual distances remain bounded when time goes to infinity, and oscillating motions, *i.e.* those motions along which the upper limit of the mutual distances goes to infinity, while the lower limit of the mutual distances remains finite. A number of bounded motions and a few oscillating motions were known, with Sitnikov's model [16] being one of the well-known examples of the latter kind.

By replacing the oscillation of the mutual distances by the oscillation of the relative velocities of the bodies, we obtain another (possible) kind of oscillating motions, which was called by C. Marchal "oscillating motions of the second kind". By consulting the criteria for velocities in Chazy's classification, we see that if such motions do exist and are not (usual) oscillating motions, then they must be bounded.

In this article, we shall investigate a particular kind of oscillating motions of the second kind of the spatial three-body problem: the *quasi-periodic almost-collision orbits*, which are, by definition, collisionless orbits along which two bodies get arbitrarily close to each other: the lower limit of their distance is zero but the upper limit is strictly positive, and they are quasi-periodic in a regularized system.

In [12], by analyzing an integrable approximating system of the lunar spatial three-body problem (in which a far away third body is added to a two-body system) near a degenerate inner elliptic orbit, C. Marchal predicted the existence of a set of positive measure of oscillating orbits of the second kind in the spatial three-body problem. More precisely, the predicted orbits

- are with incommensurable frequencies;
- arise from invariant tori of the quadrupolar system $F_{sec}^{1,2}$ (see Subsection 4.3 for its definition);
- form a possibly nowhere dense set with small but positive measure in the phase space.

Up to a change of time, the predicted orbits are exactly the quasi-periodic almost-collision orbits that we shall investigate.

The fact that such orbits form a set of positive measure has a direct astronomical significance. Indeed, as noticed by Marchal, since real bodies occupy positive volumes in the universe, the existence of a set of positive measure of quasi-periodic almost-collision motions implies a positive probability of collisions in triple star systems with one body far away from the other two, and the probability is uniform with respect to their (positive) volumes. The collision mechanism given by quasi-periodic almost-collision orbits has much larger probability, and is thus more important compared to the mechanism given by direct collisions in the particle model, especially when the sizes of the modeled real massive bodies are small.

The first rigorous mathematical study of quasi-periodic almost-collision orbits was achieved by A. Chenciner and J. Llibre in [3], where they considered the planar circular restricted three-body problem in a rotating frame with a large enough Jacobi constant which determines a Hill region with three connected components. After regularizing the dynamics near the double collisions of the astroid with one of the primaries by Levi-Civita regularization, they reduced the dynamical study to the study of the corresponding Poincaré map on a global annulus of section in the regularized system. This map is a twist map with a small twist perturbed by a much smaller perturbation, which makes it possible to apply Moser's invariant curve theorem to establish the persistence of a set of positive measure of invariant KAM tori. By adjusting the Jacobi constant, a set of positive measure of such invariant tori was shown to intersect transversally the codimension 2 *collision set* (the set in the regularized phase space corresponding to the double collision of the astroid with the primary). Such invariant tori were called invariant "punctured" tori, as in the (non-regularized) phase space, they have a finite number of punctures corresponding to collisions. As the flow is linear and ergodic on each KAM torus, most of the orbits will not pass through but will get arbitrary close to the collision set. These orbits give rise to a set of positive measure of quasi-periodic almost-collision orbits in the planar circular restricted three-body problem.

In his thesis [4] (from which prepared the article [5]), J. Féjoz generalized the study of Chenciner-Llibre to the planar three-body problem. In his study, the inner double collisions being regularized, the *secular regularized systems*, *i.e.* the normal forms one gets by averaging over the fast angles, are established with the same averaging method as the usual non-regularized ones. A careful analysis shows that the dynamics of the secular regularized system and the naturally extended (through degenerate inner ellipses) secular systems are orbitally conjugate, up to a modification of the mass of the third body which is far away from the inner pair. The persistence of a set of positive measure of invariant tori is obtained by the application of a sophisticated version of KAM theorem. After verifying the transversality of the intersections between the KAM tori and the codimension 2 collision set corresponding to collisions of the inner pair, he concluded in the same way as Chenciner-Llibre.

In this article, we generalize the studies of Chenciner-Llibre and Féjoz to the spatial three-body problem, and confirm the prediction of Marchal.

Theorem 1.1. *In the spatial three-body problem, there exists a set of positive measure of quasi-periodic almost-collision orbits on each negative energy surface. The set of quasi-periodic almost-collision orbits has positive measure in the phase space.*

1.2 Outline of the proof

We confine ourselves to the "lunar case", consisting of a two-body system together with a far-away third body. We decompose the Hamiltonian F of the three-body problem into two parts

$$F = F_{Kep} + F_{pert},$$

where F_{Kep} is the sum of two uncoupled Keplerian Hamiltonians, and F_{pert} is significantly smaller than each of the two Keplerian Hamiltonians in F_{Kep}. The dynamics of F can thus be described as a pair of almost uncoupled Keplerian motions together with the slow evolutions of the Keplerian orbits (see Section 2).

As in [3], [5], the strategy is to find a set of positive measure of irrational invariant tori in the corresponding energy level of a regularized system $\mathscr{F}$ of F. More precisely, we shall

(1) regularize the inner double collisions of F on the energy surface $\{F = -f < 0\}$ by Kustaanheimo-Stiefel regularization to obtain a Hamiltonian $\mathscr{F}$ regular at the collision set, corresponding to the inner double collisions of F (see Section 3);
(2) build an integrable truncated normal form $\mathscr{F}_{Kep} + \overline{\mathscr{F}_{sec}^{n,n'}}$ of $\mathscr{F}$ (see Section 4) and study its Lagrangian invariant tori passing near the collision set (see Section 5, where in particular, we extend Lidov-Ziglin's study [11] of the quadrupolar dynamics to degenerate inner ellipses);
(3) apply an iso-energetic KAM theorem for properly-degenerate systems to find a set of positive measure of invariant tori of $\mathscr{F}$ on its zero-energy level (only on which the dynamics of $\mathscr{F}$ extends the dynamics of F; see Section 6);
(4) show that a set of positive measure of invariant ergodic tori intersect transversely the collision set in submanifolds of codimension at least 2; conclude that there exists a set of positive measure of quasi-periodic almost-collision orbits on the energy surface $\{F = -f\}$; finally, by varying f, conclude that these orbits form a set of positive measure in the phase space of F (see Section 7).

2 Hamiltonian formalism of the three-body problem

2.1 The Hamiltonian system

The three-body problem is a Hamiltonian system defined on the phase space

$$\Pi := \left\{(q_0, q_1, q_2, p_0, p_1, p_2) \in (\mathbb{R}^3 \times \mathbb{R}^3)^3 | \forall 0 \leq j \neq k \leq 2, q_j \neq q_k\right\},$$

with the (standard) symplectic form

$$\sum_{j=0}^{2} \sum_{l=1}^{3} dp_j^l \wedge dq_j^l,$$

and the Hamiltonian function

$$F = \frac{1}{2} \sum_{0 \leq j \leq 2} \frac{\|p_j\|^2}{m_j} - \sum_{0 \leq j < k \leq 2} \frac{m_j m_k}{\|q_j - q_k\|},$$

in which q_0, q_1, q_2 denote the positions of the three particles, p_0, p_1, p_2 denote their conjugate momenta respectively, and

$$q_i = (q_i^1, q_i^2, q_i^3),\ p_i = (p_i^1, p_i^2, p_i^3),\ i = 0, 1, 2.$$

The Euclidean norm of a vector in $\mathbb{R}^3$ is denoted by $\|\cdot\|$. The gravitational constant has been set to 1.

2.2 Jacobi decomposition

The Hamiltonian F is invariant under translations in positions. To reduce the system by this symmetry, we switch to the *Jacobi coordinates* $(P_0, Q_0, P_1, Q_1, P_2, Q_2)$ defined as

$$\begin{cases} P_0 = p_0 + p_1 + p_2 \\ P_1 = p_1 + \sigma_1 p_2 \\ P_2 = p_2 \end{cases} \qquad \begin{cases} Q_0 = q_0 \\ Q_1 = q_1 - q_0 \\ Q_2 = q_2 - \sigma_0 q_0 - \sigma_1 q_1, \end{cases}$$

where

$$\frac{1}{\sigma_0} = 1 + \frac{m_1}{m_0}, \frac{1}{\sigma_1} = 1 + \frac{m_0}{m_1}.$$

Due to the translation-invariance, the Hamiltonian function is independent of Q_0. We fix the first integral P_0 (conjugate to Q_0) at $P_0 = 0$ and reduce the translation symmetry of the system by eliminate Q_0. In the reduced coordinates (P_1, Q_1, P_2, Q_2), the (reduced) Hamiltonian $F = F(P_1, Q_1, P_2, Q_2)$ describes the motions of two fictitious particles.

We further decompose the Hamiltonian $F(P_1,Q_1,P_2,Q_2)$ into two parts

$$F = F_{Kep} + F_{pert},$$

where the *Keplerian part* F_{Kep} and the *perturbing part* F_{pert} are defined respectively

$$F_{Kep} = \frac{\|P_1\|^2}{2\mu_1} + \frac{\|P_2\|^2}{2\mu_2} - \frac{\mu_1 M_1}{\|Q_1\|} - \frac{\mu_2 M_2}{\|Q_2\|},$$

$$F_{pert} = -\mu_1 m_2 \Big[\frac{1}{\sigma_o}\Big(\frac{1}{\|Q_2 - \sigma_0 Q_1\|} - \frac{1}{\|Q_2\|}\Big) + \frac{1}{\sigma_1}\Big(\frac{1}{\|Q_2 + \sigma_1 Q_1\|} - \frac{1}{\|Q_2\|}\Big)\Big],$$

with (as in [5])

$$\frac{1}{\mu_1} = \frac{1}{m_0} + \frac{1}{m_1}, \frac{1}{\mu_2} = \frac{1}{m_0 + m_1} + \frac{1}{m_2},$$
$$M_1 = m_0 + m_1, M_2 = m_0 + m_1 + m_2.$$

We shall only be interested in the region of the phase space where our system $F = F_{Kep} + F_{pert}$ is a small perturbation of a pair of Keplerian elliptic motions.

3 Regularization

We aim at carrying out a perturbative study near the inner double collisions $\{\|Q_1\| = 0\}$ where the Hamiltonian F is singular. To this end, we have to regularize the system. We shall use Kustaanheimo-Stiefel regularization (c.f. [17]) and, starting with a formula appearing in [13], we formulate this method in a quaternionic way (see [19] for more detailed discussions). Earlier slightly different quaternionic formulations can be found in [18], [15].

3.1 Kustaanheimo-Stiefel regularization

Let $\mathbb{H}\,(\cong \mathbb{R}^4)$ be the space of quaternions $z = z_0 + z_1 i + z_2 j + z_3 k$, $\mathbb{IH}\,(\cong \mathbb{R}^3)$ be the space of purely imaginary quaternions (*i.e.* those $z \in \mathbb{H}$ with $z_0 = 0$), $\bar{z} = z_0 - z_1 i - z_2 j - z_3 k$ and $|z| = \sqrt{\bar{z}z}$ be the conjugate quaternion and the quaternionic modulus of z respectively.

Let $(z,w) \in \mathbb{H} \times \mathbb{H} \cong T^*\mathbb{H}$ be a pair of quaternions. We define the Kustaanheimo-Stiefel mapping

$$K.S. : T^*(\mathbb{H} \setminus \{0\}) \to \mathbb{IH} \times \mathbb{H}$$
$$(z,w) \longmapsto \Big(Q = \bar{z}iz, P = \frac{\bar{z}iw}{2|z|^2}\Big).$$

We observe that this mapping has circle fibers

$$\{(e^{i\vartheta}z, e^{i\vartheta}w)\} \qquad \vartheta \in \mathbb{R}/2\pi\mathbb{Z}.$$

These fibers define a Hamiltonian circle action on the symplectic manifold $(T^*\mathbb{H}, Re\{d\bar{w}\wedge dz\})$ with (up to sign) the moment map

$$BL(z,w) = Re\{\bar{z}iw\} = \bar{z}iw + \overline{\bar{z}iw}$$

regarded as a function defined on $T^*\mathbb{H} \cong \mathbb{H}\times\mathbb{H}$. The equation

$$BL(z,w) = 0$$

defines a 7-dimensional quadratic cone Σ in $T^*\mathbb{H}$. By removing the point $(0,0)$ from Σ, we obtain a 7-dimensional coisotropic submanifold Σ^0 of the symplectic manifold $(T^*\mathbb{H}, Re\{d\bar{w}\wedge dz\})$, on which the above-mentioned circle action is free. Consequently, the quotient V^0 of Σ^0 by this S^1-action is a 6-dimensional symplectic manifold equipped with the induced symplectic form ω_1.

We define the 7-dimensional coisotropic submanifold Σ^1 by removing the set $\{z=0\}$ from Σ. Analogously, by passing to the quotient, it descends to a 6-dimensional symplectic manifold (V^1, ω_1).

