New Analogies between the Noisy Feigenbaum Scenario and Critical Phenomena

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The concepts of universality and scaling play an important role in various fields of theoretical physics like e.g. critical phenomena, nonlinear dynamics, hydrodynamics etc. [1]. In this note we work out further striking analogies of the the Feigenbaum route to chaos [2] in presence of weak noise [3,4] to the theory of critical phenomena.

The noisy Feigenbaum scenario is described by the one-dimensional Langevin equation for the dynamical variable x in discrete time n [3,4]

$$x_{n+1} = f_\mu(x_n) + g_\mu(x_n)\xi_n \tag{1}$$

where $f_\mu(x)$ is a map with an invariant interval I_μ with negative Schwarzian derivative [2] smoothly depending on the control parameter μ and on x with the possible exception $x = 0 \in I_\mu$ where the map has its global maximum of of order $z > 1$. The noise coupling function $g_\mu(x)$ is required to be smooth, positive, and bounded. The distribution of the uncorrelated random numbers ξ_n is given by

$$P_a(\xi_n) = u_a(\xi_n)\exp\{-h_a(\xi_n)/\sigma^a\} \tag{2}$$

where the small parameter σ determines the noise strength, the prefactor $u_a(\xi_n)$ is smooth, positive and bounded, and the exponentially leading part $h_a(\xi_n)$ has a global minimum of order $a > 0$ at $\xi_n = 0$ and is smooth everywhere else.

Iteration of (1) leads to the renormalization group transformation that can be formulated in terms of the deterministic renormalization operator [3,4] $(\hat{T}f_\mu)(x) := -\alpha f_\mu(f_\mu(-x/\alpha))$. Using an appropriate parametrization of μ and x one obtains [2]

$$(\hat{T}^k f_{\mu/\delta^k})(x) \to f^*_\mu(x) \tag{3}$$

for $k \to \infty$, $x \in I^*_\mu$. Here α and δ are the universal Feigenbaum numbers and the maps $f^*_\mu(x)$ constitute the unstable invariant manifold of the operator $\hat{T}$. For the full renormalization group this manifold yields one relevant direction in function space at the critical point $f^*_{\mu=0}(x)$. Similarly the scaling limit $g^*_\mu(x)$ of the noise coupling functions gives another relevant direction.

The renormalization group yields an approximate scaling relation for the invariant density $W_{\sigma,\mu}(x)$, $x \in I_\mu$, of the stochastic process (1) [5-7]

$$\alpha\ W_{\sigma,\mu}(x) \simeq 2\ W_{\sigma/\kappa,\mu/\delta}(-x/\alpha) \tag{4}$$

where κ is the scaling factor of the noise strength σ. Strict scaling holds only in the trivial case with vanishing noise $\sigma = 0$. However, for the invariant functions $f^*_\mu(x)$, $g^*_\mu(x)$ the deviations from exact scaling become extremely small.

As a further approximative result a Frobenius-Perron-like relation [6]

$$W_{\sigma,\mu}(x) \simeq \sum_{f_\mu(y)=x} W_{\sigma,\mu}(y)\,|f'_\mu(y)|^{-1} \tag{5}$$

is found, valid for such $x \in I_\mu$ for which there exists at least one $y \in I_\mu$ with $f_\mu(y) = x$. For $a \leq 1/2$ (5) is exact in exponentially leading order in σ but not for $a > 1/2$, although it may still yield good approximations.

Using (3)-(5) one recovers the scaling behaviour of the envelope $\bar{\lambda}$ of the Lyapunov exponent $\lambda(\sigma,\mu) = \int W_{\sigma,\mu}(x) \ln |f'_\mu(x)|\, dx$ [3,4]

$$\bar{\lambda}(\sigma,\mu) = |\mu|^{\frac{\ln 2}{\ln \delta}} L(\sigma\, |\mu|^{-\frac{\ln \kappa}{\ln \delta}}) \qquad . \tag{6}$$

where L is a scaling function.

In terms of critical phenomena [8] the noise strength σ can be identified with the relevant scaling field, $\bar{\lambda}$ is then the order parameter, and $\ln \delta$, $\ln \kappa$ are the critical exponents [3,4]. Any further critical exponent can be expressed in terms of $\ln \delta$ and $\ln \kappa$, i.e. we are dealing with a two exponent theory. The Feigenbaum number α must not be confused with a critical exponent. It rather corresponds to the spatial scaling factor in a real space renormalization theory [4]. Eq.(4) corresponds to scaling of the coarse grained partition function and (6) to the equation of state.

The scaling factor δ as well as the Feigenbaum number α depend on z [2] whereas κ depends on both z [9] and a [7] but are independent of further details of (1), (2). Thus the numbers z and a are related to the dimensionality of space and number of components of the order parameter which completely determine the universality class, fixed point and critical exponents [8].

As in (5) fluctuations are no longer taken fully into account this may be considered as the counterpart of mean field approximation for critical phenomena. Within this approximation one finds $\kappa_{MFA} = \alpha^z$ independent of a. This result follows from the full renormalization theory only if $a \leq 1$. Hence, we identify a^{-1} with the dimensionality d of space and find as its critical value $d_c = 1$. For $d < d_c$ the noise scaling factor κ is given by

$$\kappa = \alpha^z (1 + [c(z,d)/\alpha^{z-1}]^{\frac{1}{d_c - d}})^{d_c - d} \tag{7}$$

with $1 > c(z,d) > 0$ having well defined limits $d \to 0$ [7] and $z \to \infty$. Eq.(7) may be read either as $d_c - d$–expansion or as hyperscaling relation. Note that the second scaling factor δ is independent of d.

Finally we note that the whole analysis can be generalize to a large class of correlated noise including Ornstein-Uhlenbeck noise. In particular these systems belong to the same two exponent universality class independent of the correlation of the noise.

References

[1] L.Kadanoff, Physica A **163**, 1, 1990.
[2] P.Collet, J.-P. Eckmann, O.Lanford III, Comm.Math.Phys.**76**, 211, 1980.
[3] J.Crutchfield, M.Nauenberg, J.Rudnick, Phys.Rev.Lett.**46**, 933, 1981.
[4] B.Shraiman, C.Waine, P.Martin, Phys.Rev.Lett.**46**, 935, 1981.
[5] P.Reimann, P.Talkner, Phys.Rev.A **44**, 6348, 1991.
[6] P.Reimann, P.Talkner, Helv.Phys.Acta **64**, 946, 1991.
[7] A.Hamm, R.Graham, J.Stat.Phys.**66**, 689, 1992.
[8] M.Fisher, Lecture Notes in Physics **186**, Springer, 1983.
[9] F.Argoul et al., Europhys.Lett.**3**, 643, 1987.