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Angaben zur Veröffentlichung / Publication details:

Denisov, Denis, and Vitali Wachtel. 2019. "Alternative constructions of a harmonic function for a random walk in a cone." *Electronic Journal of Probability* 24: 92. https://doi.org/10.1214/19-ejp349.

Electronic Journal of Probability

Electron. J. Probab. **24** (2019), no. 92, 1–26.
ISSN: 1083-6489 https://doi.org/10.1214/19-EJP349

Alternative constructions of a harmonic function for a random walk in a cone

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Abstract

For a random walk killed at leaving a cone we suggest two new constructions of a positive harmonic function. These constructions allow one to remove a quite strong extendability assumption, which has been imposed in our previous paper (Denisov and Wachtel, 2015, Random walks in cones). As a consequence, all the limit results from that paper remain true for cones which are either convex or star-like and C^2.

Keywords: random walk; exit time; harmonic function.
AMS MSC 2010: Primary 60G50, Secondary 60G40; 60F17.
Submitted to EJP on May 3, 2018, final version accepted on July 29, 2019.
Supersedes arXiv:1805.01437.

1 Introduction and the main result

Consider a random walk $\{S(n), n \geq 1\}$ on $\mathbb{R}^d$, $d \geq 1$, where

$$S(n) = X(1) + \cdots + X(n)$$

and $\{X(n), n \geq 1\}$ is a family of independent copies of a random vector $X = (X_1, X_2, \ldots, X_d)$. We will assume that the random variables have zero mean, unit variance, and are uncorrelated, that is $\mathbf{E}[X_i] = 0, \mathbf{Var}(X_i) = 1$ for $1 \leq i \leq d$ and $\mathrm{cov}(X_i, X_j) = 0$ for $1 \leq i < j \leq d$.

Denote by $\mathbb{S}^{d-1}$ the unit sphere of $\mathbb{R}^d$ and Σ an open and connected subset of $\mathbb{S}^{d-1}$. Let K be the cone generated by the rays emanating from the origin and passing through Σ, i.e. $\Sigma = K \cap \mathbb{S}^{d-1}$. Let τ_x be the exit time from K of the random walk with starting point $x \in K$, that is,

$$\tau_x = \inf\{n \geq 1 : x + S(n) \notin K\}.$$

In the present paper we are concerned with the existence of a positive harmonic function V for a random walk killed at the exit from K, that is a function V which solves the following equation

$$\mathbf{E}[V(x + X), \tau_x > 1] = V(x), \quad x \in K.$$

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Harmonic function $V(x)$ plays a central role in our approach to study of the Markov processes confined to unbounded domains. This approach was initiated in [8], where we studied random walks in a Weyl chamber, which is an example of a cone. These studies were extended in [9], where we considered random walks in general cones. In particular, in [9] we showed that

$$\mathbf{P}(\tau_x > n) \sim C\frac{V(x)}{n^{p/2}}, \quad n \to \infty,$$

and proved global and local limit theorems for random walks conditioned on $\{\tau_x > n\}$. The approach suggested in [9] was further extended to one-dimensional random walks above the curved boundaries [11], [6], [7], integrated random walks [10], [5], products of random matrices [14], and Markov walks [13].

This approach is based on the universality ideas and heavily relies on corresponding results for Brownian motion, or, more generally, diffusion processes. Thus, an important role is played by the harmonic function of the Brownian motion killed at the boundary of K, which can be described as the minimal (up to a constant), strictly positive on K solution of the following boundary problem:

$$\Delta u(x) = 0,\ x \in K \quad \text{with boundary condition } u\big|_{\partial K} = 0.$$

The function $u(x)$ and constant p can be found as follows. If $d = 1$ then we have only one non-trivial cone $K = (0, \infty)$. In this case $u(x) = x$ and $p = 1$. Assume now that $d \geq 2$. Let $L_{\mathbb{S}^{d-1}}$ be the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$ and assume that Σ is regular with respect to $L_{\mathbb{S}^{d-1}}$. With this assumption, there exists a complete set of orthonormal eigenfunctions m_j and corresponding eigenvalues $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$ satisfying

$$\begin{aligned} L_{\mathbb{S}^{d-1}} m_j(x) &= -\lambda_j m_j(x), \quad x \in \Sigma \\ m_j(x) &= 0, \quad x \in \partial\Sigma. \end{aligned} \tag{1.1}$$

Then

$$p = \sqrt{\lambda_1 + (d/2-1)^2} - (d/2-1) > 0,$$

and the harmonic function $u(x)$ of the Brownian motion is given by

$$u(x) = |x|^p m_1\left(\frac{x}{|x|}\right), \quad x \in K. \tag{1.2}$$

We refer to [2] for further details on exit times of Brownian motion. For symmetric stable Lévy processes asymptotics for exit times and related questions have been considered in [1], [3] and [15], see also references therein.

