Angular momenta of relative equilibrium motions and real moment map geometry

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Abstract

In [8], Chenciner and Jiménez-Pérez showed that the range of the spectra of the angular momenta of all the rigid motions of a fixed central configuration in a general Euclidean space form a convex polytope. In this note we explain how this result follows from a general convexity theorem of O'Shea and Sjamaar in real moment map geometry [34]. Finally, we provide a representation-theoretic description of the pushforward of the normalized measure under the real moment map for Riemannian symmetric pairs.

1 Introduction

An n-body configuration $x = (x_1, \cdots, x_n)$ in a Euclidean space E with masses $m_1, \cdots, m_n > 0$ moving in a Newtonian *gravity* force field $F = \nabla U(x)$ with reduced center of mass $\sum m_k x_k / \sum m_k = 0$ is called *balanced* with factor Λ if

$$\nabla U(x) = -\Lambda x m$$

for $\Lambda : E \to E$ a symmetric linear operator on E and $m = \mathrm{diag}(m_1, \cdots, m_n)$ the mass matrix [1]. This is an algebraic equation with presumably an abundance of solutions for large n. It is clear that for $\mu > 0$ and $k \in \mathrm{SO}(E)$ the similar configuration $\mu k x$ is again balanced with factor $k \Lambda k^* / \mu^3$.

Let x be such a balanced configuration in E with factor Λ. If $Z : E \to E$ is a skew symmetric linear operator and satisfies $Z^2 = -\Lambda$, then the rigid motion $t \mapsto z(t) = \exp(tZ)x$ is a solution of Newton's equation

$$\ddot{z} m = \nabla U(z).$$

Chenciner and Jiménez-Pérez have shown that the range of the spectra of the angular momenta of all such rigid motions is a convex polytope [8], which is subsequently used by Chenciner in the analysis of bifurcation of relative equilibrium motions of the n-body problem [6].

In this note, we will show that this result is just an immediate consequence of a convexity theorem of O'Shea and Sjamaar in real moment map geometry [34], which will be reviewed in particular in the setting of a pair of real reductive Lie algebras. We have made an effort to write a pedestrian exposition. For this reason, we have restricted ourselves to the case of *central* configurations, for which $\Lambda = \lambda$ is just a scalar operator. Indeed, the analysis of the spectra range of the angular momentum of a balanced configuration breaks down to this case, as has been explained in [5].

Finally, one may naturally ask about the density of complex structures corresponding to the same spectrum in the range of the spectra of the angular momenta of rigid motions. An explicit description of this density requires more involved work and will be a question for future research. Nevertheless, motivated by this, and in line with O'Shea-Sjamaar's study of the real moment map, we shall give a description of the pushforward of the normalized invariant measure by the real moment map for Riemannian symmetric pairs. This provides a real version of the representation-theoretic interpretation of the pushforward measure studied in [21] (c.f. [12]).

2   The n-body problem in Euclidean space of arbitrary finite dimension

The Newtonian n-body problem in a finite dimensional Euclidean space E with inner product $(\cdot, \cdot)$ is the study of the dynamics of n point particles with positions $x_k \in E$ and masses $m_k > 0$, with time evolution according to Newton's laws of motion

$$m_k \ddot{x}_k = \sum_{j \neq k} m_j m_k (x_j - x_k)/|x_j - x_k|^3$$

for $k = 1, \cdots, n$. A configuration $x = (x_1, \cdots, x_n)$ in E^n is a row vector with entries vectors in E. Its dual configuration x^* on E^n is then a column vector with the corresponding dual vectors on E as entries. Here any vector in E gives rise to a dual vector on E by taking the inner product with that vector.

For example, with this notation x^*x is the $n \times n$ Gram matrix of the position configuration, while xmx^* is the symmetric linear operator on E sending v to $\sum_k m_k(x_k, v)x_k$. Here $m = \text{diag}(m_1, \cdots, m_n)$ is the mass matrix.

The negative of the potential energy (which is also called the *force function* by Lagrange)

$$U(x) = \sum_{j<k} m_j m_k / |x_j - x_k|$$

is a solution of the equations

$$\nabla_k U(x) = \sum_{j \neq k} m_j m_k (x_j - x_k)/|x_j - x_k|^3$$

with ∇_k the gradient with respect to the vector $x_k \in E$. If we denote $\nabla U(x) = (\nabla_1 U(x), \cdots, \nabla_n U(x)) \in E^n$, then the equations of motion can be written in the form

$$\dot{x} = y, \dot{y}m = \nabla U(x)$$

as a first order system. We denote $K(y) = \text{tr}(y^*ym)/2$ for the kinetic energy. The total energy is thus defined by $H(x, y) = K(y) - U(x)$, and is a conserved quantity: Indeed, we have $\dot{H} = \text{tr}(y^*\dot{y}m) - \text{tr}(\dot{x}^*\nabla U(x)) = 0$.

The total linear momentum $p = \sum m_k y_k \in E$ is also conserved, which in turn implies that the center of mass $c = \sum m_k x_k / \sum m_k \in E$ has uniform rectilinear motion. By the center of mass reduction we may assume that $c = p = 0$, *which will be done from now on.*

For the position-velocity pair $(x, y) \in E^n \times E^n$, the total angular momentum is defined by

$$L = ymx^* - xmy^*,$$

which is a skew symmetric linear operator on E. Since

$$\dot{L} = \dot{y}mx^* - xm\dot{y}^* = (\nabla U(x))x^* - x(\nabla U(x))^*$$

is the linear operator on E sending v to

$$\sum_{j \neq k} m_j m_k (x_k, v)\frac{x_j - x_k}{|x_j - x_k|^3} - \sum_{j \neq k} m_j m_k \frac{(x_j, v) - (x_k, v)}{|x_j - x_k|^3}x_k = 0$$

we conclude that L is conserved. The conservation of total linear momentum and of total angular momentum is a consequence of the Euclidean motion

group of E being symmetry group of the equations of motion, in accordance with the Noether theorem.

For $n \geq 3$, the system is non-integrable in the sense that there are no other independent integrals of motion than the above, a result for algebraic integrals due to Bruns in 1887 [4] (substantially completed and generalized in [24]) and for analytic integrals due to Poincaré in 1890 [37]. This work by Poincaré on the (restricted) 3-body problem reveals the great complexity of the general motion in case $n \geq 3$ [35].