Proposition 3.1. [19, Prop 3.3] *$K.S.$ induces a symplectomorphism from (V^1, ω_1) to $(T^*(\mathbb{IH}\setminus\{0\}), Re\{d\bar{P}\wedge dQ\})$.*

3.2 Regularized Hamiltonian

On the fixed negative energy surface

$$\{F = -f < 0\},$$

we make a time change (singular at inner double collisions) by passing to the new time variable τ satisfying

$$\|Q_1\|\, d\tau = dt.$$

In time τ, the corresponding motions of the particles are governed by the Hamiltonian $\|Q_1\|(F+f)$ and are lying inside its zero-energy level. We extend $K.S.$ to the mapping (the notation $K.S.$ is abusively retained for the extension)

$$(z,w,P_2,Q_2) \mapsto (Q_1 = \bar{z}iz, P_1 = \frac{\bar{z}iw}{2|z|^2}, P_2, Q_2)$$

and set

$$\mathscr{F} = K.S.^*\Big(\|Q_1\|\,(F+f)\Big).$$

This is a function on $\Sigma^0 \times T^*(\mathbb{R}^3 \setminus \{0\})$ decomposed as

$$\mathscr{F} = \mathscr{F}_{Kep} + \mathscr{F}_{pert},$$

with the *regularized Keplerian part*

$$\mathscr{F}_{Kep} = K.S.^*\Big(\|Q_1\|(F_{Kep}+f)\Big) = \frac{|w|^2}{8\mu_1} + \Big(f + \frac{\|P_2\|^2}{2\mu_2} - \frac{\mu_2 M_2}{\|Q_2\|}\Big)|z|^2 - \mu_1 M_1$$

describing the skew-product motion of the outer body moving on a Keplerian elliptic orbit, slowed-down by four "inner" harmonic oscillators in 1 : 1 : 1 : 1-resonance, and the *regularized perturbing part*

$$\mathscr{F}_{pert} = K.S.^*\Big(\|Q_1\|F_{pert}\Big);$$

both terms extend analytically through the set $\{z = 0\}$ corresponding to the inner double collisions of F.

By its expression, the function $\mathscr{F}$ can be directly regarded as a function on $T^*(\mathbb{H} \setminus \{0\}) \times T^*(\mathbb{R}^3 \setminus \{0\})$. As it is invariant under the fiber action of $K.S.$, its flow preserves $\Sigma^0 \times T^*(\mathbb{R}^3 \setminus \{0\})$. In the sequel, the relation $BL(z,w) = 0$ is always assumed to be satisfied, *i.e.* we always restrict ourselves to $\Sigma^0 \times T^*(\mathbb{R}^3 \setminus \{0\})$.

3.3 A set of action-angle coordinates

To write F_{Kep} in action-angle form, we start by defining the (symplectic) Delaunay coordinates $(L_2, l_2, G_2, g_2, H_2, h_2)$ for the outer body. Let a_2, e_2, i_2 be respectively the semi major axis, the eccentricity and the inclination of the outer ellipse. The Delaunay coordinates are defined as the follows:

$$\begin{cases} L_2 = \mu_2 \sqrt{M_2}\sqrt{a_2} & \text{circular angular momentum} \\ l_2 & \text{mean anomaly} \\ G_2 = L_2\sqrt{1-e_2^2} & \text{angular momentum} \\ g_2 & \text{argument of pericentre} \\ H_2 = G_2 \cos i_2 & \text{vertical component of the angular momentum} \\ h_2 & \text{longitude of the ascending node.} \end{cases}$$

In these coordinates, we have

$$f + \frac{\|P_2\|^2}{2\mu_2} - \frac{\mu_2 M_2}{\|Q_2\|} = f - \frac{\mu_2^3 M_2^2}{2L_2^2},$$

which is positive by hypothesis. We denote it by $f_1(L_2)$. Now

$$\mathscr{F}_{Kep} = \frac{|w|^2}{8\mu_1} + f_1(L_2)|z|^2 - \mu_1 M_1.$$

Let

$$\sqrt{8\mu_0 f_1(L_2)}\, z_i = \sqrt{2I_i}\sin\phi_i, \quad w_i = \sqrt{2I_i}\cos\phi_i, \quad i = 0,1,2,3.$$

and

$$\mathscr{P}_0 = \frac{(I_0 + I_1 + I_2 + I_3)}{2\sqrt{8\mu_0 f_1(L_2)}}, \vartheta_0 = 2\phi_0,$$
$$\mathscr{P}_i = \frac{I_i}{\sqrt{8\mu_0 f_1(L_2)}}, \qquad \vartheta_i = \phi_i - \phi_0, \quad i = 1,2,3.$$

One directly checks that

$$\mathscr{P}_0 \wedge \vartheta_0 + \mathscr{P}_1 \wedge \vartheta_1 + \mathscr{P}_2 \wedge \vartheta_2 + \mathscr{P}_3 \wedge \vartheta_3 + dL_2 \wedge dl_2' + dG_2 \wedge dg_2 + dH_2 \wedge dh_2$$
$$= Re\{d\bar{w} \wedge dz\} + dL_2 \wedge dl_2 + dG_2 \wedge dg_2 + dH_2 \wedge dh_2,$$

with $l_2' = l_2 + \dfrac{f_1'(L_2)}{2f_1(L_2)} Re\{\bar{P}_1 Q_1\}$.

We thus obtain a set of Darboux coordinates

$$(\mathscr{P}_0, \vartheta_0, \mathscr{P}_1, \vartheta_1, \mathscr{P}_2, \vartheta_2, \mathscr{P}_3, \vartheta_3, L_2, l_2', G_2, g_2, H_2, h_2),$$

in which

$$\mathscr{F}_{Kep} = \mathscr{P}_0 \sqrt{\frac{2f_1(L_2)}{\mu_1}} - \mu_1 M_1.$$

The coordinates $(\mathscr{P}_0, \vartheta_0, \mathscr{P}_1, \vartheta_1, \mathscr{P}_2, \vartheta_2, \mathscr{P}_3, \vartheta_3)$ are well-defined on the dense open set of $T^*\mathbb{H} \setminus \{(0,0)\}$ on which $I_i > 0, i = 0,1,2,3$, *i.e.* the projections of the elliptic orbits of the four harmonic oscillators in $1:1:1:1$ resonance in the four (z_i, w_i)-planes are non-degenerate, a condition which can always be satisfied by simultaneously rotating these planes properly. In the sequel, without loss of generality, we shall always assume that these conditions are satisfied.

4 Normal Forms

4.1 Physical dynamics of the regularized Keplerian Hamiltonian

The function $\mathscr{F}_{Kep}$ describes a *properly-degenerate* Hamiltonian system: it depends only on 2 of the action variables out of 7. To deduce the dynamics of $\mathscr{F}$, study of higher order is necessary.

The perturbing part $\mathscr{F}_{pert}$ describes, in the regularized phase space, the mutual interactions between two particles $Q_1 = \bar{z}iz$ and Q_2 in the physical space. Under $\mathscr{F}_{Kep}$, the particle Q_2 moves on elliptic orbits. When $\mathscr{F}_{Kep}$ is close to zero, this is also the case for Q_1:

Lemma 4.1. *Under the flow of $\mathscr{F}_{Kep}$, when $BL(z,w) = 0$, in the energy hypersurface $\mathscr{F}_{Kep} = \tilde{f}$ for any $\tilde{f} > -\mu_1 M_1$, the physical image $Q_1 = \bar{z}iz$ of z moves on a Keplerian elliptic orbit.*

Proof. The equation

$$\mathscr{F}_{Kep} = \frac{|w|^2}{8\mu_1} + f_1(L_2)\,|z|^2 - \mu_1 M_1 = \tilde{f}$$

is equivalent to

$$\|Q_1\|\Big(\frac{\|P_1\|^2}{2\mu_1} - \frac{\mu_1 M_1 + \tilde{f}}{\|Q_1\|} + f_1(L_2)\Big) = 0,$$

that is

$$\frac{\|P_1\|^2}{2\mu_1} - \frac{\mu_1 M_1 + \tilde{f}}{\|Q_1\|} = -f_1(L_2) < 0.$$

By assumption $\mu_1 M_1 + \tilde{f} > 0$. The motion of Q_1 is thus governed, up to time parametrization, by the Hamiltonian of a Kepler problem on a fixed negative energy surface; as the orbits are uniquely determined by their energy, the conclusion follows. □

We have seen from the above proof that, in the physical space, inner Keplerian ellipses are orbits of the Kepler problem with (modified) mass parameters μ_1 and $M_1 + \frac{\mathscr{F}_{Kep}}{\mu_1}$. The inner elliptic elements, *e.g.* the inner semi major axis a_1 and the inner eccentricity e_1, are the corresponding elliptic elements of the orbit of Q_1. One directly checks that $a_1 = \frac{\mathscr{P}_0}{\sqrt{2\mu_1 f_1(L_2)}}$, and ϑ_0 differs from the inner eccentric anomaly u_1 only by a phase shift.

4.2 Asynchronous elimination

Let e_1, e_2 be the eccentricities of the inner and outer ellipses respectively, a_2 be the outer semi major axis, and $\alpha = a_1/a_2$ be the ratio of semi major axes. We assume that

- the masses m_0, m_1, m_2 are (arbitrarily) fixed;
- the coordinates $(\mathscr{P}_0, \vartheta_0, \mathscr{P}_1, \vartheta_1, \mathscr{P}_2, \vartheta_2, \mathscr{P}_3, \vartheta_3, L_2, l_2, G_2, g_2, H_2, h_2)$ are all well-defined;

- three positive real numbers $e_1^\vee, e_2^\vee < e_2^\wedge$ are fixed, and
$$0 < e_1^\vee < e_1 \leq 1, \quad 0 < e_2^\vee < e_2 < e_2^\wedge < 1;$$
- two positive real numbers $a_1^\vee < a_1^\wedge$ are fixed, and
$$a_1^\vee < a_1 < a_1^\wedge;$$
- $\alpha < \alpha_0$ for some sufficiently small α_0 satisfying
$$0 < \alpha_0 < \alpha^\wedge := \min\{\frac{1-e_2^\wedge}{80}, \frac{1-e_2^\wedge}{2\sigma_0}, \frac{1-e_2^\wedge}{2\sigma_1}\}.$$

We shall take α_0 as the small parameter in this study, whose introduction is aimed to have a parameter independent of the dynamics. The required smallness of α_0 will be more precisely given in the sequel.

These assumptions determine a subset $\mathscr{P}^*$ of
$$T^*(\mathbb{H} \setminus \{0\}) \times T^*(\mathbb{R}^3 \setminus \{0\}).$$
With the coordinates defined above, we may identify $\mathscr{P}^*$ to a subset of $\mathbb{T}^7 \times \mathbb{R}^7$.

Under these assumptions, $|\mathscr{F}_{Kep}|$ is bounded, and the two Keplerian frequencies
$$\nu_1 = \frac{\partial \mathscr{F}_{Kep}}{\partial P_0} = \sqrt{\frac{2f_1(L_2)}{\mu_1}} \sim 1, \ \nu_2 = \frac{\partial \mathscr{F}_{Kep}}{\partial L_2} = \frac{\mu_2^3 M_2^2 \mathscr{P}_0}{2L_2^3\sqrt{2\mu_1 f_1(L_2)}} \sim \alpha^{\frac{3}{2}}$$
do not appear at the same order of α. This enables us to proceed, as in Jefferys-Moser [9] or Féjoz [5], by eliminating the dependence of $\mathscr{F}$ on each of the fast angles ϑ_0, l_2' without imposing any arithmetic condition on the two Keplerian frequencies.

Let $T_{\mathbb{C}} = \mathbb{C}^7/\mathbb{Z}^7 \times \mathbb{C}^7$ and T_s be the s-neighborhood of the direct product $\mathbb{T}^7 \times \mathbb{R}^7 := \mathbb{R}^7/\mathbb{Z}^7 \times \mathbb{R}^7$ in $T_{\mathbb{C}}$. Let $T_{\mathbf{A},s}$ be the s-neighborhood
$$\{z \in T_s : \exists x \in \mathbf{A}, \text{s.t. } |z-x| < s\}$$
of a set $\mathbf{A} \subset \mathbb{T}^7 \times \mathbb{R}^7$ in T_s. The complex modulus of a transformation is the maximum of the complex moduli of its components. We use $|\cdot|$ to denote the modulus of either one. A real analytic function and its complex extension are denoted by the same notation.

Proposition 4.2. *For any $n \in \mathbb{N}$, there exists an analytic Hamiltonian*
$$\mathscr{F}^n : \mathscr{P}^* \to \mathbb{R}$$

independent of the fast angles ϑ_1, l_2'*, and an analytic symplectomorphism* $\phi^n : \mathscr{P}^* \supset \tilde{\mathscr{P}}^n \to \phi^n(\tilde{\mathscr{P}}^n)$, $|\alpha|^{\frac{3}{2}}$*-close to the identity, such that*

$$|\mathscr{F} \circ \phi^n - \mathscr{F}^n| \leq C_0 |\alpha|^{\frac{3(n+2)}{2}}$$

on $T_{\tilde{\mathscr{P}}^n, s''}$ *for some open set* $\tilde{\mathscr{P}}^n \subset \mathscr{P}^*$*, and some real number* s'' *with* $0 < s'' < s$*, such that locally the relative measure of* $\tilde{\mathscr{P}}^n$ *in* $\mathscr{P}^*$ *tends to 1 when* α *tends to* 0.