In [9] we showed that one construct a harmonic function for the random walk killed at τ_x as follows

$$V(x) = \lim_{n\to\infty} \mathbf{E}[u(x + S(n), \tau_x > n].$$

The existence and positivity of V was shown under certain assumptions. The geometric assumptions in [9] can be summarised as follows,

(i) K is either starlike with Σ in C^2 or convex. We say that K is starlike if there exists $x_0 \in \Sigma$ such that $x_0 + K \subset K$ and $\operatorname{dist}(x_0 + K, \partial K) > 0$. Clearly, every convex cone is also starlike, for the proof see Remark 15 in [9].

(ii) We assume that there exists an open and connected set $\widetilde{\Sigma} \subset \mathbb{S}^{d-1}$ with $\operatorname{dist}(\partial\Sigma, \partial\widetilde{\Sigma}) > 0$ such that $\Sigma \subset \widetilde{\Sigma}$ and the function m_1 can be extended to $\widetilde{\Sigma}$ as a solution to (1.1).

Assumption (ii) is quite restrictive. For this assumption to hold it is necessary to assume that the boundary of the cone is piecewise infinitely differentiable. But this condition is not sufficient. The restriction (ii) excludes many cones which are of interest in various mathematical problems. For example, it is not clear whether (ii) holds for linear transformations of the orthant $\mathbb{R}^d_+$, $d \geq 2$ which appear often in paths enumeration problems in combinatorics. (It is worth mentioning that (ii) holds for any simply connected open cone in $\mathbb{R}^2$. This follows from the observation that $m_1(x) = \sin(C_1 + C_2 x)$ in this two-dimensional situation.)

We have shown in [9] that the condition (ii) can be dropped in the case when the random walk $\{S(n)\}$ has bounded jumps, Raschel and Tarrago [17] have recently shown that (ii) can be removed under stronger than in [9] moment restrictions on the vector X. The *main aim* of this paper is to show that this assumption can be removed without imposing any further conditions. Namely, we prove that (i) is sufficient and the following result holds.

Theorem 1.1. *Assume that either the cone K is convex or Σ is C^2 and K is starlike. If $\mathbf{E}|X|^\alpha$ is finite for $\alpha = p$ if $p > 2$ or for some $\alpha > 2$ if $p \leq 2$, then the function*

$$V(x) := \lim_{n\to\infty} \mathbf{E}\left[u(x + S(n)); \tau_x > n\right]$$

is finite and harmonic for $\{S(n)\}$ killed at leaving K, i.e.,

$$V(x) = \mathbf{E}\left[V(x + S(n)); \tau_x > n\right], \quad x \in K,\ n \geq 1.$$

Furthermore, $V(x)$ is strictly positive on the set

$$\begin{aligned} K_+ := \{x \in K :\ & \text{there exists } \gamma > 0 \text{ such that for every } R > 0 \\ & \text{there exists } n \text{ such that } \mathbf{P}(x + S(n) \in D_{R,\gamma}, \tau_x > n) > 0\}, \end{aligned}$$

where $D_{R,\gamma} := \{x \in K : |x| \geq R, \mathrm{dist}(x, \partial K) \geq \gamma|x|\}$.

We will present two very different proofs of this theorem. The first proof uses preliminary bounds for the moments of exit times of τ_x due to [16], see Lemma 2.6 below. The proof is similar to that in [9], but we use an additional idea of time-dependent shifts inside the cone. Thus the approach is reminiscent of one-dimensional random walks conditioned to stay above curved boundaries [6].

The second proof combines time-dependent shifts with an iterative procedure similar to that in [8] and [10]. The main advantage of this approach is that in principle no preliminary information on moments of exit times is needed. However, we use [16] to obtain optimal moment conditions. If we assume two additional moments then this approach becomes self-contained, see Remark 5.4 below.

A further advantage of new constructions consists in the fact that we do not use estimates for the concentration function of the random walk $\{S(n)\}$, which were important for the method used in [9].

Since the geometric assumption (ii) has been used in [9] in the construction of $V(x)$ only, Theorem 1.1 allows us to state limit theorems for random walks in cones proven in [9] and in [12] for all cones satisfying (i).

Corollary 1.2. *Under the conditions of Theorem 1.1, as $n \to \infty$,*

$$\mathbf{P}(\tau_x > n) \sim \varkappa V(x) n^{-p/2},$$

$$\mathbf{P}\left(\frac{x + S(n)}{\sqrt{n}} \in \cdot \middle| \tau_x > n\right) \to \mu \quad \textit{weakly},$$

where μ is a probability measure on K with the density $H_0 u(y) e^{-|y|^2/2}$. Furthermore, the process $\left\{\frac{x+S([nt])}{\sqrt{n}},\ t \in [0,1]\right\}$ conditioned on $\{\tau_x > n\}$ converges weakly in the space $D([0,1], \|\cdot\|_\infty)$.