For $n = 2$ the relative position $z = x_1 - x_2 \in E$ is a solution of the Kepler problem

$$\mu \ddot{z} = -\kappa z/|z|^3 \quad \Leftrightarrow \quad \ddot{z} = -\lambda z/|z|^3$$

with $\kappa = m_1 m_2, \lambda = m_1 + m_2, \mu = \kappa/\lambda > 0$. For $H = \mu|\dot{z}|^2/2 - \kappa/|z| < 0$ the motion is bounded inside the region $|z| \leq -\kappa/H$, and is either collinear or the point z moves in the Euclidean plane P spanned by z and $v = \dot{z}$ along an ellipse with a focus at the origin, according to the area law. Let i be a complex structure on P compatible with the Euclidean structure, which means that $i : P \to P$ is a skew symmetric linear operator with $i^2 = -1$. In polar coordinates (r, θ), the complex variable

$$z = re^{i\theta}$$

is a solution of the Kepler problem if and only if (r, θ) is a solution of

$$\ddot{r} - r\dot{\theta}^2 = -\lambda/r^2 \ , \ r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0 \ .$$

For $\dot{\theta} = 0$ we get the one dimensional Kepler problem $\ddot{r} = -\lambda/r^2$, which corresponds to homothetic motion of z in E. For $\dot{r} = 0$ we find $\dot{\theta}^2 = \lambda/r^3$, which corresponds to rigid uniform circular motion with angular velocity $\omega = \sqrt{\lambda/r^3}$.

For special initial configurations $x \in E^n$, there exists initial configurations of velocities $y \in E^n$ such that the above-mentioned Kepler orbits can be lifted to exact solutions of the n-body problem in E. These are the so called central configurations and give rise to homographic motions. They generalize the collinear 3-body configurations of Euler from 1767 [14] and the planar equitriangular 3-body configurations of Lagrange from 1772 [26]. Later examples were found for $n = 4$ by Lehmann-Filhés in 1891 [27], and Moulton in [32], and the abundance of planar central configurations for large n was indicated by Dziobek, who also started to use the term "central figure for such a configuration" in 1899 [13].

Planar and spatial central configurations became a renown subject in celestial mechanics, notably after the standard text book of Wintner from 1941 [39] and a crucial paper by Smale from 1970 [38]. The question of linear stability for the relative equilibrium motions of some planar central configurations was undertaken by Moeckel in the eighties and nineties, generalizing the Gascheau stability condition from 1843 for the Lagrange equilateral triangle configuration [16],[36],[29],[30]. Central configurations in a Euclidean space E of arbitrary finite dimension were considered by Albouy and Chenciner in 1998 [1]. We mention that it is not yet known to us what all central configurations are for $n = 4$ for arbitrary choice of masses, and even the finiteness problem of their number has not been completely settled for $n = 5$ (for generic choice of masses, this has been proven by Albouy and Kaloshin in [2]), and is yet largely open for $n \geq 6$. Lecture notes from 2014 by Richard Moeckel on central configurations in a Euclidean space of arbitrary finite dimension give a nice overview with many more details (also on the history of the subject), and can be found on his website [31].

3   Central configurations

We now explain the concept of central configurations in E^n and their associated homothetic, rigid and homographic motions of the Newtonian n-body problem in E.

Definition 3.1. *For given masses, an n-body configuration $x \in E^n$ is called central with constant λ if*

$$\nabla U(x) = -\lambda x m$$

for some scalar $\lambda \in \mathbb{R}$.

Since $U(x)$ is homogeneous of degree -1, we have

$$\mathrm{tr}(x^* \nabla U(x)) = \mathcal{E}U(x) = -U(x)$$

with $\mathcal{E} = \sum_k (x_k, \nabla_k)$ the Euler vector field on E^n, and therefore

$$\lambda = U(x)/\mathrm{tr}(x^* x m) > 0.$$

Note that central configurations are just the stationary points of the function $U(x)$ under the constraint $\mathrm{tr}(x^* x m)/2 = 1$. Clearly, if $x \in E^n$ is central with constant λ, then for all scalars $\mu > 0$ and all proper rigidities $k \in \mathrm{SO}(E)$, the configuration $\mu k x \in E^n$ is again central with constant λ/μ^3 .

Proposition 3.2. *If $x \in E^n$ is a central configuration with constant λ and $r(t)$ is a solution of the one dimensional Kepler problem $\ddot{r} = -\lambda/r^2$, then $z(t) = r(t)x$ is a homothetic motion of the n-body problem. Conversely, any homothetic solution $z(t) = r(t)x$ of the n-body problem can be expressed in this way for some central configuration $x \in E^n$.*

Proof. Indeed, if $x \in E^n$ is a central configuration with constant λ and the real function $r(t)$ is a solution of $\ddot{r} = -\lambda/r^2$, then the motion $z(t) = r(t)x$ satisfies
$$\ddot{z}m = \ddot{r}xm = -\lambda xm/r^2 = \nabla U(x)/r^2 = \nabla U(z),$$
since $\nabla U(x)$ is homogeneous in x of degree -2.

Conversely, suppose $z(t) = r(t)x$ is a solution of the n-body problem for some real function $r(t)$. By substitution into the equation of motion $\ddot{z}m = \nabla U(z)$, we obtain $r^2\ddot{r}xm = \nabla U(x)$. Hence $r^2\ddot{r} = -\lambda$ for some constant $\lambda \in \mathbb{R}$, and so $\nabla U(x) = -\lambda xm$, thus x is a central configuration with constant λ. $\square$

We recall that a compatible complex structure on E is a skew symmetric linear operator $J : E \to E$ with $J^2 = -1$. A neccesary and sufficient condition for such J to exist is that E has even dimension.

Proposition 3.3. *Suppose $x \in E^n$ is a central configuration in E with constant $\lambda = \omega^2 > 0$. Any compatible complex structure J on E gives rise to a rigid motion $t \mapsto z(t) = \exp(t\omega J)x$ of the n-body problem. Conversely, if E is spanned by x, then any rigid motion solution of $\ddot{z}m = \nabla U(z)$ is of this form.*

Proof. Indeed, we have $\ddot{z}m = -\omega^2 zm = \nabla U(z)$ since z is central with scalar $\lambda = \omega^2$.

