PROBABILITY DENSITIES FOR DISCRETE DYNAMICAL SYSTEMS WITH WEAK NOISE

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Problem and Model

The determination of stationary probability densities in multidimensional non-equilibrium systems in the presence of weak noise is of great importance in many cases [1] but plagued by notorious difficulties. It is widely believed that often useful predictions for the full system can be obtained by the restriction on a stroboscopic time-discretization of a single state-variable. As a model system we consider the following Langevin equation in discrete time n for the single variable x

$$x_{n+1} = f(x_n) + \xi_n = x_n + \frac{b}{2\pi}\sin(2\pi x_n) + \xi_n \tag{1}$$

where the noise term ξ_n represents an analogue of the random force in the time-continuous case. For the sake of simplicity ξ_n is assumed to be independent and identically Gaussian distributed: $P(\xi_n \in [\xi, \xi + d\xi]) = (2\pi\epsilon)^{-1/2}\, exp(-\xi^2/2\epsilon)\, d\xi$ with small noise strength $\epsilon \ll 1$ [2]. Under these assumptions the stationary probability density $W_\epsilon(x)$ satisfies the following integral equation [2]

$$W_\epsilon(x) = \frac{1}{\sqrt{2\pi\epsilon}} \int_{-\infty}^{+\infty} e^{-(x-f(y))^2/2\epsilon}\, W_\epsilon(y)\, dy \tag{2}$$

The usual analytic and numerical methods for solving integral equations fail to give the correct asymptotic behaviour for weak noise.

Results

By means of a new geometrically motivated concept we find for the stationary probability density a WKB-like result

$$W_\epsilon(x) = N_\epsilon\, Z_\epsilon(x)\, e^{-\phi(x)/\epsilon} \tag{3}$$

where the nonequilibrium potential $\phi(x)$ is ϵ-independent and $Z_\epsilon(x)$ may depend on ϵ algebraically at most. N_ϵ represents a normalization factor. For $b \le 2$ the deterministic map $f(x)$ in eq.(1) has stable fixed points at $x = 0.5\ mod\ 1$ and unstable fixed points at $x = 0\ mod\ 1$. In the neighbourhood of the stable fixed points $\phi(x)$ has a parabolic shape with curvature $1 - f'(0.5)^2$ and thus in Gaussian approximation we have $N_\epsilon = \sqrt{(1 - f'(0.5)^2)/2\pi\epsilon}$ if we choose $Z_\epsilon(0.5) = 1$. In the neighbourhood of unstable fixed points $\phi(x)$ consists of straight lines whose slopes and matching points both geometrically converge to zero at the rate $f'(0)^{-1}$ as an unstable fixed point is approached . Each straight piece touches an enveloping parabola

with curvature $1 - f'(0)^2$ from above. Analogous results for continuous time systems in more than one dimension have been obtained previously [3]. The prefactor $Z_\epsilon(x)$ in eq.(3) turns out to be ϵ-independent for most x, except for those in the vicinity of the bends of $\phi(x)$ where cusps occur with ϵ-independent heights and widths proportional to ϵ. Outside an $\epsilon^{1/2}$-neighbourhood of $x = 0$ there are plateaus between two cusps. Their heights decrease geometrically with a rate $f'(0)^{-1}$ if $x = 0$ is approached. Within the $\epsilon^{1/2}$-neighbourhood cusps and plateaus merge and the value of the prefactor at $x = 0$ becomes proportional to $\epsilon^{1/2}$. In the expression (3) for the stationary probability density the singularities of $\phi(x)$ and $Z_\epsilon(x)$ compensate each other resulting in a smooth behaviour of $W_\epsilon(x)$ for all x and ϵ. The Figure below shows $\phi(x)$ and $Z_\epsilon(x)$ for $b = 1.4$ and $\epsilon = 10^{-3}$.
Details of the novel geometrical method will be published elsewhere. Within this concept generalisations to more complicated cases as e.g. maps with periodic or chaotic attractors and to non-Gaussian white noise are straightforward.

References

[1] R.Graham, Springer Tracts in Modern Physics 60, Springer Berlin (1973)

[2] P.Talkner, P.Hänggi, E.Freidkin, D.Trautmann, J. Stat. Phys. 48, 231 (1987); P.Talkner, P.Hänggi in 'Noise in Nonlinear Dynamical Systems', Vol.2, edited by F.Moss and P.V.E.McClintock, Cambridge University Press (1989)

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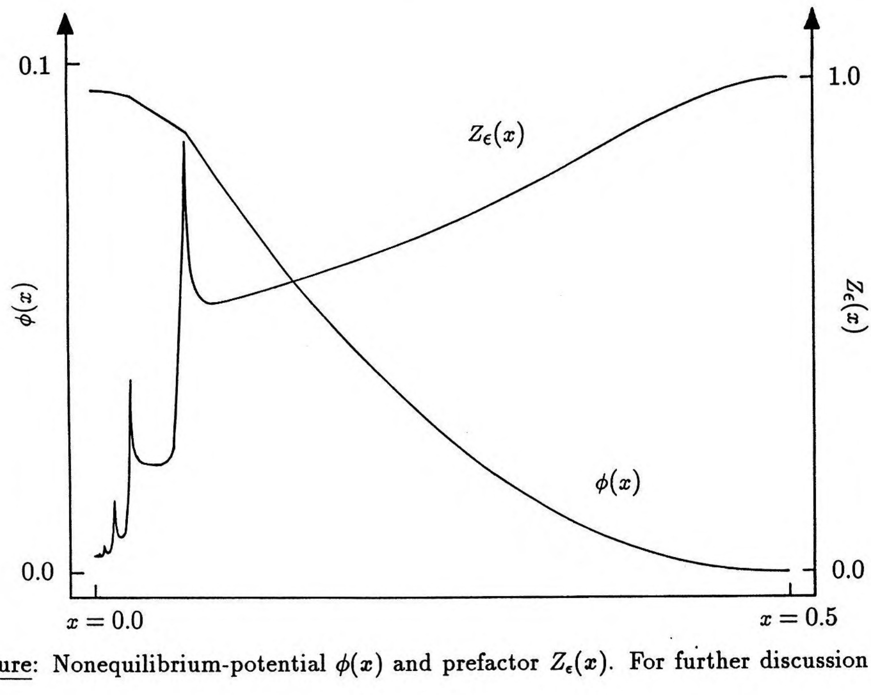

Figure: Nonequilibrium-potential $\phi(x)$ and prefactor $Z_\epsilon(x)$. For further discussion see text.