Corollary 1.3. *Assume that X takes values on a lattice R which is a non-degenerate linear transformation of $\mathbb{Z}^d$. Then, under the assumptions of Theorem 1.1,*

$$\sup_{y\in D_n(x)}\left|n^{p/2+d/2}\mathbf{P}\left(x+S(n)=y,\tau_x>n\right)-C_0V(x)u\left(\frac{y}{\sqrt{n}}\right)e^{-|y|^2/2n}\right|\to 0,$$

where

$$D_n(x):=\{y\in K:\ \mathbf{P}(x+S(n)=y)>0\}.$$

The constant C_0 is a product of the volume of the unit cell in R and of a factor, which depends on the periodicity of the distribution of X.

In the proof of Theorem 5 in [9] we have required the strong aperiodicity of X. This has been done to use the simplest version of the local limit theorem for unrestricted random walks from Spitzer's book [18]. But this standard result can be replaced by Stone's local limit theorem which is valid for all lattice walks, see [19].

2 Preliminary estimates

We first collect some useful facts about the classical harmonic function $u(x)$.

Lemma 2.1. *There exists a constant $C=C(d)$ such that for $x\in K$*

$$\begin{aligned}|\nabla u(x)|&\le C\frac{u(x)}{\mathrm{dist}(x,\partial K)},\\ |u_{x_i}|&\le C\frac{u(x)}{\mathrm{dist}(x,\partial K)},\\ \left|u_{x_ix_j}\right|&\le C\frac{u(x)}{\mathrm{dist}(x,\partial K)^2},\\ \left|u_{x_ix_jx_k}\right|&\le C\frac{u(x)}{\mathrm{dist}(x,\partial K)^3}.\end{aligned}\tag{2.1}$$

Proof. Recalling that every partial derivative u_{x_i} is harmonic and using the mean value theorem for harmonic functions, we obtain

$$u_{x_i}(x)=\frac{1}{\mathrm{Vol}(B(x,r))}\int_{B(x,r)}u_{x_i}(y)dy,$$

where $B(x,r)$ is the ball of radius r around x and $r<\mathrm{dist}(x,\partial K)$. By the Gauss-Green theorem,

$$u_{x_i}(x)=\frac{1}{\mathrm{Vol}(B(x,r))}\int_{\partial B(x,r)}u(z)(\nu(z),e_i)dz,$$

where $\nu(z)$ is the outer normal at z. Choosing $r=\mathrm{dist}(x,\partial K)/2$ and applying the Harnack inequality in the ball $B(x,\mathrm{dist}(x,\partial K))$, we conclude that

$$|u_{x_i}(x)|\le 3\cdot 2^{d-2}\frac{\mathrm{Vol}(\partial B(x,r))}{\mathrm{Vol}(B(x,r))}u(x)=3d\cdot 2^{d-1}\frac{u(x)}{\mathrm{dist}(x,\partial K)}.$$

This implies the desired estimate for $u_{x_i}(x)$. Since u_{x_j} is harmonic as well we can write

$$\begin{aligned}|u_{x_ix_j}|&=\left|\frac{1}{\mathrm{Vol}(B(x,r))}\int_{\partial B(x,r)}u_{z_i}(z)(\nu(z),e_j)dz\right|\\ &\le\frac{C(d)}{\mathrm{Vol}(B(x,r))\mathrm{dist}(x,\partial K)}\int_{\partial B(x,r)}u(z)|(\nu(z),e_j)|dz\le C(d)^2\frac{u(x)}{\mathrm{dist}(x,\partial K)^2}\end{aligned}$$

The inequality for the third derivative can be proved analogously. The inequality for the gradient immediately follows from the inequality for the first derivative. □

For every cone K one has the bound

$$\mathrm{dist}(x, \partial K) \le |x|, \quad x \in K.$$

Furthermore, it follows from (1.2) that

$$u(x) \le C|x|^p, \quad x \in K.$$

In the next lemma we derive more accurate estimates for $u(x)$.

Lemma 2.2. *Assume that either the cone K is convex or Σ is C^2 and K is starlike. Then*

$$C_1 \left(\mathrm{dist}(x, \partial K)\right)^p \le u(x) \le C_2|x|^{p-1}\mathrm{dist}(x, \partial K), \quad x \in K \tag{2.2}$$

and

$$|\nabla u(x)| \le C_3|x|^{p-1}, \quad x \in K. \tag{2.3}$$

Proof. The upper bound in (2.2) is (0.2.3) in Varopoulos [20] and the lower bound has been proved in Lemma 19 in [9]. Combining the upper bound in (2.2) with Lemma 2.1, we obtain (2.3). □

We will extend the function u by putting $u(x) = 0$ for $x \notin K$.