Conversely, if $Z : E \to E$ is a skew symmetric operator, then the rigid motion $z(t) = \exp(tZ)x$ of the central configuration x with scalar $\lambda = \omega^2$ is a solution of $\ddot{z}m = \nabla U(z)$ if and only if $Z^2x = -\lambda x$. Since by assumption E is spanned by x, we arrive at $Z = \omega J$ with J a compatible complex structure on E. $\square$

Homothetic and rigid motions of a central configuration are both special cases of the more general homographic motions.

Theorem 3.4. *Suppose $t \mapsto (r, \theta)$ is a solution of the planar Kepler problem*

$$\ddot{z} = -\lambda z/|z|^3 \ , \ z = re^{i\theta} \ \Leftrightarrow \ \ddot{r} - r\dot{\theta}^2 = -\lambda/r^2 \ , \ r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

in polar coordinates. If $x \in E^n$ is a central configuration with constant λ and J is a compatible complex structure on E, then

$$t \mapsto z(t) = r(t) \exp(\theta(t)J)x$$

is a homographic motion of the n-body problem.

Proof. This is just the standard derivation of the equations of motion for the Kepler problem in polar coordinates. Indeed, let $x \in E^n$ be a fixed central configuration with constant λ, thus $\nabla U(x) = -\lambda x m$ holds by definition. We have to check that
$$z = r(t)e^{\theta(t)J}x$$
is a solution of the equations of motion $\ddot{z}m = \nabla U(z)$ for the n-body problem. By differentiation, we have

$$\ddot{z} = r^{-1}\{(\ddot{r} - r\dot{\theta}^2) + (r\ddot{\theta} + 2\dot{r}\dot{\theta})J\}z,$$

and by assumption, we get $\ddot{z} = -\lambda r^{-3}z$. Since z is central with constant λ/r^3, we find $\nabla U(z) = -\lambda r^{-3}zm$, and hence $\ddot{z}m = \nabla U(z)$ is satisfied. $\square$

Note that the term "homographic" in the terminology "homographic motion", though commonly used by celestial mechanists, should not be confused with the term "homography" in the geometric sense, which is synonymous to projective transformations. Under a homographic motion with negative total energy
$$(\dot{r} + r\dot{\theta})^2/2 - \lambda/r < 0,$$

each point particle $x_k \in E$ traverses a Kepler ellipse in the plane spanned by $\{x_k, Jx_k\}$ with one focus at the origin according to the area law, and all n point particles traverse similar ellipses.

We end this section by showing that central configurations exist in high dimensions in abundance. Just take a (heavy) particle with mass M at the origin $x_0 = 0$ and a cloud of n (light) particles at positions $x_1, \cdots, x_n$ with equal masses m with $\sum x_i = 0$ and with a sufficient symmetry.

Theorem 3.5. *If G be a finite irreducible subgroup of the orthogonal group $\mathrm{O}(E)$, such that G acts transitively on the cloud $x_1, \cdots, x_n$ and for each $i = 1, \cdots, n$ the fixed point hyperplane in E of the stabilizer group G_i of x_i in G is equal to the line $\mathbb{R}x_i$, then the configuration $x = (x_0, x_1, \cdots, x_n)$ with masses $(M, m, \cdots, m)$ is central.*

Proof. The total force on the particle x_i is the sum of the forces expelled from the particles x_j for $j \neq i$. Hence by symmetry this total force on x_i is fixed by G_i, and therefore equal to $-\lambda_i x_i$ for some scalar λ_i. By symmetry we have $\lambda_i = \lambda_j = \lambda$ for all $i, j \geq 1, i \neq j$. We can take $\lambda_0 = \lambda$ as well, and hence we find a central configuration. $\qquad\square$

An example of such a configuration is obtained by taking for the cloud the vertices of a regular polytope in the sense of Schläfli [10]. More generally, for any finite irreducible reflection group, one can take for the cloud the orbit of a nonzero vector on an extremal ray of a positive Weyl chamber. For example, in dimension 8 one can obtain such a central configuration with a cloud of 483840 particles with Weyl group symmetry of type E_8. But there are plenty of other examples, for example the set of the minimal norm 4 vectors in the Leech lattice gives such a central configuration in dimension 24 with a cloud of 196560 particles (see [9] for explanations of these lattices).

The planar central configurations with a regular n-gon for the cloud were deeply studied by Maxwell [28], and more recently by Hall and Moeckel [19], [29]. Their rigid motion is linearly stable for $n \geq 7$ in case m/M is sufficiently small (the larger n, the smaller m/M should be). The question of linear stability of these general symmetric central configurations, in case m/M is sufficiently small and for dimension at least 4, is completely open. The motivation of Maxwell for this work was to understand the stability of the rings of Saturn. His essay, published in 1859, was highly appreciated at the time, and won him the Adams prize for the year 1856.

4   The spectra of the angular momenta

Let $x \in E^n$ be a central configuration with constant $\lambda > 0$. By a suitable positive scaling we may assume that $\lambda = 1$, *which will be assumed in this section.* For any compatible complex structure $J : E \to E$, we have discussed the rigid motion $t \mapsto z(t) = \exp(tJ)x$ of the Newtonian n-body problem.

Note that $J^* = -J = J^{-1}$, so J is both skew symmetric and orthogonal. The conserved angular momentum

$$L := \dot{z}mz^* - zm\dot{z}^* = Jxmx^* + xmx^*J$$

is a skew symmetric linear operator on E. The compatible complex structure J turns E into a finite dimensional Hilbert space (E, J) with Hermitian form whose real part is the Euclidean inner product $(\cdot, \cdot)$. Clearly L and J commute, and if we write $X = xmx^*$ for the so called *inertia operator* of the central configuration x, then

$$K := LJ^* = X + JXJ^*$$

is a nonnegative selfadjoint operator on (E, J). By definition, the *real spectrum* of L is the spectrum of K, considered as an ordered subset of $\mathbb{R}_+$ of cardinality equal to the complex dimension of the Hilbert space (E, J).