Proof. We first eliminate the dependence of $\mathscr{F}$ on ϑ_1 up to a remainder of order $O(|\alpha|^{\frac{3(n+2)}{2}})$ and then eliminate the dependence on l_2' up to a remainder of order $O(|\alpha|^{\frac{3(n+2)}{2}})$.

The elimination procedure is standard and consists in analogous successive steps. The first step is to eliminate ϑ_1 up to a $O(\alpha^{\frac{9}{2}})$-remainder. To this end we look for an auxiliary analytic Hamiltonian $\hat{H}$. We denote its Hamiltonian vector field and its flow by $X_{\hat{H}}$ and ϕ_t respectively. The required symplectic transformation is the time-1 map $\phi_1 (:= \phi_t|_{t=1})$ of $X_{\hat{H}}$. We have

$$\phi_1^* \mathscr{F} = \mathscr{F}_{Kep} + (\mathscr{F}_{pert} + X_{\hat{H}} \cdot \mathscr{F}_{Kep}) + \mathscr{F}^1_{comp,1},$$

for some remainder $\mathscr{F}^1_{comp,1}$. In the above, $X_{\hat{H}}$ is seen as a derivation operator.

Let

$$\langle \mathscr{F}_{pert} \rangle_1 = \frac{1}{2\pi} \int_0^{2\pi} \mathscr{F}_{pert} \, d\vartheta_1$$

be the average of $\mathscr{F}_{pert}$ over ϑ_1, and $\widetilde{F}_{pert,1} = \mathscr{F}_{pert} - \langle \mathscr{F}_{pert} \rangle_1$ be its zero-average part.

In order to have $\mathscr{F}^1_{comp,1} = O(\alpha^{\frac{9}{2}})$, we choose $\hat{H}$ to solve the equation

$$\nu_1 \partial_{l_1} \hat{H} = \widetilde{\mathscr{F}}_{pert,1},$$

which is satisfied if we set

$$\hat{H} = \frac{1}{\nu_1} \int_0^{l_1} \widetilde{\mathscr{F}}_{pert,1} \, dl_1,$$

which is of order $O(\alpha^3)$ in $T_{\mathscr{P}^*, s}$ for sufficiently small s. By Cauchy inequality, $|X_{\hat{H}}| = O(\alpha^3)$ in $T_{\mathscr{P}^*, s-s_0}$ for $0 < s_0 < s/2$. We shrink the domain from $T_{\mathscr{P}^*, s-s_0}$ to $T_{\mathscr{P}^{**}, s-s_0-s_1}$, where $\mathscr{P}^{**}$ is an open subset of $\mathscr{P}^*$, so that

$\phi_1(T_{\mathscr{P}^{**},s-s_0-s_1}) \subset T_{\mathscr{P}^*,s-s_0}$, with $s-s_0-s_1>0$. The time-1 map ϕ_1 of X_H thus satisfies $|\phi_1 - Id| \leq \text{Cst } |\alpha|^3$ in $T_{\mathscr{P}^{**},s-s_0-s_1}$, and hence

$$\phi_1^*\mathscr{F} = \mathscr{F}_{Kep} + \langle \mathscr{F}_{pert}\rangle_1 + \mathscr{F}^1_{comp,1},$$

is defined on $T_{\mathscr{P}^{**},s-s_0-s_1}$ and satisfies

$$\begin{aligned}\mathscr{F}^1_{comp,1} &= \int_0^1 (1-t)\phi_t^*(X^2_{\hat{H}} \cdot \mathscr{F}_{Kep})dt + \int_0^1 \phi_t^*(X_{\hat{H}} \cdot \mathscr{F}_{pert})dt - \nu_2 \frac{\partial \hat{H}}{\partial l_2}\\ &\leq \text{Cst } |X_{\hat{H}}|(|\widetilde{\mathscr{F}_{pert,1}}| + |\mathscr{F}_{pert}|) + \nu_2|\hat{H}| \leq \text{Cst } |\alpha|^{\frac{9}{2}}.\end{aligned}$$

The first step of eliminating ϑ_0 is completed.

Analogously, we may eliminate the dependence of the Hamiltonian on the angle ϑ_1 up to order $O(\alpha^{\frac{3(n+2)}{2}})$ for any chosen $n \in \mathbb{Z}_+$. The Hamiltonian $\mathscr{F}$ is then analytically conjugate to

$$\mathscr{F}_{Kep} + \langle \mathscr{F}_{pert}\rangle_1 + \langle \mathscr{F}^1_{comp,1}\rangle_1 + \cdots + \langle \mathscr{F}^1_{comp,n-1}\rangle_1 + \mathscr{F}^1_{comp,n},$$

in which the expression $\mathscr{F}_{Kep} + \langle \mathscr{F}_{pert}\rangle_1 + \langle \mathscr{F}^1_{comp,1}\rangle_1 + \cdots + \langle \mathscr{F}^1_{comp,n-1}\rangle_1$ is independent of ϑ_1, and $\mathscr{F}^1_{comp,n}$ is of order $O\left(\alpha^{\frac{3(n+2)}{2}}\right)$.

The elimination of l_2' in

$$\mathscr{F}_{Kep} + \langle \mathscr{F}_{pert}\rangle_1 + \langle \mathscr{F}^1_{comp,1}\rangle_1 + \cdots + \langle \mathscr{F}^1_{comp,n-1}\rangle_1 + \mathscr{F}^1_{comp,n}$$

is analogous. Let $\langle\cdot\rangle$ denotes the averaging of a function over ϑ_1 and l_2'. The Hamiltonian generating the transformation of the first step of eliminating l_2' is

$$\frac{1}{\nu_2}\int_0^{l_2} (\langle F_{pert}\rangle_1 - \langle F_{pert}\rangle)dl_2 = O(\alpha^{\frac{3}{2}}).$$

This implies that the transformation is $|\alpha|^{\frac{3}{2}}$-close to the identity.

The Hamiltonian $\mathscr{F}$ is thus conjugate to

$$\mathscr{F}_{Kep} + \langle \mathscr{F}_{pert}\rangle + \langle \mathscr{F}_{comp,1}\rangle + \cdots + \langle \mathscr{F}_{comp,n-1}\rangle + \mathscr{F}_{comp,n},$$

in which

$$\langle \mathscr{F}_{pert}\rangle = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \mathscr{F}_{pert} d\vartheta_1 dl_2'.$$

The *n-th order secular system*

$$\mathscr{F}^n_{sec} := \langle \mathscr{F}_{pert}\rangle + \langle \mathscr{F}_{comp,1}\rangle + \cdots + \langle \mathscr{F}_{comp,n-1}\rangle$$

(by convention $\mathscr{F}_{pert} = \mathscr{F}_{comp,0}$) is independent of ϑ_1, l_2', and the remainder $\mathscr{F}_{comp,n}$ is of order $O(\alpha^{\frac{3(n+2)}{2}})$ in $T_{\tilde{\mathscr{P}}^n,s''}$ for some open subset $\tilde{\mathscr{P}}^n \subset \mathscr{P}^*$ and some $0 < s'' < s$ both of which are obtained by a finite number of steps of eliminations. In particular, the set $\tilde{\mathscr{P}}^n$ is obtained by shrinking $\mathscr{P}^*$ from its boundary by a distance of $O(\alpha^{\frac{3}{2}})$. We may thus set

$$\mathscr{F}^n := \mathscr{F}_{Kep} + \mathscr{F}^n_{sec}.$$

□

4.3 Elimination of g_2

The Hamiltonian $\mathscr{F}^n$ has 7 degrees of freedom. It is invariant under the action of the group $\mathbb{T}^3 \times SO(3)$ consisting in the fiber circle action of $K.S.$, the $\mathbb{T}^2$-action of the fast angles, and the induced $SO(3)$-action by the simultaneous rotations of positions and momenta in the phase space. Standard symplectic reduction procedure leads to a 2-degrees-of-freedom reduced system with no other obvious continuous symmetries. It is *a priori* not integrable.

In light of [8], [11], to obtain an integrable approximating system of $\mathscr{F}$, we proceed in the following way: The function $\mathscr{F}_{pert}$ is naturally an analytic function of $a_1, a_2, \frac{Q_1}{a_1}, \frac{Q_2}{a_2}$ (by replacing Q_i by $a_i \frac{Q_i}{a_i}, i = 1, 2$). Through the relation $a_2 = \frac{a_1}{\alpha}$, it is also an analytic function of $a_1, \alpha, \frac{Q_1}{a_1}, \frac{Q_2}{a_2}$. The calculation of F^n_{sec} from the power series of $\mathscr{F}_{pert}$ in α naturally leads to the expansion

$$\mathscr{F}^n_{sec} = \sum_{i=2}^{\infty} \mathscr{F}^{n,i}_{sec} \alpha^{i+1}.$$

By construction,

$$\mathscr{F}^n_{sec} - \mathscr{F}^1_{sec} = \langle \mathscr{F}_{comp,1} \rangle + \cdots + \langle \mathscr{F}_{comp,n-1} \rangle + \mathscr{F}_{comp,n} = O(\alpha^{\frac{9}{2}}),$$

which implies

$$\mathscr{F}^{n,2}_{sec} = \mathscr{F}^{1,2}_{sec}, \ \mathscr{F}^{n,3}_{sec} = \mathscr{F}^{1,3}_{sec}, \quad \forall n \in \mathbb{Z}_+.$$

The following lemma shows in particular that $\mathscr{F}_{quad} = \mathscr{F}^{1,2}_{sec}$ has an additional circle symmetry, and is thus integrable.

Lemma 4.3. *The function $\mathscr{F}^{1,2}_{sec}$ depends non-trivally on G_2, but is independent of g_2. The function $\mathscr{F}^{1,3}_{sec}$ depends non-trivially on g_2.*

Proof. Consider the function F_{pert}, which is naturally a function of l_1 (resp. u_1), the mean anomaly (resp. eccentric anomaly) of the inner Keplerian ellipse, when l_1 (resp. u_1) is well defined. We define

$$F^1_{sec} = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} F_{pert}\, dl_1 dl_2.$$

and develop F^1_{sec} in powers of α:

$$F^1_{sec} = \sum_{i=2}^{\infty} F^{1,i}_{sec}\, \alpha^{i+1}.$$

We see in [10] that $F^{1,2}_{sec}$ depends non-trivially on G_2 (through e_2), but is independent of g_2, and $F^{1,3}_{sec}$ depends non-trivially on g_2.

To conclude, it suffices to notice that, aside from degenerate inner ellipses, we have

$$\mathscr{F}^{1,2}_{sec} = K.S.^*\left(a_1 \cdot F^{1,2}_{sec}\right),\ \mathscr{F}^{1,3}_{sec} = K.S.^*\left(a_1 \cdot F^{1,3}_{sec}\right),$$

which are deduced from

$$\begin{aligned} K.S.^*\left(a_1 \cdot F^1_{sec}\right) &= K.S.^*\left(\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \|Q_1\|\, F_{pert}\, du_1 dl_2\right) \\ &= K.S.^*\left(\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \|Q_1\|\, F_{pert}\, d\vartheta_0 dl'_2\right) \\ &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} K.S.^*\left(\|Q_1\|\, F_{pert}\right)\, d\vartheta_0 dl'_2 \\ &= \mathscr{F}^1_{sec}. \end{aligned}$$

In the above, we have used the following facts:

- $a_1\, dl_1 = \|Q_1\|\, du_1$;
- ϑ_0, l'_2 differs from u_1, l_2 only by some phase shifts depending on neither of these angles.

□

Better integrable approximating systems are obtained by eliminating the dependence of g_2 in $\mathscr{F}^n_{sec}$. Let $\nu_{quad,2} = \frac{\partial \mathscr{F}_{quad}}{\partial g_2}$ be the frequency of g_2 in the system $\mathscr{F}_{quad}$. As a non-constant analytic function, $\nu_{quad,2}$ is non-zero almost everywhere in $\tilde{\mathscr{P}}^n$, and the set $\check{\mathscr{P}}^n_{\varepsilon_0} \subset \tilde{\mathscr{P}}^n$ characterized by the condition $|\nu_{quad,2}| > \varepsilon_0$ has relative measure tending to 1 in $\tilde{\mathscr{P}}^n$ when $\varepsilon_0 \to 0$. We shall show in Subsection 5.3 that, for sufficiently small ε_0, the set $\check{\mathscr{P}}^n_{\varepsilon_0}$ contains the region of the phase space that we are interested in. After fixing ε_0, there exists an open subset $\hat{\mathscr{P}}^n_{\varepsilon_0} \subset \check{\mathscr{P}}^n_{\varepsilon_0}$ whose relative measure in $\check{\mathscr{P}}^n_{\varepsilon_0}$ tends to 1 when $\alpha \to 0$, and a symplectomorphism

$$\psi^{n'} : \hat{\mathscr{P}}^n_{\varepsilon_0} \to \psi^{n'}(\hat{\mathscr{P}}^n_{\varepsilon_0})$$

which is $|\alpha|$-close to the identity, such that

$$\psi^{n'*}\phi^{n*}\mathscr{F} = \mathscr{F}_{Kep} + \overline{\mathscr{F}^{n,n'}_{sec}} + \mathscr{F}^{n,n'}_{pert},$$

with $\overline{\mathscr{F}^{n,n'}_{sec}} = \alpha^3 F_{quad} + O(\alpha^4)$ invariant under the $SO(3)$-symmetry and independent of ϑ_0, l'_2, g_2 (thus integrable), and $\mathscr{F}^{n,n'}_{pert} = O(\alpha^{\min\{n'+1, \frac{3(n+2)}{2}\}})$. For any $(n, n'-2) \in \mathbb{Z}^2_+$, the function $\mathscr{F}_{Kep} + \overline{\mathscr{F}^{n,n'}_{sec}}$ is always (conjugate to) an integrable approximating system of $\mathscr{F}$.