Lemma 2.3. *Assume that either the cone K is convex or Σ is C^2 and K is starlike. Let $x \in K$. Then,*

$$|u(x+y) - u(x)| \le C|y|\left(|x|^{p-1} + |y|^{p-1}\right) \tag{2.4}$$

and, for $|y| \le |x|/2$,

$$|u(x+y) - u(x)| \le C|y||x|^{p-1}. \tag{2.5}$$

For $p < 1$ and $x \in K$,

$$|u(x+y) - u(x)| \le C|y|^p. \tag{2.6}$$

Proof. Consider first the case $p \ge 1$. To prove (2.4) consider first the case when the interval $[x, x+y]$ lies in K. Then,

$$|u(x+y) - u(x)| = \left|\int_0^1 (\nabla u(x+ty), y)dt\right| \le |y|\int_0^1 |\nabla u(x+ty)|dt.$$

Hence, by (2.3),

$$|u(x+y) - u(x)| \le C_3|y|\int_0^1 |x+ty|^{p-1}dt \le C2^{p-1}|y|\big(|x|^{p-1} + |y|^{p-1}\big),$$

as required. Now if $[x, x+y]$ does not belong to K then we have two cases: $x+y \in K$ or $x+y \notin K$. If $x+y \in K$ then there exist $t_1, t_2 : 0 < t_1 < t_2 < 1$ such that $[x, x+t_1y) \subset K$ and $(x+t_2y, x+y] \subset K$ and $x+t_1y, x+t_2y \in \partial K$. If $x+y \notin K$ then there exists $t_1 : 0 < t_1$ such that $[x, x+t_1y) \subset K$ and we put $t_2 = 1$. Since in both cases $x+t_1y, x+t_2y \notin K$ and $u = 0$ outside of K we obtain

$$\begin{aligned}
|u(x) - u(x+y)| &= |u(x) - u(x+t_1y) + u(x+t_2y) - u(x+y)| \\
&\le |u(x) - u(x+t_1y)| + |u(x+t_2y) - u(x+y)| \\
&= \left|\int_0^{t_1} (\nabla u(x+ty), y)dt\right| + \left|\int_{t_2}^1 (\nabla u(x+ty), y)dt\right| \\
\text{(by (2.3))} &\le C_4|y|\left(\int_0^{t_1} + \int_{t_2}^1\right)|x+ty|^{p-1}dt \le C|y|\big(|x|^{p-1} + |y|^{p-1}\big),
\end{aligned}$$

as required. If $p \ge 1$ then (2.5) is immediate from (2.4).

For $p < 1$ we will prove a stronger statement (2.6) which clearly implies (2.4). Consider first again the case when the interval $[x, x+y]$ lies in K. If $|x| \ge 2|y|$ then

$$\begin{aligned}|u(x+y) - u(x)| &\le |y| \int_0^1 |\nabla u(x+ty)| dt \\ &\le C|y| \int_0^1 |x+ty|^{p-1} dt \le C|y|(2|y| - |y|)^{p-1} \le C|y|^p\end{aligned}$$

and

$$\begin{aligned}|u(x+y) - u(x)| &\le |y| \int_0^1 |\nabla u(x+ty)| dt \\ &\le C|y| \int_0^1 |x+ty|^{p-1} dt \le C|y|(|x| - |x|/2)^{p-1} \le C|y||x|^{p-1}.\end{aligned}$$

Therefore, we have (2.5) and (2.6) for $|y| \le |x|/2$. Furthermore, for $|x| < 2|y|$ one has

$$|u(x+y) - u(x)| \le C(|x+y|^p + |y|^p) \le C(3^p + 1)|y|^p,$$

which completes the proof (2.6) in the case when $[x, x+y] \subset K$. The case when $[x, x+y]$ does not belong to K can be considered in the same way as for $p \ge 1$. □

For $x \in K$ let

$$f(x) = \mathbf{E}[u(x+X)] - u(x). \tag{2.7}$$

Next we require a bound on $f(x)$.

Lemma 2.4. *Let the assumptions of Theorem 1.1 hold and f be defined by* (2.7). *Then, for some $\delta > 0$,*

$$|f(x)| \le C \frac{|x|^p}{\operatorname{dist}(x, \partial K)^{2+\delta}} \quad \textit{for all } x \in K \textit{ with } |x| \ge 1.$$

Furthermore,

$$|f(x)| \le C \quad \textit{for all } x \in K \textit{ with } |x| \le 1.$$

Proof. Let $x \in K$ be such that $|x| \ge 1$. Put $g(x) = \operatorname{dist}(x, \partial K)$, and let $\eta \in (0,1)$. Then, for any $y \in B(0, \eta g(x))$, the interval $[x, x+y] \subset K$. By the Taylor theorem,

$$\left| u(x+y) - u(x) - \nabla u \cdot y - \frac{1}{2} \sum_{i,j} u_{x_i x_j} y_i y_j \right| \le R_3(x)|y|^3.$$

The remainder $R_3(x)$ can be estimated by Lemma 2.1,

$$R_3(x) = C_d \max_{z \in B(x, \eta g(x))} \max_{i,j,k} |u_{x_i x_j x_k}(z)| \le C \frac{(1+\eta)^p}{(1-\eta)^3} \frac{|x|^p}{g(x)^3},$$

which will give us

$$\left| u(x+y) - u(x) - \nabla u \cdot y - \frac{1}{2} \sum_{i,j} u_{x_i x_j} y_i y_j \right| \le C \frac{|x|^p}{g(x)^3} |y|^3. \tag{2.8}$$

