Continuous-Time Limit for Maps with Colored Noise

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Nonlinear dynamical systems under the influence of colored noise have attained considerable interest in recent years. In spite of the great effort important properties of this problem are still not yet understood [1].

As a particular model we consider a particle with coordinate x moving in an external potential $V(x)$ under the influence of an exponentially correlated random force

$$\dot{x} = -V'(x) + \tau^{-1/2}\, y \qquad , \qquad \dot{y} = -\tau^{-1} y + \sqrt{2D\tau^{-1}}\, \xi(t) \tag{1}$$

where $\xi(t)$ is Gaussian white noise, τ denotes the correlation time, and D the strength of the colored noise $\tau^{-1/2}y$. For $V(x)$ we choose a continuously differentiable piecewise parabolic symmetric double well potential [2] with wells at $x = \pm 1$ and a barrier at $x = 0$. The well and barrier frequencies are denoted by $\omega_0^2 = V''(\pm 1)$ and $\omega_b^2 = -V''(0)$, respectively.

By discretization of the time t we obtain a two-dimensional noisy map

$$x_{n+1} = x_n - aV'(x_n) + y_n \qquad , \qquad y_{n+1} = A\, y_n + \sqrt{\epsilon/2}\, \xi_n \tag{2}$$

where a denotes the time step and ξ_n are independent Gaussian random numbers with vanishing mean and unit correlation. Further we defined $x_n := x(na)$, $y_n := \tau^{-1/2} a y(na)$, $A := 1 - a\tau^{-1}$, and $\epsilon := 4D\tau^{-2}a^3$.

In this note we consider the invariant density $W(x,y)$ of the process (2) in the limit of weak noise ϵ and vanishing time step a. In the limit of weak noise $W(x,y)$ becomes

$$W(x,y) = Z(x,y)\, \exp\{-\Phi(x,y)/\epsilon\} \tag{3}$$

where $\Phi(x,y)$ denotes the generalized potential and $Z(x,y)$ the prefactor. For an arbitrary but finite a, the generalized potential $\Phi(x,y)$ may be determined by means of an implicit variational principle [3].

With

$$\phi(x,y)/D := \lim_{a\to 0} \Phi(x, y{\cdot}a/\sqrt{\tau})/\epsilon \tag{4}$$

we find the central result

$$\phi(x,y) = \frac{1}{2} \min_{\lambda \geq 0, \Theta = \pm 1} \{ b_\lambda (x e^{-\lambda\tau\omega_b^2} + \Theta)^2 + (1 - b_\lambda c_\lambda^2)^{-1} (y - b_\lambda c_\lambda (x e^{-\lambda\tau\omega_b^2} + \Theta))^2 \} \quad . \tag{5}$$

The coefficients b_λ and c_λ are defined by

$$b_\lambda = \left(\frac{1}{2\,\Delta\phi} + \frac{2\tau^2\,(\omega_b^2+\omega_0^2)\,e^{-\lambda(1+\tau\omega_b^2)}}{(1+\tau\omega_b^2)(1-\tau\omega_b^2)(1+\tau\omega_0^2)} - \frac{e^{-2\lambda\tau\omega_b^2}}{\omega_b^2(1-\tau\omega_b^2)}\right)^{-1}$$

$$c_\lambda = \frac{\tau^{1/2}\,e^{-\lambda\tau\omega_b^2}}{1-\tau\omega_b^2} - \frac{\tau^{3/2}\,(\omega_b^2+\omega_0^2)\,e^{-\lambda}}{(1-\tau\omega_b^2)(1+\tau\omega_0^2)} \tag{6}$$

where $\Delta\phi$ denotes the difference of the generalized potential at the barrier and the well

$$\Delta\phi = (V(0)-V(1))\left(1+\frac{\tau^2\,\omega_b^2\,\omega_0^2}{1+\tau\,(\omega_b^2+\omega_0^2)}\right) \quad . \tag{7}$$

Whereas this result (7) agrees with [2], the reduced potential $\phi(x)$ derived from (5) does slightly differ from the findings in [2]. Because of singularities in the higher derivatives of the potential $V(x)$ Eq.(7) cannot be conferred to the results of [4] as far as the corrections to the leading order expressions in the limiting cases of small and large correlation time are concerned.

The generalized potential (5) approaches the correct parabolic behavior near the stable fixed point [5]

$$\phi(1+x,y)-\phi(1,0) = \frac{1}{2}(1+\tau\omega_0^2)(\omega_0^2x^2+(\sqrt{\tau}\omega_0^2x-y)^2) \quad . \tag{8}$$

As long as $\tau\omega_b^2 < 1$ the same form of $\phi(x,y)$ holds also near the unstable fixed point with ω_0^2 replaced by $-\omega_b^2$. When $\tau\omega_b^2 > 1$ the generalized potential becomes non-analytic. For the reduced potential $\phi(x)$ one obtains from (5) for small x-values $\phi(0)-\phi(x) \propto |x|^{1+\omega_b^2\tau}$ in agreement with [2], [4] and with [5] in the large τ limit. In the limit of small correlation time the known form of $\phi(x)$ [6] is recovered from (5).

The prefactor $Z(x,y)$ is constant for $\tau = 0$ and has a minimum at the unstable fixed point for $\tau > 0$. From this fact in combination with the above mentioned behavior of the generalized potential it is possible to show that for $\omega_b^2\tau > 1$ the invariant density has no longer a saddle at the unstable fixed point but a relative minimum for small enough noise strengths D. This explains the phenomenon known as 'holes in the invariant density' observed in numerical simulations [7].

It can be shown that the above results qualitatively hold true for more general potentials $V(x)$.

References

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