What are the possible real spectra of L when J varies over all the possible compatible complex structures on E? This question was posed by Chenciner, who conjectured it to be a convex polytope [5], which was subsequently shown by an indirect argument by Chenciner and Jiménez-Pérez [8] by realizing this real spectrum range between two Horn-type convex polytopes, and observe that a combinatorial lemma by Fomin–Fulton–Li–Poon [15] affirms the coincidence of these two convex polytopes.

The curious convexity property of this real spectrum range raised the question of finding for it a direct, conceptual proof, which is a question posed by Chenciner and Leclerc [7]. To present a direct proof of this convexity property, let us rephrase the question.

Let $j : E \to E$ be a fixed compatible complex structure. Any compatible complex structure on E is of the form $J = k^*jk$ for some $k \in \mathrm{O}(E)$, and therefore

$$M := kKk^* = (kXk^*) + j(kXk^*)j^*$$

is a nonnegative selfadjoint operator on the fixed Hilbert space (E, j). Let us write $\mathfrak{s}(E)$ for the space of symmetric operators on E, and write $\mathfrak{s}(E, j)$ for its linear subspace of selfadjoint operators on (E, j). We consider $\mathfrak{s}(E)$ as Euclidean space with respect to the trace form $(Y, Z) = \mathrm{tr}(YZ)$, and observe that $\mathrm{O}(E)$ acts on $\mathfrak{s}(E)$ by conjugation as orthogonal linear transformations. Note that the map

$$\mathfrak{s}(E) \to \mathfrak{s}(E, j), \ Y \mapsto (Y + jYj^*)/2$$

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is nothing else but the orthogonal projection of $\mathfrak{s}(E)$ onto $\mathfrak{s}(E, j)$. Clearly this map is equivariant for the conjugation action of the unitary group $\mathrm{U}(E, j)$. Therefore the question on the range of the spectra of the selfadjoint operator K on the Hilbert space (E, J) as J varies over the space of all compatible complex structures on E boils down to the determination of the image under the so called *real moment map*

$$\mu : \mathfrak{X} \to \mathfrak{s}(E, j), \ \mu(Y) = (Y + jYj^*)/2$$

for the *real Hamiltonian action* of the unitary group $\mathrm{U}(E, j)$ on the connected isospectral class $\mathfrak{X} = \{kXk^*; k \in \mathrm{O}(E)\}$ in $\mathfrak{s}(E)$ of the inertia operator $X = xmx^*$ of the central configuration x.

With these settings, the convexity result of Chenciner and Jiménez-Pérez will be an immediate consequence of a convexity theorem for the real moment polytope of O'Shea and Sjamaar [34] for real reductive Lie algebras. Their result will be explained in the next section.

5   The convexity theorem

The real general linear Lie algebra $\mathfrak{gl}(E)$ of a Euclidean vector space E has the standard Cartan involution $\theta : \mathfrak{gl}(E) \to \mathfrak{gl}(E)$ given by $\theta(X) = -X^*$, and the corresponding Cartan decomposition

$$\mathfrak{gl}(E) = \mathfrak{so}(E) \oplus \mathfrak{s}(E)$$

as sum of $+1$ and -1 eigenspaces of θ. The commutator bracket turns $\mathfrak{so}(E)$ in a Lie algebra, and $\mathfrak{s}(E)$ in a representation space for $\mathfrak{so}(E)$. The trace form $(X, Y) = \mathrm{tr}(XY)$ on $\mathfrak{gl}(E)$ is a nondegenerate symmetric bilinear form, which is negative definite on $\mathfrak{so}(E)$ and positive definite on $\mathfrak{s}(E)$. The conjugation representation of $\mathrm{O}(E)$ on $\mathfrak{s}(E)$ is an orthogonal representation.

Definition 5.1. *A real reductive Lie algebra with Cartan involution is a pair $(\mathfrak{g}, \theta)$ with Lie subalgebra $\mathfrak{g} < \mathfrak{gl}(E)$ that is invariant under the standard Cartan involution θ of $\mathfrak{gl}(E)$. By abuse of notation, the restriction of θ to $\mathfrak{g}$ is again denoted by θ, and is called the Cartan involution of $\mathfrak{g}$. We have a corresponding Cartan decomposition*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}, \ \mathfrak{k} = \mathfrak{g} \cap \mathfrak{so}(E), \ \mathfrak{s} = \mathfrak{g} \cap \mathfrak{s}(E)$$

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of $\mathfrak{g}$ as sum of $+1$ and -1 eigenspaces of θ. The restriction of the trace form to $\mathfrak{g}$ is called the trace form of $\mathfrak{g}$. It is a nondegenerate symmetric bilinear form, which is negative on $\mathfrak{k}$ and positive on $\mathfrak{s}$. The connected Lie subgroup $K < \mathrm{SO}(E)$ with Lie algebra $\mathfrak{k}$ has a representation on $\mathfrak{s}$ by conjugation. Finally, we shall assume that $K < \mathrm{SO}(E)$ is compact, so as to exclude the case of quasi-periodic subgroups. The connected Lie subgroup $G < \mathrm{GL}(E)$ with Lie algebra $\mathfrak{g}$ is a real reductive Lie group with K as a maximal compact subgroup.

Example 5.2. *If $j : E \to E$ is a fixed complex structure on E then the complex general linear Lie algebra $\mathfrak{gl}(E, j)$ gives, by restriction of scalars, a real reductive Lie algebra with Cartan involution.*

Definition 5.3. *A real reductive Lie algebra $(\mathfrak{g}, \theta)$ is called complex if there is a complex structure $j : \mathfrak{g} \to \mathfrak{g}$ making $\mathfrak{g}$ into a complex Lie algebra, such that $j\theta = -\theta j$. This means that θ is an antilinear involution of $(\mathfrak{g}, j)$. Note that multiplication by j interchanges $\mathfrak{k}$ and $\mathfrak{s}$.*

The complex general linear Lie algebra $(\mathfrak{gl}(E, j), \theta)$ is a natural example of a complex reductive Lie algebra with Cartan involution.

