Probability Densities for the Noisy Feigenbaum Scenario

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The influence of noise on the period doubling route to chaos is the subject of many investigations concernig e.g. scaling of the Lyapunov exponent and the appearance of a bifurcation gap at the onset of chaos [1] as well as the determination of the escape rate from attracors [2].

The standard equation describing the noisy Feigenbaum route to chaos reads

$$x_{n+1} = f_r(x_n) + \sqrt{\epsilon}\,\xi_n \tag{1}$$

We assume independent identically distributed Gaussian noise $P(\xi_n \in [\xi,\, \xi + d\xi]) = \pi^{-1/2} \cdot \exp\{-\xi^2\}\, d\xi$ of strenth ϵ. Further on $f_r(x)$ is supposed to map some finite iterval $I_r = [-x_r,\, x_r]$ onto itself possessing a single quadratic maximum in I_r and to depend smoothly on the control parameter r. Thus after fixing the of scales by $f_0(0) = 1$ and $f_1(0) = x_1$ the functions $f_r(x)$ are uniquely determined through the recursion

$$f_{\delta \cdot r}(x) = -\alpha\, f_r(f_r(-x/\alpha)) \tag{2}$$

where $\alpha = 2.5029...$ and $\delta = 4.6692...$ are the universal Feigenbaum constants. This definition yields a suitably idealized version of the logistic map with qualitatively identical features. Outside I_r the detais of $f_r(x)$ are of no relevance to our problem and one may simply think of a constant continuation there.

We are interested in the properties of the stationary probability density $W_\epsilon^r(x)$ [3],[4] satisfying

$$W_\epsilon^r(x) = \frac{1}{\sqrt{\pi\epsilon}} \int_{-\infty}^{+\infty} dy\, W_\epsilon^r(y)\, e^{-(x-f_r(y))^2/\epsilon} \tag{3}$$

in the case of weak noise $\epsilon << 1$. Using the WKB ansatz [3],[4]

$$W_\epsilon^r(x) = Z_\epsilon^r(x)\, e^{-\phi_r(x)/\epsilon} \tag{4}$$

an approximate renormalization treatment of Eq.(3) yields the following results

$$2\, W_\epsilon^r(x) = \alpha\, W_{\lambda\cdot\epsilon}^{\delta\cdot r}(-\alpha\, x) \qquad \text{and} \qquad W_\epsilon^r(x) = W_\epsilon^r(f_r(x)) \cdot |f_r'(x)| \tag{5}$$

where the first equation holds for $-x_r < x < x_{\delta\cdot r}/\alpha$ and the second for $x_{\delta\cdot r}/\alpha < x < x_r$. Here $\lambda = 43.81...$ denotes a third universal constant [1] and $x_{\delta\cdot r}/\alpha$ equals the unstable fixed point of $f_r(x)$ in I_r which follows from (2). This scaling behavior of which certain aspects can be found in [1],[5] is compared with the result of a recently introduced numerical method [3] by which $\phi_r(x)$ and $Z_\epsilon^r(x)$ can separately be determined (see Fig.). We find deviations from (5) that depend on x and are rather small both for $\phi_r(x)$ and $Z_\epsilon^r(x)$.

Our renormalization procedure for the generalized potential $\phi_r(x)$ allows one to estimate these deviations analytically in good agreement with the numerical findings. Further one finds analytically some local properties of $\phi_r(x)$ and $Z_\epsilon^r(x)$ as e.g. the behavior of $\phi_r(x)$ in the vicinity of the attractor and the repellors as well as the location, heigth , and width of the most dominant peaks of $Z_\epsilon^r(x)$ [4].

Within the above described limitations the central result (5) is universal in the same sense as the deterministic Feigenbaum scenario: qualitatively it applies for other maps than $f_r(x)$ with quadratic maximum, and for nonvanishing multiplicative noise, too. For non-Gaussian but still white noise different values of the scaling parameter λ may appear.

Finally we note that the qualitative behavior of $\phi_r(x)$ is well understood from the fact that it is a Lyapunov function of the deterministic map $f_r(x)$ [4] and that it obeys the same scaling relation (5) as the probability density $W_\epsilon^r(x)$.

References
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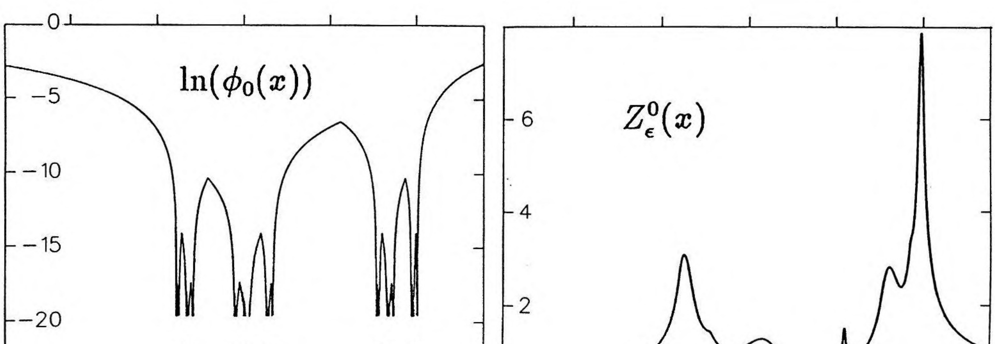

Figure: Logarithm of $\phi_0(x)$ and $Z_\epsilon^0(x)$ at the onset of chaos at $r = 0$ for $\epsilon = 4 \cdot 10^{-4}$. The Feigenbaum attractor is indicated above the x-axis. Deviations from Eq.(5) are beyond graphical resolution.