The symplectomorphism $\psi^{n'}$ is constructed by successive steps of elimination of g_2 analogous to the proof of Prop 4.2, and is dominated by ψ^3 when α is small enough. We shall describe the choice of ψ^3 more precisely when needed (Section 7).

5 Dynamics of the integrable approximating system

For sufficiently small α_0 and large enough n, n', the system $\psi^{n'*}\phi^{n*}\mathscr{F}$ (to which $\mathscr{F}$ is conjugate) is a small perturbation of the integrable approximating systems $\mathscr{F}_{Kep} + \overline{\mathscr{F}^{n,n'}_{sec}}$ in which the fast motion is dominated by $\mathscr{F}_{Kep}$, while secular evolution of the (physical regularized) ellipses is governed by $\overline{\mathscr{F}^{n,n'}_{sec}}$.

5.1 Local reduction procedure

To prove Theorem 1.1, we shall be only interested in those invariant tori of $\mathscr{F}_{Kep} + \overline{\mathscr{F}^{n,n'}_{sec}}$ close to the double inner collisions, their geometry and their torsion (of the frequencies). We first reduce the system by its known continuous symmetries.

After fixing $\mathscr{P}_0, L_2 > 0$ (*i.e.* fixing $a_1, a_2 > 0$) and being reduced by the Keplerian $\mathbb{T}^2$-action, the functions $\mathscr{F}_{Kep} + \overline{\mathscr{F}^{n,n'}_{sec}}$ are naturally defined on a subset of the direct product of the space of (inner) centered ellipses with fixed semi major axis with the space of (outer) Keplerian ellipses (*i.e.* bounded orbits of the Kepler problem, which are possibly degenerate or circular ellipses with one focus at the origin) with fixed semi major axis. The constant term $\mathscr{F}_{Kep}$ plays no role in the reduced dynamics and is omitted from now on. The circle fibers of $K.S.$ are further reduced out by considering the functions $\overline{\mathscr{F}_{sec}^{n,n'}}$ as defined on subsets of the *secular space*, *i. e.* the space of pairs of (possibly degenerate) Keplerian ellipses with fixed semi major axes. A construction originated from W. Pauli [14] (See also [1, Lecture 4]) shows that, when no further restrictions are imposed on the two ellipses, the secular space is homeomorphic to $S^2 \times S^2 \times S^2 \times S^2$.

By fixing the direction of the total angular momentum $\vec{C}$ vertical (which implies in particular that the two node lines of the two Keplerian ellipses coincide), we restrict the $SO(3) \times SO(2)$-symmetry of $\overline{\mathscr{F}_{sec}^{n,n'}}$ to a (Hamiltonian) $\mathbb{T}^2$-symmetry with moment mapping $(C := \|\vec{C}\|, G_2)$. Fixing C and G_2 and then reducing by the $\mathbb{T}^2$-symmetry accomplishes the corresponding symplectic reduction procedure. We assume that C and G_2 are fixed properly so that the reduced space is 2-dimensional.

By the triangle inequality, the norm of the angular momentum of the inner Keplerian ellipse G_1 satisfies

$$G_1 \geq |C - G_2| \triangleq G_{1,min}.$$

When $C \neq G_2$, this inequality bounds e_1 away from 1. The inequality becomes equality exactly when $G_1 = C - G_2$ or $G_1 = G_2 - C$, corresponding respectively to direct and retrograde coplanar motions.

A local analysis of the reduced space near coplanar motions suffices for our purpose (c.f. Figure 5.1). The corresponding reduction procedure of the (free) $SO(2) \times SO(2)$-symmetry for non-coplanar pairs of ellipses is just a combination of Jacobi's reduction of the nodes together with the identification of all the outer ellipses with the same angular momentum but different pericentre directions; the coplanar pairs with fixed inner and outer angular momenta are reduced to a point by identifying all the pericentre directions of the inner and outer ellipses. We thus obtain the following:

When $C \neq G_2$, locally near the set $\{G_1 = G_{1,min}\}$, the reduced space is a disc containing the point corresponding to $\{G_1 = G_{1,min}\}$. The rest of the disc is foliated by the closed level curves of G_1 (for $G_1 > G_{1,min}$).

When $C = G_2$, for small G_1, the two ellipses are coplanar only if the inner ellipse degenerates (to a line segment), corresponding to a point after reduction. The reduced space is a disc containing this point; it also contains a line segment corresponding to degenerate inner Keplerian ellipses slightly inclined with respect to the outer ellipse.

5.2 Coordinates on the reduced spaces

To analyze the reduced dynamics of $\overline{\mathscr{F}_{sec}^{n,n'}}$, we need to find appropriate coordinates in the reduced space. For this purpose, to start with the coordinates defined in Subsection 3.3 for the inner motion is not convenient, since they do not naturally descend to Darboux coordinates in the quotient. Instead, we use Delaunay coordinates $(L_1, l_1, G_1, g_1, H_1, h_1)$ for the inner (physical) Keplerian ellipse (with modified masses), which may be equally seen as a set of Darboux coordinates on an open subset of V^0 where all these elements are well-defined for the inner Keplerian ellipse.

We observe that fixing $\mathscr{P}_0$ (defined in Subsection 3.3) and L_2 is equivalent to fixing $L_1(\mathscr{P}_0, L_2)$ and L_2, and defines a 10-dimensional submanifold of $V^0 \times T^*\mathbb{R}^3$, on which the symplectic form

$$dL_1 \wedge dl_1 + dG_1 \wedge dg_1 + dH_1 \wedge dh_1 + dL_2 \wedge dl_2 + dG_2 \wedge dg_2 + dH_2 \wedge dh_2$$

restricts to

$$dG_1 \wedge dg_1 + dH_1 \wedge dh_1 + dG_2 \wedge dg_2 + dH_2 \wedge dh_2$$

with, thanks to the modification of the masses, the latter's kernel containing exactly the vectors tangent to the (regularized) inner orbits at each point (c.f. Lem 4.1), and thus descends to the quotient by the Keplerian $\mathbb{T}^2$-action. We thus obtain a set of Darboux coordinates on a dense open subset of the secular space.

To reduce the $SO(3)$-symmetry, we use Jacobi's elimination of the nodes: we fix $\vec{C}$ vertical[1] (which implies that $h_1 = h_2 + \pi$ and $H_1 + H_2 = C$) and reduce by the conjugate $SO(2)$-symmetry to get a set of Darboux coordinates (G_1, g_1, G_2, g_2) in the quotient space. Due to the lack of the node lines, the angles g_1, g_2 are not well-defined when the inner ellipse degenerates. Nevertheless, these coordinates are sufficient for what follows. The $SO(2)$-symmetry of rotating the outer ellipse in its orbital plane is symplectically reduced by identifying all the outer ellipses having the same orbital plane while differing only by their pericentre directions and fixing G_2. The pair (G_1, g_1) forms a set of Darboux coordinates in an open subset of the 2-dimensional quotient space.

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[1] This choice of direction of $\vec{C}$ is convenient, but not essential: the reduced dynamics is the same regardless of the direction of $\vec{C}$.

5.3 The quadrupolar system and its dynamics

The system $\overline{\mathscr{F}_{sec}^{n,n'}}$ is an $O(\alpha^4)$-perturbation of $\alpha^3 \mathscr{F}_{quad}$. Let us first analyze the quadrupolar dynamics, *i.e.* the dynamics of $\mathscr{F}_{quad}$.

Let $\mu_{quad} = \frac{m_0 m_1 m_2}{m_0+m_1}$. In coordinates (G_1, g_1) with parameters L_1, L_2, C, G_2, the function

$$\mathscr{F}_{quad}(G_1, g_1; L_1, L_2, C, G_2)$$

is equal to

$$-\frac{\mu_{quad} L_2^3}{8G_2^3}\Big\{3\frac{G_1^2}{L_1^2}\Big[1+\frac{(C^2-G_1^2-G_2^2)^2}{4G_1^2G_2^2}\Big] +15\Big(1-\frac{G_1^2}{L_1^2}\Big)\Big[\cos^2 g_1+\sin^2 g_1\frac{(C^2-G_1^2-G_2^2)^2}{4G_1^2G_2^2}\Big]-6\Big(1-\frac{G_1^2}{L_1^2}\Big)-4\Big\},$$

which only differs from F_{quad} by a non-essential factor a_1. Note that we have separated the variables from the parameters of a system by a semi-colon. The dynamics of F_{quad} has been extensively studied by Lidov and Ziglin in [11], from which the dynamics of $\mathscr{F}_{quad}$ is deduced directly.

Remark 5.1. The relationship between $\mathscr{F}_{quad}$ and F_{quad} (c.f. the proof of Lem 4.3) also justifies the fact that F_{quad} can be extended analytically through degenerate inner ellipses.

For $|C-G_2|$ positive and small, locally the reduced secular space is foliated by closed curves around the point $\{G_1 = G_{1,min}\}$ corresponding to the case when the inner and outer ellipses are coplanar. This is deduced from [11] by noticing that (G_1, g_1) are regular coordinates outside the point $\{G_1 = G_{1,min}\}$. (c.f. Figure 5.1)

When $C = G_2$, the Hamiltonian $\mathscr{F}_{quad}$ takes the form

$$\mathscr{F}_{quad} = -\frac{\mu_{quad} L_2^3}{8G_2^3}\Big\{3\frac{G_1^2}{L_1^2}\Big[1+\frac{G_1^2}{4G_2^2}\Big] +15\Big(1-\frac{G_1^2}{L_1^2}\Big)\Big[\cos^2 g_1+\sin^2 g_1\frac{G_1^2}{4G_2^2}\Big]-6\Big(1-\frac{G_1^2}{L_1^2}\Big)-4\Big\}$$

which admits the discrete symmetry $(G_1, g_1) \sim (-G_1, \pi - g_1)$ and is a well-defined analytic function on the cylinder

$$\mathscr{D} := \{(G_1, g_1) \in \mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z} : -\min\{L_1, C+G_2\} < G_1 < \min\{L_1, C+G_2\}\},$$

which is a (branched) double cover of a neighborhood of the line segment $\{G_1 = 0\}$ in the reduced space. Moreover, the 2-form $dG_1 \wedge dg_1$

extends uniquely to a (non-degenerate) 2-form invariant under the symmetry $(G_1, g_1) \sim (-G_1, \pi - g_1)$ on $\mathscr{D}$. The corresponding Hamiltonian flow of the function $\mathscr{F}_{quad}(G_1, g_1; C = G_2, L_1, L_2)$ in $\mathscr{D}$ is thus interpreted as the lift of the quadrupolar flow in the reduced space. Therefore, rather than choosing coordinates in the reduced space and studying the quadrupolar flow directly, we shall study the dynamics of $\mathscr{F}_{quad}(G_1, g_1; C = G_2, L_1, L_2)$ in $\mathscr{D}$ on which we have global Darboux coordinates (G_1, g_1) (c.f. Figure 5.2).

Let I be the mutual inclination of two Keplerian orbits. The condition $C = G_2$ implies $\cos I = -\dfrac{G_1}{2C}$. In particular, when $G_1 = 0$, the limiting orbital plane of the inner Keplerian ellipse is perpendicular to the outer orbital plane. The coplanar case is thus represented by

$$(G_1, g_1) = (0, 0) \text{ and } (G_1, g_1) = (0, \pi),$$

which are two elliptic equilibria for the lifted flow in $\mathscr{D}$ surrounded by periodic orbits. These periodic orbits meet the line segment $\{G_1 = 0\}$ transversely with an angle independent of α. Being reduced by the discrete symmetry $(G_1, g_1) \sim (-G_1, \pi - g_1)$, the two elliptic equilibria in $\mathscr{D}$ descend to an elliptic equilibrium E surrounded by periodic orbits in the reduced space, and these periodic orbits meet the set $\{G_1 = 0\}$ transversely. We observe that the $\mathbb{Z}_2$-action $(G_1, g_1) \sim (-G_1, \pi - g_1)$ is free everywhere except for the two points $(G_0 = 0, g_1 = \pm\dfrac{\pi}{2})$. These two points descend to two singular points in the quotient space. (c.f. Figure 5.1)

In Subsection 4.3, we have defined the set $\check{\mathscr{P}}^n_{\varepsilon_0} \subset \tilde{\mathscr{P}}^n$ by the condition $|\nu_{quad,2}| = \left|\frac{\partial \mathscr{F}_{quad}}{\partial G_2}\right| > \varepsilon_0$. The function $\left.\frac{\partial \mathscr{F}_{quad}}{\partial G_2}\right|_{C=G_2}$ being regarded as a function on $\mathscr{D}$, we find by setting $G_1 = 0$ that

$$\left.\frac{\partial \mathscr{F}_{quad}}{\partial G_2}\right|_{C=G_2, G_1=0} = -\frac{15\mu_{quad} L_2^3}{8 G_2^4}(3 - 4\cos^2 g_1).$$

This shows (by passing to the quotient of the discrete symmetric relation $(G_1, g_1) \sim (-G_1, \pi - g_1)$) that for ε_0 small enough, after being reduced by the $\mathbb{T}^2 \times SO(3) \times SO(2)$-symmetry of the quadrupolar system, $\check{\mathscr{P}}^n_{\varepsilon_0}$ contains a neighborhood of E (whose size is independent of α).