Definition 5.4. *A real reductive Lie algebra $(\mathfrak{g}, \theta)$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ has a natural complexification $(\mathfrak{g}_c, \theta)$ defined by*

$$\mathfrak{g}_c = \mathfrak{g} + i\mathfrak{g}, \ i = \sqrt{-1}$$

with Cartan decomposion

$$\mathfrak{g}_c = \mathfrak{u} \oplus \mathfrak{p}, \ \mathfrak{u} = \mathfrak{k} + i\mathfrak{s}, \ \mathfrak{p} = \mathfrak{s} + i\mathfrak{k}$$

for the natural antilinear Cartan involution θ on $\mathfrak{g}_c$. The homogeneous spaces G/K and U/K are dual (in the sense of Élie Cartan) Riemannian symmetric spaces of noncompact and compact type respectively. Both spaces are different real forms of the complex symmetric space G_c/K_c with transversal intersection at the base point eK. Here G_c is the complex Lie subgroup of $\mathrm{GL}(E_c)$ with Lie algebra $\mathfrak{g}_c$ (with E_c the complexification of E), and with K_c the complex Lie subgroup of G_c Lie algebra $\mathfrak{k}_c = \mathfrak{k} + i\mathfrak{k}$.

The following theorem collects the standard structure theory for real reductive Lie algebras with Cartan involution [22].

Theorem 5.5. *Let $(\mathfrak{g}, \theta)$ be a real reductive Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. Any two maximal commutative linear subspaces in $\mathfrak{s}$ are conjugated under K. If $\mathfrak{a} < \mathfrak{s}$ is a fixed maximal commutative linear subspace, then the Weyl group $W = N_K(\mathfrak{a})/C_K(\mathfrak{a})$ (normalizer modulo centralizer of $\mathfrak{a}$ in K) acts by conjugation on $\mathfrak{a}$ as a finite reflection group. Let $\mathfrak{a}_+$ denote the closure of a fixed connected component of the complement $\mathfrak{a}^{\circ}$ of all the mirrors in $\mathfrak{a}$, and call it the (closed) positive Weyl chamber. Then $\mathfrak{a}_+$ is a strict fundamental domain for the action of W on $\mathfrak{a}$, and likewise for the conjugation action of K on $\mathfrak{s}$.*

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ be a real reductive Lie algebra with complexification $\mathfrak{g}_c = \mathfrak{u} \oplus \mathfrak{p}$ as above. For $X \in \mathfrak{a}_+$ we shall denote

$$\mathfrak{X} = \{kXk^*; k \in K\} \subset \mathfrak{s}$$

and call it the isospectral class of X in $\mathfrak{s}$. By construction, $\mathfrak{X}$ is connected, and $X = \mathfrak{X} \cap \mathfrak{a}_+$ by the above theorem. If we denote

$$\mathfrak{X}_c = \{uXu^*; u \in U\} \subset \mathfrak{p}$$

then $\mathfrak{X}_c$ has the structure of a complex manifold with real form $\mathfrak{X} = \mathfrak{X}_c \cap \mathfrak{s}$. Moreover $\mathfrak{X}_c$ has a Kähler metric, whose imaginary part is the Kirillov-Kostant–Souriau symplectic form ω on $\mathfrak{X}_c$. The action of U on $\mathfrak{X}_c$ is Hamiltonian with moment map the inclusion $\mathfrak{X}_c \hookrightarrow \mathfrak{p}$. For this reason, we shall call the action of K on the real form $\mathfrak{X}$ a real Hamiltonian action with real moment map the inclusion $\mathfrak{X} \hookrightarrow \mathfrak{s}$.

We now have set up all the notations in order to formulate the convexity theorem of O'Shea and Sjamaar [34] in case of a real reductive Lie algebra.

Theorem 5.6. *Suppose $(\mathfrak{g}, \theta) < (\mathfrak{g}', \theta')$ is a comparable pair of real reductive Lie algebras with Cartan involution. For $\mathfrak{X}' \subset \mathfrak{s}'$ a fixed isospectral class the orthogonal projection $\mu : \mathfrak{X}' \to \mathfrak{s}$ is clearly equivariant for the conjugation action of K, and is called the real moment map for the real Hamiltonian action of K on $\mathfrak{X}'$. Under all these assumptions, the intersection*

$$\mu(\mathfrak{X}') \cap \mathfrak{a}_+$$

is a convex polytope, called the moment polytope of the real Hamiltonian action of K on $\mathfrak{X}'$.

This theorem has a long history, and we shall mention just a few selected references. In case $(\mathfrak{g}, \theta) < (\mathfrak{g}', \theta')$ are both complex reductive Lie algebras with Cartan involution the theorem is due to Heckman [21]. The result was generalized by Guillemin and Sternberg, who replaced the coadjoint orbit $\mathfrak{X}'$ of the overgroup K' by a complex projective manifold with a Fubini–Study metric h with a holomorphic linearizable action of K, which leaves the symplectic form $\omega = \Im h$ invariant, and μ the moment map for this Hamiltonian action of K [18]. This result was also obtained by Mumford, published in the appendix of a paper by Ness [33]. This is the non-abelian convexity theorem in the Kähler case, which generalizes the former Abelian convexity theorem of Atiyah [3], and that of Guillemin and Sternberg [17]. The proof of the general case without assuming the symplectic manifold to be Kähler was found by Kirwan [25]. These works were all done in the early eighties with many more exciting developments in moment map geometry.

It took almost two decades before O'Shea and Sjamaar discovered the natural real setting of the convexity theorem, which generalizes the Abelian real convexity theorem of Duistermaat [11].

Indeed, consider the commutative diagram

$$
\begin{array}{ccc}
\mathfrak{s}' \supset \mathfrak{X}' & \longrightarrow & \mathfrak{X}'_c \subset \mathfrak{p}' \\
\big\downarrow{\scriptstyle \mu} & & {\scriptstyle \mu}\big\downarrow \\
\mathfrak{s} \supset \mu(\mathfrak{X}') & \longrightarrow & \mu(\mathfrak{X}'_c) \subset \mathfrak{p}
\end{array}
$$

with $\mathfrak{X}'_c = \{uXu^*; u \in U'\}$. As before, $\mathfrak{X}'_c$ can be canonically identified with a coadjoint orbit of U'. Therefore it has a natural symplectic form ω', for which the action of U' by conjugation is Hamiltonian with moment map the inclusion $\mathfrak{X}'_c \hookrightarrow \mathfrak{p}'$. The restriction of the action from U' to U gives a moment map $\mu : \mathfrak{X}'_c \to \mathfrak{p}$, which is just the restriction of the orthogonal projection $\mathfrak{p}' \to \mathfrak{p}$.