Then we can proceed as follows

$$\begin{aligned}|f(x)| &= |\mathbf{E}\,(u(x+X)-u(x))\,\mathbf{1}(|X|\le \eta g(x))| \\ &\quad + |\mathbf{E}\,(u(x+X)-u(x))\,\mathbf{1}(|X|> \eta g(x))| \\ &\le \left|\mathbf{E}\left[\left(\nabla u\cdot X+\frac{1}{2}\sum_{i,j}u_{x_ix_j}X_iX_j\right)\mathbf{1}(|X|\le \eta g(x))\right]\right| \\ &\quad + C\frac{|x|^p}{g(x)^3}\mathbf{E}\left[|X|^3\mathbf{1}(|X|\le \eta g(x))\right] \\ &\quad + C\mathbf{E}\left[(|x|^p+|X|^p)\mathbf{1}(|X|>\eta g(x))\right].\end{aligned}$$

Here we used also the bounds $|u(x+y)-u(x)|\le C(|x+y|^p+|x|^p)\le C(|x|^p+|y|^p)$ valid for all x and y. After rearranging the terms we obtain

$$\begin{aligned}|f(x)| &\le \left|\mathbf{E}\left[\nabla u\cdot X+\frac{1}{2}\sum_{i,j}u_{x_ix_j}X_iX_j\right]\right| \\ &\quad + \left|\mathbf{E}\left[\left(\nabla u\cdot X+\frac{1}{2}\sum_{i,j}u_{x_ix_j}X_iX_j\right)\mathbf{1}(|X|>\eta g(x))\right]\right| \\ &\quad + C\frac{|x|^p}{g(x)^3}\mathbf{E}\left[|X|^3\mathbf{1}(|X|\le \eta g(x))\right] \\ &\quad + C\mathbf{E}\left[(|x|^p+|X|^p)\mathbf{1}(|X|>\eta g(x))\right].\end{aligned}$$

Now note that the first term is 0 due to $\mathbf{E}X_i=0$, $\mathrm{cov}(X_i,X_j)=\delta_{ij}$ and $\Delta u=0$. The partial derivatives of the function u in the second term can be estimated via Lemma 2.1, which results in the following estimate

$$\begin{aligned}|f(x)| \le C\Bigg(&\frac{|x|^p}{g(x)}\mathbf{E}\left[|X|;|X|>\eta g(x)\right]+\frac{|x|^p}{g(x)^2}\mathbf{E}\left[|X|^2;|X|>\eta g(x)\right] \\ &+\frac{|x|^p}{g(x)^3}\mathbf{E}\left[|X|^3;|X|\le\eta g(x)\right]+|x|^p\mathbf{P}(|X|>\eta g(x)) \\ &+\mathbf{E}\left[|X|^p;|X|>\eta g(x)\right]\Bigg).\end{aligned}$$

Hence, from the Markov inequality we conclude

$$\begin{aligned}|f(x)| &\le C\frac{|x|^p}{\eta^2 g^2(x)}\mathbf{E}\left[|X|^2;|X|>\eta g(x)\right]+C\frac{|x|^p}{g^3(x)}\mathbf{E}\left[|X|^3;|X|\le\eta g(x)\right] \\ &\quad + C\mathbf{E}\left[|X|^p;|X|>\eta g(x)\right].\end{aligned}\tag{2.9}$$

Now recall the moment assumption that $\mathbf{E}|X|^{2+\delta}<\infty$ for some $\delta>0$. The first term is estimated via the Chebyshev inequality,

$$\frac{|x|^p}{\eta^2 g^2(x)}\mathbf{E}\left[|X|^2;|X|>\eta g(x)\right]\le\frac{|x|^p}{\eta^{2+\delta}g^{2+\delta}(x)}\mathbf{E}|X|^{2+\delta}.$$

The second term can be estimated similarly,

$$\frac{|x|^p}{g^3(x)}\mathbf{E}\left[|X|^3;|X|\le\eta g(x)\right]\le\frac{|x|^p}{\eta^2 g^3(x)}\eta^{1-\delta}g^{1-\delta}(x)\mathbf{E}|X|^{2+\delta}.$$

In order to bound the last term in (2.9) we have to distinguish between $p\le 2$ and $p>2$.

If $p \le 2$, then, by the Chebyshev inequality,

$$\mathbf{E}\left[|X|^p; |X| > \eta g(x)\right] \le \frac{1}{(\eta g(x))^{2+\delta-p}} \mathbf{E}\left[|X|^{2+\delta}\right] \le C \frac{|x|^p}{g^{2+\delta}(x)},$$

as $g(x) = \mathrm{dist}(x, \partial K) \le |x|$.