The following lemma enables us to deduce the local dynamics of $\mathscr{F}^{n,n'}_{sec}$ from that of $\mathscr{F}_{quad}$.

Lemma 5.2. *The equilibrium E is non-degenerate.*

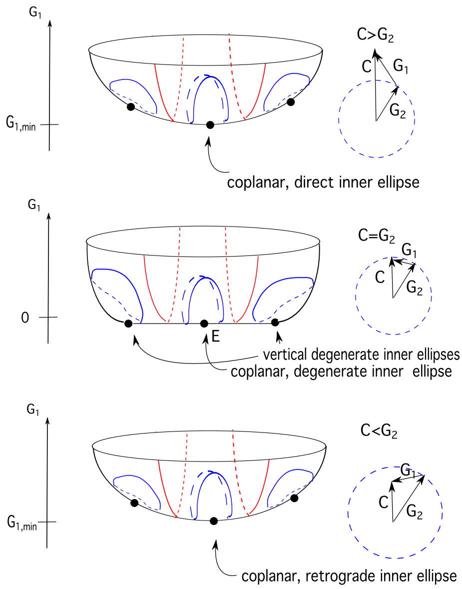

FIGURE 5.1. The reduced quadrupolar flow near the "bottom" $\{G_1 = G_{1,min}\}$. We are interested only in the regions near the depicted coplanar equilibria inside the separatrices.

Proof. It is enough to investigate the equilibrium $(G_1, g_1) = (0,0)$ of the lifted flow on $\mathscr{D}$, at which the Hessian of $\mathscr{F}_{quad}(G_1, g_1; C = G_2, L_1, L_2)$ equals to $\dfrac{45}{8}\dfrac{\mu_{quad}^2 L_2^6}{G_2^6 L_1^2} \neq 0$. □

By continuity, the coplanar equilibria $\{G_1 = G_{1,min}\}$ are non-degenerate for small positive $G_{1,min} = |C - G_2|$. Consequently, $\overline{\mathscr{F}_{sec}^{n,n'}}$ is orbitally conjugate to $\mathscr{F}_{quad}$ in some small neighborhoods of these coplanar equilibria when α_0 is small enough.

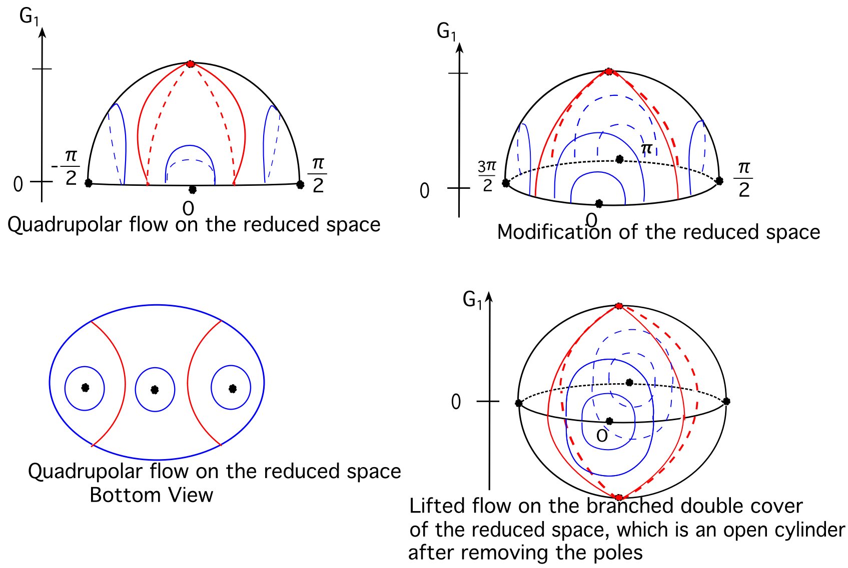

FIGURE 5.2. The flow foliation on the reduced space with $C = G_2$ and its branched double cover.

6 Application of a KAM theorem

6.1 Iso-energetic KAM theorem

For $p \geq 1$, consider the phase space $\mathbb{R}^p \times \mathbb{T}^p = \{(I, \theta)\}$ endowed with the standard symplectic form $dI \wedge d\theta$. All mappings are assumed to be analytic except when explicitly mentioned otherwise.

Let $\delta > 0$, $\varpi \in \mathbb{R}^p$. Let B^p_δ be the p-dimensional closed ball with radius δ centered at the origin in $\mathbb{R}^p$, and $N_\varpi = N_\varpi(\delta)$ be the space of Hamiltonians $N \in C^\omega(\mathbb{T}^p \times B^p_\delta, \mathbb{R})$ of the form

$$N = c + \langle \varpi, I \rangle + \langle A(\theta), I \otimes I \rangle + O(|I|^3),$$

with $c \in \mathbb{R}$ and $A \in C^\omega(\mathbb{T}^p, \mathbb{R}^p \otimes \mathbb{R}^p)$; the Lagrangian torus $\mathbb{T}^p \times \{0\}$ is an invariant Lagrangian ϖ-quasi-periodic torus of N with energy c.

Let $\bar{\gamma} > 0$ and $\bar{\tau} > p - 1$, and let $|\cdot|_2$ be the ℓ^2-norm on $\mathbb{Z}^p$. Let $HD_{\bar{\gamma},\bar{\tau}}$ be the set of vectors ϖ satisfying the following homogeneous Diophantine conditions:

$$\forall k \in \mathbb{Z}^p \setminus \{0\}, \quad |k \cdot \varpi| \geq \bar{\gamma} |k|_2^{-\bar{\tau}}.$$

Let $\|\cdot\|_s$ be the s-analytic norm of an analytic function, *i.e.*, the supremum norm of its analytic extension to

$$\{x \in \mathbb{C}^p \times \mathbb{C}^p/\mathbb{Z}^p : \exists y \in \mathbb{R}^p \times \mathbb{T}^p, \text{ such that } |x-y| \leq s\}$$

of its (real) domain in the complexified space $\mathbb{C}^p \times \mathbb{C}^p/\mathbb{Z}^p$ with "radius" s.

Theorem 6.1. *Let $\varpi^o \in HD_{\bar{\gamma},\bar{\tau}}$, $N^o \in N_{\varpi^o}$. For some $d > 0$ small enough, there exists $\varepsilon > 0$, such that for every Hamiltonian $N \in C^\omega(\mathbb{T}^p \times B^p_\delta)$ such that*

$$\|N - N^o\|_d \leq \varepsilon,$$

there exists a vector (ϖ, c) satisfying the following properties:

- *the map $N \mapsto (\varpi, c)$ is of class C^∞ and, in the C^∞-topology, is ε-close to $(\varpi^o, c^o) = (\varpi(N^o), c(N^o))$;*
- *if $\varpi(N) \in HD_{\bar{\gamma},\bar{\tau}}$, N is symplectically analytically conjugate to a Hamiltonian*

$$c(N) + \langle \varpi(N), I\rangle + \langle A(\theta)(N), I \otimes I\rangle + O(|I|^3) \in N_\varpi.$$

Moreover, ε can be chosen to be of the form $Cst\,\bar{\gamma}^k$ (for some $Cst > 0$, $k \geq 1$) when $\bar{\gamma}$ is small.

This theorem is an analytic version of the C^∞ "hypothetical conjugacy theorem" (Theorem 42) of [6] (for Lagrangian tori)[2]. We refer to [7] for its complete proof.

We now consider families of Hamiltonians N^o_ι and N_ι depending analytically [3] on some parameter $\iota \in B^p_1$. Recall that for each ι, N^o_ι is of the form

$$N^o_\iota = c^o_\iota + \langle \varpi^o_\iota, I\rangle + \langle A_\iota(\theta), I \otimes I\rangle + O(|I|^3).$$

With the aim of finding zero-energy invariant tori of $\mathscr{F}$ (recall that it is only on $\{\mathscr{F} = 0\}$ that the dynamics of $\mathscr{F}$ extends that of F), we now deduce an iso-energetic KAM theorem from Theorem 6.1. Denote by $[\cdot]$ the projective class of a vector. Let

$$D^o = \left\{(c^o_\iota, [\varpi^o_\iota]) : c^o_\iota = c^o_0 = c^o,\, \varpi^o_\iota \in HD_{2\bar{\gamma},\bar{\tau}},\, \iota \in B^p_{1/2}\right\};$$

note that the factor 2 in the Diophantine constant $2\bar{\gamma}$ is meant to take into account the fact that along a given projective class, locally the constant $\bar{\gamma}$

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[2] We remark that, aside from the analyticity of the conjugation (which is not used in the sequel), Theorem 6.1 follows directly from [6, Theorem 42] by treating analytic functions as C^∞-smooth.

[3] Actually C^1-smoothly would suffice.

may worsen a little bit (we will apply Theorem 6.1 with Diophantine constants $(\bar{\gamma}, \bar{\tau})$). By Theorem 6.1, the mapping $\iota \mapsto (\varpi_\iota, c_\iota) = (\varpi(N_\iota), c(N_\iota))$ is C^∞ and is ε-close to $(\varpi_\iota^o, C_\iota^o)$.

Corollary 6.2 (Iso-energetic KAM theorem)**.** *Assume that the map*

$$B_1^p \to \mathbb{R} \times \mathbf{P}(\mathbb{R}^p), \quad \iota \mapsto (c_\iota^o, [\varpi_\iota^o])$$

is a diffeomorphism onto its image. If ε is small enough and if for some $d > 0$, we have $\|N_\iota - N_\iota^o\|_d < \varepsilon$ for each ι, then the following holds:

For every $(c^o, \nu^o) \in D^o$, there exists a unique $\iota \in B_1^p$ such that

$$(c_\iota, [\varpi_\iota]) = (c^o, \nu^o),$$

and N_ι is symplectically (analytically) conjugate to some $N'_\iota \in N_{\varpi_\iota, \beta_\iota}$ of the form

$$N'_\iota = c^o + \langle \varpi_\iota, I \rangle + \langle A_\iota(\theta), I \otimes I \rangle + O(|I|^3).$$

Moreover, there exists $\bar{\gamma} > 0, \bar{\tau} > p - 1$, such that the set

$$\{\iota \in B_{1/2}^p : c_\iota = c^o, \varpi_\iota \in HD^o\}$$

has positive $(p-1)$-dimensional Lebesgue measure.

Proof. By hypothesis, the image of the restriction to $\{\iota : c_\iota^o = c^o\}$ of the mapping $\iota \mapsto \varpi_\iota^o$ is a $(p-1)$-dimensional smooth manifold, diffeomorphic to a subset of $\mathbf{P}(\mathbb{R}^p)$ with non-empty interior, hence it contains a positive measure set of Diophantine vectors. Therefore there exists $\bar{\gamma} > 0, \bar{\tau} > p-1$, such that the set D^o has positive $(p-1)$-dimensional Lebesgue measure.

Moreover, $D^o \subset D' = \left\{(c^o, [\varpi_\iota]) : \varpi_\iota \in HD_{\bar{\gamma},\bar{\tau}}, \iota \in B_{2/3}^p\right\}$. Indeed, if $(c^o, [\varpi_{\iota^o}^o]) \in D^o, \iota^o \in B_{1/2}$, then there exists $\iota' \in B_{2/3}$ such that we have $(c^o, [\varpi_{\iota^o}^o]) = (c^o, [\varpi_{\iota'}])$. When ε is small enough, $\varpi_{\iota'}$ is close enough to $\varpi_{\iota^o}^o \in HD_{2\bar{\gamma},\bar{\tau}}$, hence belongs to $HD_{\bar{\gamma},\bar{\tau}}$, and $(c^o, [\varpi_{\iota'}]) \in D'$. If ε is small enough, the map $\iota \mapsto (c_\iota, [\varpi_\iota])$ is C^1-close to $\iota \mapsto (c_\iota^o, [\varpi_\iota^o])$, hence it is a diffeomorphism, and the image of its restriction to $B_{2/3}^p$ contains the set D'.