The space $\mathfrak{X}'_c$ has a natural antisymplectic involution τ, which is just the restriction of the antiinvolution $-\theta'$ of $\mathfrak{p}' = \mathfrak{s}' \oplus i\mathfrak{k}'$, taken $+1$ on $\mathfrak{s}'$ and -1 on $i\mathfrak{k}'$. In turn, the fixed point locus of τ on $\mathfrak{X}'_c$ is just $\mathfrak{X}' = \mathfrak{X}'_c \cap \mathfrak{s}'$. Hence the map $\mu : \mathfrak{X}' \to \mathfrak{s}$ is nothing but the restriction of $\mu : \mathfrak{X}'_c \to \mathfrak{p}$ to the real form $\mathfrak{X}'$. This explains our use of the terms *real moment map* and *real Hamiltonian action*.

If $\mathfrak{h} \subset \mathfrak{p}$ is a maximal commutative subspace with $\mathfrak{h} \cap \mathfrak{s} = \mathfrak{a}$ and $\mathfrak{h}_+$ is an adapted positive Weyl chamber, in the sense that $\mathfrak{h}_+ \cap \mathfrak{a} = \mathfrak{a}_+$, then

$$
\mu(\mathfrak{X}'_c) \cap \mathfrak{a}_+ = (\mu(\mathfrak{X}'_c) \cap \mathfrak{h}_+) \cap \mathfrak{a}
$$

is a convex polytope by the convexity theorem of Heckman.

Theorem 5.6 is therefore a direct consequence of the following result, which is also due to O'Shea and Sjamaar.

Theorem 5.7. *We have $\mu(\mathfrak{X}') \cap \mathfrak{a}_+ = \mu(\mathfrak{X}'_c) \cap \mathfrak{a}_+$.*

We have restricted ourselves to the case of (co)adjoint orbits for a real reductive Lie algebra, which both suffices for our purpose and keeps the exposition as concrete as possible. In their paper, O'Shea and Sjamaar formulated everything in the general setting of a Hamiltonian action of a connected compact Lie group U on a connected symplectic manifold (M, ω). Suppose that the group U has an involution θ with fixed point group K, and the space (M, ω) has an antisymplectic involution τ with M^τ not empty. These two structures are assumed to be compatible, in the sense that

$$\tau(ux) = \theta(u)\tau(x) \text{ and } \mu(\tau(x)) = -\theta(\mu(x))$$

for all $u \in U$ and $x \in M$. Under these conditions, O'Shea and Sjamaar obtained the following general result

Theorem 5.8. *We have $\mu(M^\tau) \cap \mathfrak{a}_+ = \mu(M) \cap \mathfrak{a}_+$ and the right hand side*

$$\mu(M) \cap \mathfrak{a}_+ = (\mu(M) \cap \mathfrak{h}_+) \cap \mathfrak{a}$$

is indeed a convex polytope by the convexity theorem of Kirwan.

It is readily seen that this implies Theorem 5.7.

6 Pushforward of the normalized measure by the real moment map

We start this section by explaining the notion of Gelfand pairs, and their associated harmonic analysis.

Harmonic analysis for Gelfand pairs

Definition 6.1. *A locally compact unimodular topological group G with a compact subgroup $K < G$ is called a* Gelfand pair *if the natural unitary representation of G on $L^2(G/K, dx)$ decomposes in a multiplicity free way.*

It can be shown that this definition is equivalent to the following one:

Definition 6.2. *A pair $K < G$ is called a* Gelfand pair*, if for any irreducible unitary representation $(V, \langle \cdot, \cdot \rangle)$ of G, the restriction from G to K contains the trivial representation of K with multiplicity at most 1.*

Definition 6.3. *For a Gelfand pair $K < G$, an irreducible unitary representation $(V, \langle \cdot, \cdot \rangle)$ of G is called* spherical *if $V^K = \mathbb{C}v$ has dimension 1 for some $v \in V$ with $\langle v, v \rangle = 1$. The function*

$$G \ni g \mapsto \phi_V(g) = \langle gv, v \rangle$$

is called the elementary spherical function *associated with the spherical representation V.*

Note that elementary spherical functions are normalized by $\phi_V(e) = 1$.

Definition 6.4. *Any function on G that is both left and right invariant under K is called a* spherical function.

Yet, a third equivalent definition for a Gelfand pair is the following:

Definition 6.5. *The pair $K < G$ is a* Gelfand pair *if the Hecke algebra $\mathcal{H}(G/K)$ of continuous spherical functions on G with compact support is commutative with respect to the convolution product.*

The elementary spherical functions are the simultaneous eigenfunctions for the commutative algebra $\mathcal{H}(G/K)$, acting as convolution integral operators on the space of spherical functions.

Finally, in case that G is a connected Lie group, there is a fourth equivalent definition for a Gelfand pair:

Definition 6.6. *For a connected Lie group G, the pair $K < G$ is a* Gelfand pair *if and only if the algebra $\mathcal{D}(G/K)$ of linear differential operators on G/K, which are invariant under G, is commutative.*

Similarly, the elementary spherical functions are the simultaneous eigenfunctions for the commutative algebra $\mathcal{D}(G/K)$, acting as invariant differential operators on the space of spherical functions.

Under all these equivalent conditions, the abstract spherical inversion theorem gives the existence of a unique positive measure μ_{P} on the set $\widehat{G/K}$ of

equivalence classes of unitary irreducible spherical representations of $K < G$, called the *spherical Plancherel measure*, such that

$$\phi(x) = \int_{\widehat{G/K}} \hat{\phi}(V)\phi_V(x)d\mu_{\mathrm{P}}(V)$$

for all $\phi \in \mathcal{H}(G/K)$, with

$$\hat{\phi}(V) = \int_{G/K} \phi(x)\overline{\phi_V(x)}dx$$

the so called *spherical Fourier transform* of $\phi \in \mathcal{H}(G/K)$.