In case $p > 2$ we have, according to our moment condition, $\mathbf{E}[|X|^p] < \infty$. Consequently,

$$\mathbf{E}\left[|X|^p; |X| > \eta g(x)\right] \le C.$$

The second statement follows easily from the fact that $u(x)$ is bounded on $|x| \le 1$ and the inequality $\mathbf{E}[u(x+X)] \le C(1 + \mathbf{E}[|X|^p])$. □

We derive next an estimate for the maximum

$$M(n) := \max_{k \le n} |S(k)|,$$

which will be used several times in the proofs of our main results.

Lemma 2.5. *If $\mathbf{E}|X|^t < \infty$ for some $t \ge 2$ then, uniformly in x, as $n \to \infty$,*

$$\mathbf{E}\left[M^t(n); \tau_x > n, M(n) > n^{1/2+\varepsilon/2}\right] = o\left(\mathbf{E}[\tau_x \wedge n] \vee 1\right).$$

Proof. For every fixed $a > 0$ one has

$$\begin{aligned} &\mathbf{P}\left(M(n) > r, \tau_x > n\right) \\ &\quad \le \mathbf{P}\left(M(n) > r, \max_{j \le n} |X(j)| \le ar\right) + \mathbf{P}\left(\max_{j \le n} |X(j)| > ar, \tau_x > n\right). \end{aligned} \tag{2.10}$$

Using first the standard union bound and then the Fuk-Nagaev-type inequality from Corollary 23 in [9], one gets

$$\mathbf{P}\left(M(n) > r, \max_{j \le n} |X(j)| \le ar\right) \le 2dn \left(\frac{\sqrt{d}e}{a}\right)^{1/a} \left(\frac{n}{r^2}\right)^{1/(a\sqrt{d})}. \tag{2.11}$$

Furthermore,

$$\begin{aligned} \mathbf{P}\left(\max_{j \le n} |X(j)| > ar, \tau_x > n\right) &\le \sum_{j=1}^n \mathbf{P}\left(|X(j)| > ar, \tau_x > n\right) \\ &\le \sum_{j=1}^n \mathbf{P}\left(|X(j)| > ar, \tau_x > j-1\right) \\ &= \mathbf{E}[\tau_x \wedge n]\mathbf{P}(|X| > ar). \end{aligned} \tag{2.12}$$

Combining (2.10)–(2.12), we conclude that

$$\mathbf{P}\left(M(n) > r, \tau_x > n\right) \le 2dn \left(\frac{\sqrt{d}e}{a}\right)^{1/a} \left(\frac{n}{r^2}\right)^{1/(a\sqrt{d})} + \mathbf{E}[\tau_x \wedge n]\mathbf{P}(|X| > ar).$$

Choosing here $a = \frac{2\varepsilon}{\sqrt{d}((1+\varepsilon)t+5)}$ and integrating the latter bound, one easily gets the bound

$$\begin{aligned} &\mathbf{E}\left[(M(n))^t; \tau_x > n, M(n) > n^{1/2+\varepsilon/2}\right] \\ &\qquad \le C(a)\left(n^{-3/2} + \mathbf{E}[\tau_x \wedge n]\mathbf{E}\left[|X|^t; |X| > an^{1/2+\varepsilon/2}\right]\right). \end{aligned} \tag{2.13}$$

Thus, the proof is complete. □

Finally, we will require the following results from [16].

Lemma 2.6. *For every $\beta < p$ we have*

$$\mathbf{E}[\tau_x^{\beta/2}] \le C(1+|x|^\beta) \tag{2.14}$$

and

$$\mathbf{E}[M^\beta(\tau_x)] \le C(1+|x|^\beta), \tag{2.15}$$

where $M(\tau_x) := \max_{k\le\tau_x} |x+S(k)|$.

This is the statement of Theorem 3.1 of [16]. One has only to notice that $e(\Gamma, R)$ in that theorem is denoted by p in our paper.

3 First proof of Theorem 1.1

Since K is starlike there exists $x_0 \in K$ with $|x_0| = 1$, $x_0 + K \subset K$ and R_0 such that $\mathrm{dist}(R_0x_0 + K, \partial K) > 1$. For $k \ge 0$ set

$$g_k = k^{1/2-\gamma} R_0 x_0,$$

where $\gamma \in (0, \min(1/2, p))$. First we will show that it is sufficient to show convergence of

$$\mathbf{E}[u(x + g_k + S(k)); \tau_x > k]$$

as k to infinity.