The first assertion then follows from Theorem 6.1. Since the mapping $\iota \mapsto (c_\iota, [\varpi_\iota])$ is smooth, the pre-image of a set of positive $(p-1)$-Lebesgue measure has positive $(p-1)$-dimensional Lebesgue measure. □

Condition 6.3. When an integrable Hamiltonian $K^o = K^o(I)$ depends only on the action variables I, we may set $N_\iota^o(I) := K^o(\iota + I)$. The iso-energetic non-degeneracy of N^o_ι is just the non-degeneracy of the bordered Hessian

$$\mathscr{H}^B(K^o)(I) = \begin{bmatrix} 0 & {K^o}'_{I_1} & \cdots & {K^o}'_{I_p} \\ {K^o}'_{I_1} & {K^o}''_{I_1,I_1} & \cdots & {K^o}''_{I_1,I_p} \\ \vdots & \vdots & \ddots & \vdots \\ {K^o}'_{I_p} & {K^o}''_{I_p,I_1} & \cdots & {K^o}''_{I_p,I_p} \end{bmatrix}$$

(in which ${K^o}'_{I_i} = \dfrac{\partial K^o}{\partial I_i}, {K^o}''_{I_i,I_j} = \dfrac{\partial^2 K^o}{\partial I_i \partial I_j}$), i.e.

$$|\mathscr{H}^B(K^o)(I)| \neq 0.$$

When this is satisfied, Corollary 6.2 asserts the persistence under sufficiently small perturbation of a set of Lagrangian invariant tori (with fixed energy c_0) of $N^o = K^o(I)$ parametrized by a positive $(p-1)$-Lebesgue measure set in the action space. These invariant tori form a set of positive measure in the energy surface of the perturbed system with energy c_0.

Moreover, if the system $K^o(I)$ is properly-degenerate, say for

$$I = (I^{(1)}, I^{(2)}, \cdots, I^{(N)}), 0 < d_1 < d_2, \cdots, < d_N,$$

we have

$$K^o(I) = K^o_1(I^{(1)}) + \varepsilon^{d_1} K^o_2(I^{(1)}, I^{(2)}) + \cdots + \varepsilon^{d_N} K^o_N(I),$$

then, by replacing entries of the matrix $\mathscr{H}^B(K^o)(I)$ by their orders in ε, we obtain

$$\begin{bmatrix} 0 & 1 & \varepsilon^{d_1} & \cdots & \varepsilon^{d_N} \\ 1 & 1 & \varepsilon^{d_1} & \cdots & \varepsilon^{d_N} \\ \varepsilon^{d_1} & \varepsilon^{d_1} & \varepsilon^{d_1} & \cdots & \varepsilon^{d_N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \varepsilon^{d_N} & \varepsilon^{d_N} & \varepsilon^{d_N} & \cdots & \varepsilon^{d_N} \end{bmatrix}$$

which in particular implies that in order to have

$$|\mathscr{H}^B(K^o)(I)| \neq 0, \ \forall 0 < \varepsilon << 1,$$

it is sufficient to have

$$|\mathscr{H}^B(K^o_1)(I^{(1)})| \neq 0, |\mathscr{H}(K^o_i)(I^{(i)})| \neq 0, \forall (i) = (2), \cdots, (N).$$

The smallest frequency of $K^o(I)$ is of order ε^{d_N}. If the function $K^o(I)$ is iso-energetically non-degenerate, then for any $0 < \varepsilon << 1$, there exists a set of positive $(p-1)$-Lebesgue measure of the action space, such that the set of the projective classes of their frequencies contains a set of positive measure of the projective classes of the homogeneous Diophantine vectors in

$$HD_{\varepsilon^{d_N}\bar{\gamma},\bar{\tau}} := \{\varpi \in \mathbb{R}^p : \forall k \in \mathbb{Z}^p \setminus \{0\}, \quad |k \cdot \varpi| \geq \varepsilon^{d_N} \bar{\gamma} |k|_2^{-\bar{\tau}}\}$$

whose measure is uniformly bounded from below for $0 < \varepsilon << 1$, since for any vector $\nu' \in \mathbb{R}^p$, we have

$$\varepsilon^{d_N} \nu' \in HD_{\varepsilon^{d_N}\bar{\gamma},\bar{\tau}} \Leftrightarrow \nu' \in HD_{\bar{\gamma},\bar{\tau}},$$

while for ε sufficiently small, the measure of the projective classes of those Diophantine frequencies of $K^o(I)$ in $HD_{\varepsilon^{d_N}\bar{\gamma},\bar{\tau}}$ is at least the measure of the projective classes of the Diophantine frequencies of

$$K_1^o(I^{(1)}) + K_2^o(I^{(1)}, I^{(2)}) + \cdots + K_N^o(I)$$

in $HD_{\bar{\gamma},\bar{\tau}}$, which is independent of ε.

Thus following Theorem 6.1, we may set $\varepsilon = \mathrm{Cst}\,(\varepsilon^{d_N}\,\bar{\gamma})^k$ for the size of allowed perturbations, for some positive constant Cst and some $k \geq 1$, provided $\bar{\gamma}$ is small enough.

6.2 Application of the iso-energetic KAM theorem

Let us first show the existence of torsion near the set $\{G_1 = G_{1,min}\}$, for $G_{1,min} > 0$ small enough in the system $\mathscr{F}_{Kep} + \overline{\mathscr{F}_{sec}^{n,n'}}$ after symplectic reduction by the $SO(3)$-symmetry and the S^1-fiber action of $K.S.$.

In view of Condition 6.3, for $0 < \alpha < \alpha_0$ with some sufficiently small α_0, we verify the iso-energetic non-degeneracy condition by verifying the corresponding non-degeneracy conditions separately for the Keplerian part (with respect to $\mathscr{P}_0$ and L_2) and for the system $\overline{\mathscr{F}_{sec}^{n,n'}}$ reduced further by the Keplerian $\mathbb{T}^2$-symmetry.

Keplerian part

The bordered Hessian of

$$\mathscr{F}_{Kep} = P_0 \sqrt{\frac{2 f_1(L_2)}{\mu_1}} - \mu_1 M_1$$

with respect to $\mathscr{P}_0$ and L_2 is non-degenerate.

Secular non-degeneracy

Keeping unreduced only the $SO(2)$-symmetry conjugate to G_2, the periodic orbits in the corresponding completely reduced 1-degree-of-freedom system are lifted to invariant 2-tori of $\overline{\mathscr{F}_{sec}^{n,n'}}$ whose frequencies differ from that of the invariant 2-tori of $\alpha^3 F_{quad}$ only by quantities of order $O(\alpha^4)$. For small enough α_0, the existence of torsion of these invariant 2-tori of $\overline{\mathscr{F}_{sec}^{n,n'}}$ for any n, n' thus follows from the existence of torsion of invariant 2-tori of $\mathscr{F}_{quad}$. For $|C - G_2|$ small enough, we shall verify in Appendix B the existence of torsion of almost coplanar invariant 2-tori of $\mathscr{F}_{quad}$ close enough to $\{G_1 = G_{1,min}\}$ (in particular, it does not vanish when $C - G_2 \to 0$).

Application of the iso-energetic KAM theorem

We fix n, n' large enough, so that $\mathscr{F}_{pert}^{n,n'}$ is of order $O(\alpha^{4k+1})$ (the order is chosen so as to fit Condition 6.3 when α_0 is sufficiently small).

The invariant tori of $\mathscr{F}_{Kep} + \overline{\mathscr{F}_{sec}^{n,n'}}$ near $\{G_1 = G_{1,min}\}$ are smoothly parametrized by $(\mathscr{P}_0, L_2, \mathscr{J}_1, G_2)$ where $\mathscr{J}_1$ designates the area of the region (containing the point $\{G_1 = G_{1,min}\}$) enclosed by the corresponding periodic orbit of the invariant torus after further reducing by the Keplerian $\mathbb{T}^2$-symmetry and the $SO(2)$-symmetry conjugate to G_2. For small enough α_0, the above non-degeneracies ensure the existence of a neighborhood Ω of $\{G_1 = G_{1,min}\}$ for small enough $G_{1,min} = |C - G_2|$, in which the mapping

$$(\mathscr{P}_0, L_2, \mathscr{J}_1, G_2) \mapsto \left(\mathscr{F}_{Kep} + \overline{\mathscr{F}_{sec}^{n,n'}}, [\nu_{n,n'}]\right)$$

is a local diffeomorphism (with energy containing a neighborhood of 0), where we have denoted by $\nu_{n,n'}$ the frequencies of the invariant 4-tori of $\mathscr{F}_{Kep} + \overline{\mathscr{F}_{sec}^{n,n'}}$. Therefore, there exist $\bar{\gamma} > 0, \bar{\tau} \geq 3$, and a set Ω' of positive measure (whose measure is uniformly bounded from below for all $0 < \alpha < \alpha_0$), consisting of $(\alpha^3\bar{\gamma}, \bar{\tau})$-Diophantine invariant Lagrangian tori of $\mathscr{F}_{Kep} + \overline{\mathscr{F}_{sec}^{n,n'}}$. For any such torus with parameter $(\mathscr{P}_0^o, L_2^o, \mathscr{J}_1^o, G_2^o)$, there exists $\lambda > 0$, such that for $(\overline{\mathscr{P}_0}, \overline{L_2}, \overline{\mathscr{J}_1}, \overline{G_2}) \in B_1^4$, the mapping

$$\begin{aligned}\Phi_\lambda(\overline{\mathscr{P}_0}, \overline{L_2}, \overline{\mathscr{J}_1}, \overline{G_2}) :=& (\mathscr{P}_0^o + \lambda\,\overline{\mathscr{P}_0}, L_2^o + \lambda\,\overline{L_2}, \mathscr{J}_1^o + \lambda\,\overline{\mathscr{J}_1}, G_2^o + \lambda\,\overline{G_2})\\ &\mapsto \left(\mathscr{F}_{Kep} + \overline{\mathscr{F}_{sec}^{n,n'}}, [\nu_{n,n'}]\right)\end{aligned}$$

is a diffeomorphism. We assume that the invariant torus $(\mathscr{P}_0^o, L_2^o, \mathscr{J}_1^o, G_2^o)$ of $\mathscr{F}_{Kep} + \overline{\mathscr{F}_{sec}^{n,n'}}$ has zero energy.

We may now apply Corollary 6.2 with

$$N^o = \Phi_\lambda^*(\mathscr{F}_{Kep} + \overline{\mathscr{F}_{sec}^{n,n'}}), N = \Phi_\lambda^*(\mathscr{F}_{Kep} + \overline{\mathscr{F}_{sec}^{n,n'}} + \mathscr{F}_{pert}^{n,n'})$$

and

$$\iota = (\mathscr{P}_0^o, L_2^o, \mathscr{J}_1^o, G_2^o)$$

provided α_0 is small enough.

We thus obtain a set of invariant 4-tori of $\mathscr{F}$ reduced by the $SO(3)$-symmetry having positive measure on the energy level $\{\mathscr{F} = 0\}$. By rotating around $\vec{C}$, these 4-tori give rise to a set of invariant 5-tori of $\mathscr{F}$ (being reduced by the S^1-fiber symmetry of $K.S.$) with fixed (vertical) direction of $\vec{C}$ in $\Pi_{reg} := V^0 \times T^*(\mathbb{R}^3 \setminus \{0\})$. Finally, by rotating $\vec{C}$, we obtain a set of invariant 5-tori of $\mathscr{F}$ in $V^0 \times T^*(\mathbb{R}^3 \setminus \{0\})$ having positive measure on the energy level $\{\mathscr{F} = 0\}$. Depending on the commensurability of the frequencies, the flows on these invariant 5-dimensional tori may either be ergodic or be non-ergodic but only ergodic on some invariant 4-dimensional subtori.

7 Transversality

7.1 Transversality of the ergodic tori with the collision set

The S^1-fiber action of $K.S.$ is free on the codimension-3 submanifold $\{(0, w, Q_2, P_2) \in T^*\mathbb{H} \setminus \{(0,0)\} \times T^*(\mathbb{R}^3 \setminus \{0\})\}$ of Σ^0 corresponding to inner double collisions of F. The quotient $\mathscr{C}ol$ is thus a codimension-3 submanifold of $\Pi_{reg} := V^0 \times T^*(\mathbb{R}^3 \setminus \{0\})$.

We aim to show that after being reduced by the S^1-fiber symmetry of $K.S.$, the invariant ergodic tori of $\mathscr{F}$ intersecting $\mathscr{C}ol$ transversely form a set of positive measure in the energy level $\{\mathscr{F} = 0\}$ in Π_{reg}.

In Subsection 4.3, we have shown the existence of a symplectic transformation

$$\phi^n \circ \psi^{n'} : \hat{\mathscr{P}}_{\varepsilon_0}^n \to \phi^n \circ \psi^{n'}(\hat{\mathscr{P}}_{\varepsilon_0}^n),$$

dominated by ψ^3 for small α, such that

$$\psi^{n'*}\phi^{n*}\mathscr{F} = \mathscr{F}_{Kep} + \overline{\mathscr{F}_{sec}^{n,n'}} + \mathscr{F}_{pert}^{n,n'}.$$

Let us first show that when α_0 is sufficiently small, the invariant 5-tori of $\mathscr{F}_{Kep} + \overline{\mathscr{F}_{sec}^{n,n'}}$ intersecting $\mathscr{C}ol' = (\phi^3)^{-1}(\mathscr{C}ol)$ transversely form an open set in the energy level $\{\mathscr{F}_{Kep} + \overline{\mathscr{F}_{sec}^{n,n'}} = 0\}$ in Π_{reg}.