The case that K is the trivial subgroup of $G = \mathbb{R}_+$ or $G = \{z \in \mathbb{C}^\times; |z| = 1\}$ gives the classical inversion formula for Fourier integrals and Fourier series respectively.

Harmonic analysis for Riemannian symmetric pairs

After a brief exposition of the harmonic analysis for general Gelfand pairs, we now come to certain particular cases of our interest. In the notation of the previous section, these are the Riemannian symmetric space G/K of noncompact type, its compact dual Riemannian symmetric space U/K and, finally, the intermediate flat tangent space $\mathfrak{s}$, considered as homogeneous space for the so called *Cartan motion group* $\mathfrak{s} \rtimes K$.

The spherical inversion formula was made explicit in the symmetric space case G/K by Harish-Chandra [20] with simplifications by Helgason, Gangolli and Rosenberg (cf. [23]). Harish-Chandra enlarged the set $\widehat{G/K}$ of equivalence classes of spherical irreducible unitary representations of G to the set $\widetilde{G/K}$ of equivalence classes of spherical continuous irreducible representations of G on a Hilbert space, which are only unitary for the subgroup K. He showed that $\widetilde{G/K} \cong \mathfrak{a}_c/W$ and derived the *Harish-Chandra isomorphism*

$$\mathcal{D}(G/K) \cong S\mathfrak{a}_c^W, D \mapsto \gamma_D,$$

in which $S\mathfrak{a}_c^W$ denotes the symmetric algebra of $\mathfrak{a}_c^W$. The associated elementary spherical functions are given by the *Harish-Chandra integral formula*

$$\phi_\lambda(g) = \int_K a(gk)^{\lambda-\rho}dk = \int_K e^{(\lambda-\rho,A(gk))}dk$$

with Iwasawa decomposition $G = KAN, g = k(g)a(g)n(g)$, Iwasawa projection $A(g) = \log a(g)$, the restricted Weyl vector ρ (half sum of positive restricted roots counting multiplicities) and the normalized Haar measure dk on K.

These elementary spherical functions are solutions of the system of differential equations

$$D\phi_\lambda = \gamma_D(\lambda)\phi_\lambda, \quad D \in \mathcal{D}(G/K)$$

with normalization $\phi_\lambda(e) = 1$ as before. The spherical inversion theorem now takes the form

$$\phi(x) = \frac{1}{|W|} \int_{i\mathfrak{a}} \hat{\phi}(\lambda)\phi_\lambda(x)\frac{d\mu_L(\lambda)}{|c(\lambda)|^2}$$

with spherical Fourier transform

$$\hat{\phi}(\lambda) = \int_{G/K} \phi(x)\overline{\phi_\lambda(x)}dx,$$

the Lebesgue measure μ_L on $i\mathfrak{a}$ and the Harish-Chandra c-function $\lambda \mapsto c(\lambda)$, given as an explicit product of Γ-factors by the Gindikin–Karpelevic formula.

The pair $K < \mathfrak{s} \rtimes K$, with the semidirect product $\mathfrak{s} \rtimes K$ acting on $\mathfrak{s}$ via rotations and translations, is a Gelfand pair as well, and the group $\mathfrak{s} \rtimes K$ is called the Cartan motion group of the space $\mathfrak{s}$. The algebra $\mathcal{D}(\mathfrak{s})$ of invariant linear differential operators is isomorphic to the algebra $S\mathfrak{s}_c^K \cong S\mathfrak{a}_c^W$ of K-invariant linear differential operators on $\mathfrak{s}$ with constant coefficients. Its simultaneous eigenfunctions are the symmetrized plane waves

$$\psi_\lambda(X) = \int_K e^{(\lambda, kXk^*)}dk$$

normalized by $\psi_\lambda(0) = 1$ for all $\lambda \in \mathfrak{a}_c$ and $X \in \mathfrak{s}$. The spherical inversion theorem is a direct consequence of the classical inversion theorem for the Euclidean Fourier transform on $\mathfrak{s}$, applied for functions invariant under K. In a sense, we can consider this theory on the flat space $\mathfrak{s}$ as the confluent limit of the above Harish-Chandra theory for the curved space G/K, by the help of the following formula

Proposition 6.7. *We have*

$$\psi_\lambda(X) = \lim_{n\to\infty} \phi_{n\lambda}(\exp(X/n))$$

for all $\lambda \in \mathfrak{a}_c$ and $X \in \mathfrak{s}$.

Proof. By Harish-Chandra's integral formula

$$\phi_\lambda(\exp X) = \int_K e^{(\lambda-\rho, A(\exp X \cdot k))} dk = \int_K e^{(\lambda-\rho, A(\exp(Ad(k)X)))} dk,$$

we have

$$\phi_{n\lambda}\big(\exp(X/n)\big) = \int_K e^{(n\lambda-\rho, A(\exp(Ad(k)X/n)))} dk.$$

On $\mathfrak{s}$, the infinitesimal Iwasawa projection $\mathfrak{s} \to \mathfrak{a}$ coincides with the orthogonal projection $\mathfrak{s} \to \mathfrak{a}$. Indeed, if $X \in \mathfrak{s}$ has infinitesimal Iwasawa decomposition $X = Y+H+Z$, for which $Y \in \mathfrak{k}, H \in \mathfrak{a}, Z \in \mathfrak{n}$, then $X = H+(Z-\theta Z)/2$, which means that H is also the orthogonal projection of X on $\mathfrak{a}$, as we have the orthogonal decomposition $\mathfrak{s} = \mathfrak{a} \oplus (\mathfrak{s} \cap (\mathfrak{n} \oplus \theta\mathfrak{n}))$. We therefore deduce that

$$\lim_{n\to\infty} \phi_{n\lambda}\big(\exp(X/n)\big) = \int_K e^{(\lambda, \lim_{n\to\infty} nA(exp(Ad(k)X/n)))} dk = \int_K e^{(\lambda, Ad(k)X)} dk,$$

which by definition is equal to $\psi_\lambda(X)$, for all $\lambda \in \mathfrak{a}$, and thus for all $\lambda \in \mathfrak{a}_c$. $\square$