Lemma 3.1. *For any $x \in K$, as $k \to \infty$,*

$$\mathbf{E}[u(x + g_k + S(k)); \tau_x > k] - \mathbf{E}[u(x + S(k)); \tau_x > k] \to 0. \tag{3.1}$$

Proof. Consider first the case $p \ge 1$. Using (2.4), we obtain

$$\begin{aligned}
&\big|\mathbf{E}[u(x + g_k + S(k)); \tau_x > k] - \mathbf{E}[u(x + S(k)); \tau_x > k]\big| \\
&\quad = |\mathbf{E}[u(x + g_k + S(k)) - u(x + S(k)); \tau_x > k]| \\
&\quad \le C|g_k|\mathbf{E}[|x + S(k)|^{p-1}; \tau_x > k] + C|g_k|^p\mathbf{P}(\tau_x > k) \\
&\quad \le C|g_k|\mathbf{E}[|S(k)|^{p-1}; \tau_x > k] + C(1 + |x|^{p-1})|g_k|^p\mathbf{P}(\tau_x > k). \qquad (3.2)
\end{aligned}$$

Using the Markov inequality and (2.14) with $\beta = p - p\gamma$, we get

$$|g_k|^p\mathbf{P}(\tau_x > k) \le Ck^{p/2-p\gamma}\frac{\mathbf{E}[\tau_x^{p/2-p\gamma/2}]}{k^{p/2-p\gamma/2}} \to 0, \quad k \to \infty. \tag{3.3}$$

Furthermore,

$$\begin{aligned}
&\mathbf{E}[|S(k)|^{p-1}; \tau_x > k] \\
&\quad \le k^{(1+\varepsilon)(p-1)/2}\mathbf{P}(\tau_x > k) + \mathbf{E}[|S(k)|^{p-1}; \tau_x > k, |S(k)| > k^{(1+\varepsilon)/2}] \\
&\quad \le k^{(1+\varepsilon)(p-1)/2}\mathbf{P}(\tau_x > k) + k^{-(1+\varepsilon)/2}\mathbf{E}[|S(k)|^p; \tau_x > k, |S(k)| > k^{(1+\varepsilon)/2}]. \qquad (3.4)
\end{aligned}$$

Choosing $\varepsilon < \gamma/(p-1)$, applying the Markov inequality and using (2.14) with $\beta = p - \varepsilon(p-1)$, we conclude that

$$|g_k|k^{(1+\varepsilon)(p-1)/2}\mathbf{P}(\tau_x > k) \le |g_k|k^{(1+\varepsilon)(p-1)/2}\frac{\mathbf{E}[\tau_x^{p/2-\varepsilon(p-1)/2}]}{k^{p/2-\varepsilon(p-1)/2}} \to 0. \tag{3.5}$$

If $p > 2$ then $\mathbf{E}\tau_x$ is finite and, by Lemma 2.5,

$$|g_k|k^{-(1+\varepsilon)/2}\mathbf{E}[|S(k)|^p; \tau_x > k, |S(k)| > k^{(1+\varepsilon)/2}] \to 0. \tag{3.6}$$

If $p \le 2$ then, using (2.14) once again, we have

$$\mathbf{E}[\tau_x \wedge k] \le k^{1-p/2+\delta/2}\mathbf{E}[\tau_x^{p/2-\delta/2}] \le C(1+|x|^p)k^{1-p/2+\delta/2}.$$

Combining this estimate with Lemma 2.5, we obtain

$$\mathbf{E}\left[|S(k)|^p; \tau_x > k, |S(k)| > k^{(1+\varepsilon)/2}\right] \le k^{-(2+\delta-p)(1+\varepsilon)/2}\mathbf{E}\left[|S(k)|^{2+\delta}; \tau_x > k, |S(k)| > k^{(1+\varepsilon)/2}\right] \to 0.$$

Therefore, (3.6) remains valid for $p \le 2$. Combining (3.5) and (3.6), we conclude that

$$|g_k|\mathbf{E}[|S(k)|^{p-1}; \tau_x > k] \to 0.$$

Applying this and (3.3) to the right hand side in (3.2), we have (3.1).

We are left to consider the case $p < 1$. By (2.6), we immediately arrive at

$$|\mathbf{E}[u(x+g_k+S(k)) - u(x+S(k)); \tau_x > k]| \le C|g_k|^p\mathbf{P}(\tau_x > k) \le C|g_k|^p \frac{\mathbf{E}[\tau_x^{p/2-p\gamma/2}]}{k^{p/2-p\gamma/2}} \to 0. \qquad \square$$

Now we prove the existence of the limit of the sequence $\mathbf{E}[u(x+g_k+S(k)); \tau_x > k]$.