Denote by $\Pi_{reg}^{\tilde{C}}$ the 11-dimensional submanifold of Π_{reg} with $C=\tilde{C}>0$ and by $\underline{\mathscr{C}ol}$ the (transverse) intersection of $\mathscr{C}ol$ and $\Pi_{reg}^{\tilde{C}}$. The intersection of $\{C=G_2\}$ with $\Pi_{reg}^{\tilde{C}}$ is denoted by $\underline{\{C=G_2\}}$.

As we are interested in invariant tori in $\{\mathscr{F}=0\}$, we could fix $\mathscr{F}_{Kep}=0$ (which implies $\mathscr{F}=O(\alpha^3)$) and then adjust the energy properly. In the sequel, unless otherwise stated, an invariant torus is always meant to be an invariant 5-torus (on which $\vec{C}$ is conserved) of $\mathscr{F}_{Kep}+\overline{\mathscr{F}_{sec}^{n,n'}}$ on which $\mathscr{F}_{Kep}=0$. In addition, we suppose that $\vec{C}$ is sufficiently inclined and $\alpha<\alpha_0$ with sufficiently small α_0, so that the Delaunay coordinates are well-defined for the outer body. We take any convenient coordinates on V^0 for the inner body.

Lemma 7.1. *When α_0 is small enough, any invariant torus in $\underline{\{C=G_2\}}$ intersects $\underline{\mathscr{C}ol}$ transversely in $\underline{\{C=G_2\}}$.*

Proof. Any such invariant torus is an $O(\alpha)$-deformation of an invariant torus of the system $\mathscr{F}_{Kep}+\alpha^3\mathscr{F}_{quad}$ in $\underline{\{C=G_2\}}$. After being reduced by the $\mathbb{T}^2\times SO(3)\times SO(2)$-symmetry, this invariant torus of the system $\mathscr{F}_{Kep}+\alpha^3\mathscr{F}_{quad}$ descends to a closed orbit intersecting the line segment $\{G_1=0\}$ transversely (c.f. Figure 5.2), therefore it intersects transversely the codimension-1 submanifold of $\underline{\{C=G_2\}}$ consisting of degenerate inner ellipses in $\underline{\{C=G_2\}}$; moreover, being foliated by the S^1-orbits of the inner particle of $\mathscr{F}_{Kep}$ (parametrized by u_1), this torus also intersects the codimension-2 submanifold $\underline{\mathscr{C}ol}$ (in which $u_1=0$) of $\underline{\{C=G_2\}}$ transversely in $\underline{\{C=G_2\}}$. The conclusion thus follows for all $\alpha<\alpha_0$ with α_0 small enough. □

At any intersection point $\tilde{p}_0$ of an invariant torus $\bar{A}$ with $\underline{\mathscr{C}ol}$, we have the direct sum decomposition

$$T_{\tilde{p}_0}\Pi_{reg}^{\tilde{C}}=E^9\oplus E_{G_2,g_2},$$

in which we have denoted by E^9 the 9-dimensional subspace tangent to $\{C=G_2=\tilde{C},g_2=g_2(\tilde{p}_0)\}$, and E_{G_2,g_2} is the 2-dimensional subspace generated by $\frac{\partial}{\partial G_2}(\tilde{p}_0)$ and $\frac{\partial}{\partial g_2}(\tilde{p}_0)$. We observe that

- $E^9\subset T_{\tilde{p}_0}\underline{\mathscr{C}ol}+T_{\tilde{p}_0}\bar{A}=T_{\tilde{p}_0}\underline{\{C=G_2\}}$, and
- $\frac{\partial}{\partial g_2}(\tilde{p}_0)\in T_{\tilde{p}_0}\bar{A}\cap T_{\tilde{p}_0}\underline{\mathscr{C}ol}$;

the first assertion comes from the transversality of $\bar{A}$ with $\underline{\mathscr{C}ol}$ in $\underline{\{C = G_2\}}$, while the second assertion holds since G_2 is a first integral of $\mathscr{F}_{Kep} + \overline{\mathscr{F}_{sec}^{n,n'}}$, and $\bar{A}$ is obtained from an invariant torus in the reduced system by the symmetry of shifting g_2.

The transformation ψ^3 is the time 1-map of a function $\hat{\mathscr{H}}$ satisfying the cohomological equation:

$$\nu_{quad,2} \frac{\partial \hat{\mathscr{H}}}{\partial g_2} = \alpha \left(\mathscr{F}_{sec}^{1,3} - \frac{1}{2\pi} \int_0^{2\pi} \mathscr{F}_{sec}^{1,3} d\bar{g}_2 \right);$$

recall that $\nu_{quad,2}$ denotes the frequency of g_2 in the system $\mathscr{F}_{quad}$.

Lemma 7.2. *There exists a small real number $\tilde{\varepsilon} > 0$ independent of α, and a non empty open subset $\underline{\mathscr{C}ol_0}$ of $\underline{\mathscr{C}ol}$ whose relative measure tends to 1 locally in $\underline{\mathscr{C}ol}$ when $\tilde{\varepsilon} \to 0$, such that*

$$\left| \frac{\partial^2 \hat{\mathscr{H}}}{\partial g_2^2} \Big|_{\underline{\mathscr{C}ol_0}} \right| > 2\alpha \cdot \tilde{\varepsilon}.$$

Proof. It suffices to show that the function $\mathscr{F}_{sec}^{1,3}\Big|_{\underline{\mathscr{C}ol}}$ (and thus $\frac{\partial \hat{\mathscr{H}}}{\partial g_2}\Big|_{\underline{\mathscr{C}ol}}$) depends non-trivially on g_2. Indeed, this implies that the analytic function $\frac{1}{\alpha} \frac{\partial^2 \hat{\mathscr{H}}}{\partial g_2^2}$ is not identically zero on $\underline{\mathscr{C}ol}$, therefore there exists $\tilde{\varepsilon} > 0$ which bounds the absolute value of this function from below on an open set whose relative measure tends to 1 locally in $\underline{\mathscr{C}ol}$ when $\tilde{\varepsilon} \to 0$.

To deduce that $\mathscr{F}_{sec}^{1,3}\Big|_{\underline{\mathscr{C}ol}}$ depends non-trivially on g_2, it is sufficient to observe from [10] that when the two Keplerian ellipses are coplanar,

$$\mathscr{F}_{sec}^{1,3} = -\frac{15}{64} \frac{(4e_1 + 3e_1^3)e_2}{(1-e_2^2)^{\frac{5}{2}}} \cos(g_1 - g_2),$$

which depends non-trivially on g_2 when further restricted to $\{e_1 = 1\}$. □

We now determine the transformation ϕ^3 more precisely: we require this transformation to preserve C. For this, we require $\hat{\mathscr{H}}$ to be $SO(3)$-invariant. Notice that the function $\nu_{quad,2}$ is invariant under rotations. From [10], we see that on a dense open subset of Π_{reg} where the angle g_2 is well-defined, the function $\mathscr{F}_{sec}^{1,3}(g_2)$ is a linear combination of $\cos g_2$ and $\sin g_2$, with coefficients independent of g_2. We may thus choose

$$\hat{\mathscr{H}} = -\frac{\alpha}{\nu_{quad,2}} \mathscr{F}_{sec}^{1,3} \left(g_2 + \frac{\pi}{2} \right).$$

Let $\underline{\mathscr{C}ol'_0} = (\phi^3)^{-1}(\underline{\mathscr{C}ol_0})$. This is an open subset of

$$\underline{\mathscr{C}ol'} = (\phi^3)^{-1}(\underline{\mathscr{C}ol}) \subset \Pi^{\tilde{C}}_{reg}$$

whose relative measure tends to 1 locally in $\underline{\mathscr{C}ol'}$ when $\tilde{\varepsilon} \to 0$.

Lemma 7.3. *For small enough α_0, any invariant torus of $\mathscr{F}_{Kep} + \overline{\mathscr{F}^{n,n'}_{sec}}$ intersecting $\underline{\mathscr{C}ol'_0}$ is transverse to $\mathscr{C}ol'$ in Π_{reg}.*

Proof. Any $\tilde{p} \in \underline{\mathscr{C}ol'_0}$ can be written as $\tilde{p} = (\phi^3)^{-1}(\tilde{p}_0)$ for some $\tilde{p}_0 \in \underline{\mathscr{C}ol_0}$. Let $\tilde{A}$ be the invariant torus which intersects $\underline{\mathscr{C}ol'_0}$ at $\tilde{p}$. Since the transversality of E^9 and E_{G_2,g_2} is independent of α, for $\alpha < \alpha_0$ with α_0 sufficiently small, we may decompose $T_{\tilde{p}}\Pi^{\tilde{C}}_{reg}$ as

$$T_{\tilde{p}_0}\Pi^{\tilde{C}}_{reg} = (\phi^3)^{-1}_* E^9 \oplus E'_{G_2,g_2},$$

in which E'_{G_2,g_2} is the 2-dimensional space generated by $\frac{\partial}{\partial G_2}(\tilde{p})$ and $\frac{\partial}{\partial g_2}(\tilde{p})$. We choose a basis $(\mathbf{e}_1, \cdots, \mathbf{e}_9)$ of $(\phi^3)^{-1}_* E^9$, and 9 vectors $(\mathbf{v}_1, \cdots, \mathbf{v}_9)$ in $T_{\tilde{p}}\mathscr{C}ol' + T_{\tilde{p}}\tilde{A}$ such that $\mathbf{v}_i = \mathbf{e}_i + O(\alpha), i = 1, \cdots, 9$. The vectors

$$\left\{\frac{\partial}{\partial g_2}(\tilde{p}), \frac{\partial}{\partial G_2}(\tilde{p}), \mathbf{e}_1, \cdots, \mathbf{e}_9\right\}$$

form a basis of $T_{\tilde{p}}\Pi^{\tilde{C}}_{reg}$.

By Lem 7.2, for α_0 small enough, $\left|\frac{\partial^2 \mathscr{H}}{\partial \tilde{g}_2^2}\right| > \alpha \cdot \tilde{\varepsilon}$ is satisfied in some $O(\alpha)$-neighborhood of $\tilde{p}_0$ containing $\tilde{p}$. Hence we may write the vector $(\phi^3)^{-1}_* \frac{\partial}{\partial g_2}(\tilde{p}) \in T_{\tilde{p}}Col'$ as

$$(1 + O(\alpha), \tilde{\alpha}, O(\alpha), \cdots, O(\alpha)),$$

in which $|\tilde{\alpha}| > \alpha \cdot \tilde{\varepsilon}$.

In such a way, we have obtained 11 vectors $\left\{\frac{\partial}{\partial g_2}(\tilde{p}), \frac{\partial}{\partial G_2}(\tilde{p}), \mathbf{v}_1, \cdots, \mathbf{v}_9\right\}$ in $T_{\tilde{p}}\underline{\mathscr{C}ol'} + T_{\tilde{p}}\tilde{A}$, which, written as row vectors, form a matrix of the form

$$\begin{pmatrix} 1 & 0 & \vec{0}_9 \\ 1 + O(\alpha) & \tilde{\alpha} & O(\alpha)_9 \\ O(\alpha)_9^T & O(\alpha)_9^T & Id_{9,9} + O(\alpha)_{9,9} \end{pmatrix},$$

in which $\vec{0}_9$ is the 1×9 zero matrix, $O(\alpha)_9$ (resp. $O(\alpha)_{9,9}$) is a 1×9 (resp. 9×9) matrix with only $O(\alpha)$ entries, and $Id_{9,9}$ is the 9×9 identity matrix.

The determinant of this matrix is $\tilde{\alpha}+O(\alpha^2)$, which is non-zero provided α is small enough. This implies $T_{\tilde{p}}\underline{\mathscr{C}ol'}+T_{\tilde{p}}\tilde{A}=T_{\tilde{p}}\Pi^{\tilde{C}}_{reg}$, *i.e.* $\underline{\mathscr{C}ol'}$ is transverse to $\tilde{A}$ at $\tilde{p}$ in $\Pi^{\tilde{C}}_{reg}$.

The vector $\frac{\partial}{\partial G_2}(\tilde{p}_0)$ being tangent to $\mathscr{C}ol$, the space $T_{\tilde{p}}\mathscr{C}ol'$ must contain a vector of the form $(O(\alpha),1+O(\alpha),O(\alpha)_9)$. Since $\frac{\partial}{\partial G_2}(\tilde{p}_0)$ is transverse to $T_{(\tilde{p}_0)}\Pi^{\tilde{C}}_{reg}$ in $T_{(\tilde{p}_0)}\Pi_{reg}$, any vector of the form $(O(\alpha),1+O(\alpha),O(\alpha)_9)$ is also transverse to $\Pi^{\tilde{C}}_{reg}$, provided α is small enough. Therefore $\tilde{A}$ is transverse to $\mathscr{C}ol'$ at $\tilde{p}$ in Π_{reg}. □

Since $(\phi^3)^{-1}$ preserves $\mathscr{P}_0$ and L_2, it may only change the energy of a system at order $O(\alpha^3)$. By hypothesis, The invariant tori intersecting $\mathscr{C}ol'$ transversely we have obtained have energy $O(\alpha^3)$. We may then make proper $O(\alpha^3)$-modifications of $\mathscr{L}_1$ to obtain an open set of invariant tori on the energy level $\{\mathscr{F}_{Kep}+\overline{\mathscr{F}^{n,n'}_{sec}}=0\}$ intersecting the set $\mathscr{C}ol'$ transversely in Π_{reg}.