The elementary spherical function $(\lambda, x) \mapsto \phi_\lambda(x)$ is holomorphic and Weyl group invariant in the spectral variable $\lambda \in \mathfrak{a}_c$, and real analytic in the space variable $x \in G/K$, or, in other words, holomorphic in the space variable x taken from a suitable tubular neighborhood of G/K in the complexified space G_c/K_c. It has a holomorphic extension to all of G_c/K_c if and only if $(\lambda - \rho)$ lies in the intersection $L \cap \mathfrak{a}_+$ of a suitable lattice L with the positive chamber $\mathfrak{a}_+ \subset \mathfrak{a}$, given in explicit terms of the restricted root system by the Cartan–Helgason theorem ([23, Ch.V, Theorem 4.1]). The corresponding irreducible spherical representation $V(\lambda)$ (with highest weight $(\lambda-\rho)$) is then finite dimensional, and unitary for the compact form U of G_c. If $v \in V(\lambda)^K$ is a normalized spherical vector, then $\phi_\lambda(u) = \langle uv, v \rangle$ for $u \in U$ with $\langle \cdot, \cdot \rangle$ the invariant Hermitian form on $V(\lambda)$.

Pushforward of the normalized measure

After this survey of the theory of spherical functions, we can finally explain the meaning of the pushforward under the real moment map $\mu : \mathfrak{X}' \to \mathfrak{s}$ of the normalized invariant measure on $\mathfrak{X}'$ in the notation of Theorem 5.6 in terms of spherical representation theory. Let $\lambda' \in L'_+ = (L' \cap \mathfrak{a}'_+) + \rho'$ and $(V(\lambda'), \langle \cdot, \cdot \rangle)$ be the associated finite dimensional spherical irreducible

unitary representation of U' with normalized spherical vector $v' \in V(\lambda')^{K'}$. Let $\phi_{\lambda'}(u') = \langle u'v', v' \rangle$ be the associated elementary spherical function on U'/K'. Its restriction to the totally geodesic submanifold $U/K < U'/K'$ is given by

$$\phi_{\lambda'}(u) = \sum_{\lambda \in L_+} m_{\lambda'}(\lambda)\phi_\lambda(u)$$

with $m_{\lambda'}(\lambda) = \langle v_\lambda, v_\lambda \rangle$ if $v' = \sum_\lambda v_\lambda$ is the primary decomposition of v' into components v_λ for $\lambda \in L_+$ of spherical vectors for the Gelfand pair $K < U$.

Due to the linearity and continuity of the real moment map, it is enough to analyze those $\lambda' \in L'_+$. For $\lambda' \in L'_+$, let

$$\mu : \mathfrak{X}'_{\lambda'} = \{k'\lambda'k'^*; k' \in K'\} \to \mathfrak{s}$$

be the real moment map for the real Hamiltonian action of K, and let $d\mathfrak{x}'$ be the normalized K'-invariant measure on $\mathfrak{X}'_{\lambda'}$, so that $\int d\mathfrak{x}' = 1$.

Theorem 6.8. *Let $\nu \mapsto \delta(\nu - \lambda)$ be the Dirac delta distribution on $\mathfrak{a}$ with unit mass at λ. The probability measure $\mu_{\lambda'}$ on $\mathfrak{a}_+$ given by*

$$d\mu_{\lambda'}(\nu) = \lim_{n\to\infty} \sum_{\lambda \in L_+} m_{n\lambda'}(\lambda)\delta(\nu - \lambda/n)$$

describes the pushforward measure $\mu_*(d\mathfrak{x}')$ on $\mathfrak{s}$ by the relation

$$\int_\mathfrak{s} f(\lambda)\mu_*(d\mathfrak{x}')(\lambda) = \int_{\mathfrak{a}_+} f(\lambda)d\mu_{\lambda'}(\lambda)$$

for all continuous functions f on $\mathfrak{s}$, which are invariant under K.

Proof. For $n \in \mathbb{N}$, $\lambda' \in L'_+$ and $X \in \mathfrak{s}$ we have

$$\phi_{n\lambda'}(\exp(X/n)) = \sum_{\lambda \in L_+} m_{n\lambda'}(\lambda)\phi_\lambda(\exp(X/n))$$

$$= \sum_{\lambda \in L_+/n} m_{n\lambda'}(n\lambda)\phi_{n\lambda}(\exp(X/n)),$$

which in turn implies

$$\psi_{\lambda'}(X) = \int_{\mathfrak{a}_+} \psi_\lambda(X)d\mu_{\lambda'}(\lambda)$$

for all $X \in \mathfrak{s}$. Hence the desired formula for $\mu_{\lambda'}$ as the pushforward measure $\mu_*(d\mathfrak{x}')$ follows from the Euclidean inversion theorem for the flat space $\mathfrak{s}$ and the Fubini theorem. $\square$

This theorem generalizes the result of [21] on the relation between the asymptotic behaviour of branching multiplicities and the pushforward of the Liouville measure under the moment map in case $(\mathfrak{g}, \theta) < (\mathfrak{g}', \theta')$ are both complex reductive Lie algebras with a Cartan involution. In that paper, the convexity theorem was derived from the above theorem together with a simple representation-theoretic property.

Some questions

We end this section and the paper with some questions.

Question 6.9. *For $\lambda \in L_+$ and $\lambda' \in L'_+$, does the spherical irreducible representation $V(\lambda)$ of $(\mathfrak{g}, \theta)$ occur as subrepresentation of the spherical irreducible representation $V(\lambda')$ of $(\mathfrak{g}', \theta')$ if and only $m_{\lambda'}(\lambda) > 0$?*

Question 6.10. *Is it possible to generalize the results of this section to the general Hamiltonian Kählerian setting, in line with O'Shea and Sjamaar?*

Question 6.11. *Is there a localization formula for the pushforward of the normalized Riemannian measure under the real moment map?*

Question 6.12. *In the case studied by Chenciner and Jimenez, the convex spectra polytope can be explicitly described. Is it yet possible to give an explicit description of the pushforward measure?*

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Gert Heckman, Radboud University Nijmegen: g.heckman@math.ru.nl
Lei Zhao, Chern Institute of Mathematics, Nankai University: l.zhao@nankai.edu.cn