Therefore, those invariant 5-tori of $\mathscr{F}$ obtained in Subsection 6.2 intersecting $\mathscr{C}ol$ transversely form a set of positive measure in the energy level $\{\mathscr{F}=0\}$. Consequently, the intersection has codimension 3 in these 5-dimensional tori. If such a 5-dimensional torus is not ergodic, then it is foliated by 4-dimensional ergodic subtori obtained from one another by a rotation around $\vec{C}$. This gives a free SO(2)-action on the intersection of *Col* with the 5-dimensional tori, hence the intersection of $\mathscr{C}ol$ with each 4-dimensional ergodic torus is also of codimension 3.

7.2 Conclusion

Lemma 7.4. *Let* $\mathbf{K}$ *be a submanifold of the n-dimensional torus* $\mathbb{T}^n$ *having codimension at least 2 in* $\mathbb{T}^n$. *Let* $\tilde{\theta}=(\tilde{\theta}_1,\cdots,\tilde{\theta}_n)$ *be the angular coordinates on* $\mathbb{T}^n$*; then almost all the orbits of the linear flow* $\dfrac{d}{dt}\tilde{\theta}=\tilde{v}$, $\tilde{v}\in\mathbb{R}^n$ *do not intersect* $\mathbf{K}$.

Proof. By hypothesis, the set $\mathbf{K}\times\mathbb{R}\subset\mathbb{T}^n\times\mathbb{R}$ has Hausdorff dimension at most $n-1$. The set $\mathbf{K}'$ formed by orbits intersecting $\mathbf{K}$ is the image of $\mathbf{K}\times\mathbb{R}$ under the smooth mapping

$$\mathbb{T}^n\times\mathbb{R}\to\mathbb{T}^n\ \ (\tilde{\theta}(0),t)\mapsto\tilde{\theta}(t),$$

in which $\tilde{\theta}(t)$ denotes the solution of this linear system with initial condition $\tilde{\theta}(0)$ when $t=0$. Therefore $\mathbf{K}'$ has zero measure in $\mathbb{T}^n$. □

This lemma confirms that almost all trajectories on those ergodic tori (on which the flow is linear) intersecting $\mathscr{C}ol$ transversely do not intersect $\mathscr{C}ol$. Moreover, since the flow is irrational on these invariant ergodic tori of $\mathscr{F}$, almost all the trajectories pass arbitrarily close to $\mathscr{C}ol$ without intersection. Each of such trajectories give rise (via $K.S.$) to a collisionless orbit of F which pass arbitrarily close to the set $\{Q_1 = 0\}$ of the inner double collisions. Such orbits form a set of positive measure on the energy level $\{F = -f\}$. By varying f and applying Fubini theorem, the collisionless orbits of F along which the inner pair pass arbitrarily close to each other form a set of positive measure in the phase space Π. Theorem 1.1 is thus proved.

Remark 7.5. We have concentrated ourselves to those quasi-periodic almost-collision orbits along which the two instantaneous physical elliptic orbits are close to be coplanar. By analyticity of the system, the required non-degeneracy condition, and therefore the result, can be improved to include more inclined cases as well.

Appendix A: Estimation of the perturbing function

The following lemma is just a reformulation of [5, Lem 1.1].

Lemma A.1. *When $\|Q_1\| \neq 0$, the expansion*

$$\begin{aligned}\mathscr{F}_{pert} &= K.S.^* \left(-\mu_1 m_2 \sum_{n\geq 2} \sigma_n P_n(\cos\zeta) \frac{\|Q_1\|^{n+1}}{\|Q_2\|^{n+1}} \right) \\ &= -\mu_1 m_2 \sum_{n\geq 2} \sigma_n K.S.^* \left(P_n(\cos\zeta) \frac{(1-e_1\cos u_1)^{n+1}}{(1-e_2\cos u_2)^{n+1}} \right) \alpha^{n+1}\end{aligned}$$

is convergent in $\dfrac{\|Q_1\|}{\|Q_2\|} \leq \dfrac{1}{\hat{\sigma}}$, *where*

- P_n *is the n-th Legendre polynomial,*
- ζ *is the angle between vectors* Q_1 *and* Q_2,
- e_1, e_2 *are respectively the eccentricities of the two elliptic orbits,*
- u_1, u_2 *are respectively the eccentric anomalies of* Q_1, Q_2 *on their orbits,*
- $\hat{\sigma} = max\{\sigma_0, \sigma_1\}$ *and* $\sigma_n = \sigma_0^{n-1} + (-1)^n \sigma_1^{n-1}$.

Lemma A.2. *Under the assumptions of Subsection 4.2, there exists $s > 0$, such that in the s-neighborhood $T_{\mathscr{P}^*,s}$ of $\mathscr{P}^*$, we have $|\mathscr{F}_{pert}| \leq Cst|\alpha|^3$ for some constant Cst independent of α.*

Proof. By continuity, there exists a positive number s, such that in a dense open set of $T_{\mathscr{P}^*,s}$ defined by the condition $\|Q_1\| \neq 0$, we have

$$|\cos\zeta| \leq 2; \ \left|\frac{\|Q_1\|}{\|Q_2\|}\right| \leq \frac{4|\alpha|}{1-e_2^\wedge}.$$

in which $\cos\zeta$, $\|Q_1\|$ and $\|Q_2\|$ are considered as the corresponding analytical extensions of the original functions.

Using Bonnet's recursion formula of Legendre polynomials

$$(n+1)P_{n+1}(\cos\zeta) = (2n+1)\cos\zeta\, P_n(\cos\zeta) - nP_{n-1}(\cos\zeta),$$

by induction on n, we obtain $|P_n(\cos\zeta)| \leq 5^n$.

Thus

$$\begin{aligned}|\mathscr{F}_{pert}| &= \mu_1 m_2 \left|\sum_{n\geq 2} \sigma_n P_n(\cos\zeta)\frac{\|Q_1\|^{n+1}}{\|Q_2\|^{n+1}}\right| \\ &\leq \mu_1 m_2 \sum_{n\geq 2} 5^n \left|\frac{\|Q_1\|}{\|Q_2\|}\right|^{n+1} \\ &\leq \frac{\mu_1 m_2}{5}\sum_{n\geq 2}\frac{5^{n+1}4^{n+1}|\alpha|^{n+1}}{(1-e_2^\wedge)^{n+1}} \\ &\leq \frac{\mu_1 m_2}{5}\frac{20^3|\alpha|^3}{(1-e_2^\wedge)^2}\frac{1}{1-e_2^\wedge-20|\alpha|}.\end{aligned}$$

It is then sufficient to impose $\alpha \leq \alpha^\wedge$ and make s sufficiently small to ensure that $|\alpha| \leq \dfrac{1-e_2^\wedge}{40}$.

By continuity of the function $\mathscr{F}_{pert}$, the estimation holds in $T_{\mathscr{P}^*,s}$. □

Appendix B: Torsion of the quadrupolar tori

We fix $\vec{C}$ vertical. After Jacobi's elimination of nodes, we further normalize the coordinates (G_1, g_1, G_2) and parameters C as in [11] by setting

$$\mathfrak{a} = \frac{C}{L_1}, \mathfrak{b} = \frac{G_2}{L_1}, \mathfrak{d} = \frac{G_1}{L_1}, \mathfrak{w} = g_1.$$

In these coordinates, we have

$$\mathscr{F}_{quad} = k\,\mathfrak{b}^{-3}(\mathscr{W} + \frac{5}{3}),$$

in which k is a irrelevant non-zero constant, and

$$\mathscr{W} = -2\delta^2 + \frac{(\alpha^2-\beta^2-\delta^2)^2}{4\beta^2} + 5(1-\delta^2)\sin^2\omega\left(\frac{(\alpha^2-\beta^2-\delta^2)^2}{4\beta^2\delta^2} - 1\right).$$

Let $\overline{\mathscr{W}}(\delta,\omega,\beta;\alpha) = \beta^{-3}(\mathscr{W}+\frac{5}{3})$. This defines a 2-degrees-of-freedom Hamiltonian systems in coordinates $(\delta,\omega,\beta,g_2)$ with a parameter α. We shall formulate our results in terms of $\overline{\mathscr{W}}$, from which the corresponding results for $\mathscr{F}_{quad}$ follow directly.

In the forthcoming proof, we deduce the existence of torsion for $\overline{\mathscr{W}}$ from a local approximating system $\overline{\mathscr{W}}'(\delta,\omega,\beta;\alpha)$ near $\{\delta = \delta_{min} := |\alpha-\beta| > 0\}$ whose flow, for fixed β, is linear in the (δ,ω)-plane. Note that when $\beta \neq \alpha$, the expression of $\overline{\mathscr{W}}$ is analytic at $\delta = \delta_{min}$. The local approximating system is thus obtained by developing $\overline{\mathscr{W}}$ into Taylor series of δ at $\delta = \delta_{min}$. Finally, we show that the torsion does not vanish when $\alpha-\beta \to 0$, which ensures the existence of torsion for quadrupolar tori at which $\alpha = \beta$ and close enough to the coplanar equilibrium with a degenerate inner ellipse. This is allowed since locally in this region, the symplectically reduced secular space by the $SO(3)$-symmetry is smooth. By doing so, we avoid choosing local coordinates near these tori.

Lemma B.1. *The torsion of the invariant tori of* $\widetilde{\mathscr{W}}$ *near the lower boundary* $\{\delta = \delta_{min} := |\alpha-\beta|\} > 0$ *exists and does not vanish when* $\alpha-\beta \to 0$.

Proof. We develop $\widetilde{\mathscr{W}}$ into Taylor series with respect to δ at $\delta = \delta_{min}$. We set $\delta_1 = \delta - \delta_{min}$, and obtain

$$\widetilde{\mathscr{W}} = \bar{\Phi}(\alpha,\beta) + \bar{\Xi}(\alpha,\beta,\omega)\delta_1 + O(\delta_1^2),$$

in which

$$\bar{\Xi}(\alpha,\beta,\omega) = -\frac{2\left((9\alpha^2\beta-6\alpha\beta^2+\beta^3-4\alpha^3+5\alpha)+(-5\alpha+5\alpha^3-10\alpha^2\beta+5\alpha\beta^2)\cos^2\omega\right)}{\beta^4|\alpha-\beta|}.$$

We eliminate the dependence of ω in the linearized Hamiltonian function $\bar{\Phi}(\alpha,\beta)+\bar{\Xi}(\alpha,\beta,\omega)\delta_1$ by computing action-angle coordinates. The value of the action variable $\overline{\mathscr{I}}_1$ on the level curve

$$E_f : \bar{\Phi}(\alpha,\beta) + \bar{\Xi}(\alpha,\beta,\omega)\delta_1 = f$$

is computed from the area between this curve and $\{\delta_1 = 0\}$, that is

$$\overline{\mathscr{I}}_1 = \frac{1}{2\pi}\int_{E_f}\delta_1 d\omega = \frac{f-\bar{\Phi}(\alpha,\beta)}{2\pi}\int_0^{2\pi}\frac{1}{\bar{\Xi}(\alpha,\beta,\omega)}d\omega = \overline{\mathscr{I}}_1.$$

We have then

$$\widetilde{\mathscr{W}} = \bar{\Phi}(\alpha,\beta) + 2\pi \left(\int_0^{2\pi} \frac{1}{\bar{\bar{\Xi}}(\alpha,\beta,\omega)} d\omega \right)^{-1} \overline{\mathscr{I}}_1 + O(\overline{\mathscr{I}}_1^2).$$

For $\overline{\mathscr{I}}_1$ small enough, the torsion of $\widetilde{\mathscr{W}}$ is dominated by the torsion of the term linear in $\overline{\mathscr{I}}_1$, which, represented by the absolute value of the determinant of the corresponding Hessian function with respect to $\overline{\mathscr{I}}_1$ and β, is

$$\mathscr{H}_s = \left[2\pi \frac{d}{d\beta} \left(\int_0^{2\pi} \frac{1}{\bar{\bar{\Xi}}(\alpha,\beta,\omega)} d\omega \right)^{-1} \right]^2 .$$

Using the formula

$$\int_0^{2\pi} \frac{d\omega}{a + b\cos\omega} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

we obtain

$$2\pi \left(\int_0^{2\pi} \frac{1}{\bar{\bar{\Xi}}(\alpha,\beta,\omega)} d\omega \right)^{-1} = -\frac{2\sqrt{\alpha+\beta}\sqrt{9\alpha^2\beta - 6\alpha\beta^2 + \beta^3 - 4\alpha^3 + 5\alpha}}{\beta^4},$$

which depends non-trivially on β.

Moreover, at the limit $\alpha = \beta$, the function $\mathscr{H}_s$ equals to $\frac{1125}{2\beta^8}$. By continuity, this proves the non-vanishing of the torsion for those invariant tori of $\widetilde{\mathscr{W}}$ at which $\alpha = \beta$ and close enough to the coplanar equilibrium with a degenerate inner ellipse. □

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