Proposition 3.2. *There exist a finite function $V(x)$ such that*

$$\lim_{k\to\infty} \mathbf{E}[u(x+g_k+S(k)); \tau_x > k] = V(x).$$

We shall split the proof of this proposition into several steps. To this end we shall use the following decomposition:

$$\begin{aligned}
&u(x+g_k+S(k))\mathbb{I}\{\tau_x > k\} \\
&= u(x) + \sum_{l=1}^{k} [u(x+g_l+S(l))\mathbb{I}\{\tau_x > l\} - u(x+g_{l-1}+S(l-1))\mathbb{I}\{\tau_x > l-1\}] \\
&= u(x) - \sum_{l=1}^{k} u(x+g_l+S(l))\mathbb{I}\{\tau_x = l\} \\
&\quad + \sum_{l=1}^{k} [u(x+g_l+S(l)) - u(x+g_{l-1}+S(l-1))]\,\mathbb{I}\{\tau_x > l-1\} \\
&= u(x) - u(x+g_{\tau_x}+S(\tau_x))\mathbb{I}\{\tau_x \le k\} \\
&\quad + \sum_{l=1}^{k} [u(x+g_l+S(l)) - u(x+g_{l-1}+S(l))]\,\mathbb{I}\{\tau_x > l-1\} \\
&\quad + \sum_{l=1}^{k} [u(x+g_{l-1}+S(l)) - u(x+g_{l-1}+S(l-1))]\,\mathbb{I}\{\tau_x > l-1\} \\
&=: u(x) - W_k^{(1)}(x) + W_k^{(2)}(x) + W_k^{(3)}(x).
\end{aligned} \tag{3.7}$$

The proposition will follow if we show that the expectations of all three random variables in (3.7) converge, as $k \to \infty$, to finite limits.

Lemma 3.3. *The sequence $W_k^{(1)}(x)$ converges almost surely and in L^1 towards $u(x+g_{\tau_x}+S(\tau_x))$. Furthermore,*

$$\mathbf{E}u(x+g_{\tau_x}+S(\tau_x)) \le C(1+|x|^{p-\gamma}). \tag{3.8}$$

Proof. The almost sure convergence is immediate from the fact that the sequence $W_k^{(1)}$ is increasing. Thus, it remains to show that (3.8) holds.

Since $x+S(\tau_x)\notin K$, $\mathrm{dist}(x+g_{\tau_x}+S(\tau_x),\partial K)\le |g_{\tau_x}|$ in the case when $x+g_{\tau_x}+S(\tau_x)\in K$.

Assume first that $p<1$. Combining (2.6) and (2.14), we obtain

$$\mathbf{E}[u(x+g_{\tau_x}+S(\tau_x))]\le C\mathbf{E}[|g_{\tau_x}|^p]\le C\mathbf{E}[\tau_x^{p/2-p\gamma}]\le C(1+|x|^{p-\gamma}).$$

Consider now the case $p\ge 1$. Then, using the upper bound (2.2), we obtain

$$\begin{aligned}\mathbf{E}[u(x+g_{\tau_x}+S(\tau_x))]&\le C\mathbf{E}[|g_{\tau_x}|(|x|+|g_{\tau_x}|+|S(\tau_x)|))^{p-1}]\\&\le C\left(|x|^{p-1}\mathbf{E}|g_{\tau_x}|+\mathbf{E}|g_{\tau_x}|^p+\mathbf{E}[|g_{\tau_x}|M(\tau_x)^{p-1}]\right).\end{aligned}$$

Recalling the definition of the sequence g_k, we have

$$\mathbf{E}[u(x+g_{\tau_x}+S(\tau_x))]\le C|x|^{p-1}\mathbf{E}\tau_x^{1/2-\gamma}+C\mathbf{E}\tau_x^{p/2-p\gamma}+C\mathbf{E}\left[\tau_x^{1/2-\gamma}M(\tau_x)^{p-1}\right].$$

Using (2.14), we conclude that the first two summands are bounded from above by $C(1+|x|^{p-2\gamma})$. Applying the Hölder inequality with some $p'\in(p,p+p\gamma)$ to the third summand, we get

$$\mathbf{E}\left[\tau_x^{1/2-\gamma}M(\tau_x)^{p-1}\right]\le\left(\mathbf{E}\tau_x^{p'/2-p'\gamma}\right)^{1/p'}\left(\mathbf{E}M^{(p-1)p'/(p'-1)}(\tau_x)\right)^{(p'-1)/p'}$$

By (2.15), $\mathbf{E}M^{\beta}(\tau_x)\le C_\beta(1+|x|^\beta)$, $\beta<p$. From this inequality and from (2.14), we infer that

$$\mathbf{E}\left[\tau_x^{1/2-\gamma}M(\tau_x)^{p-1}\right]\le C(1+|x|^{1-\gamma})(1+|x|^{p-1}).$$

As a result, (3.8) holds also for $p\ge 1$. □

Lemma 3.4. *There exists $W^{(2)}(x)$ such that $W_k^{(2)}(x)$ converges a.s. and in L^1 towards $W^{(2)}(x)$. Moreover,*

$$\mathbf{E}[W^{(2)}(x)]\le C(1+|x|^{p-\gamma}). \tag{3.9}$$

Proof. It is clear that all claims in the lemma will follow from

$$\sum_{l=1}^{\infty}\mathbf{E}[|u(x+g_l+S(l))-u(x+g_{l-1}+S(l))|;\tau_x>l-1]\le C(1+|x|^{p-\gamma}). \tag{3.10}$$

First we will consider the case $p\ge 1$. Note that if $x+g_{l-1}+S(l)\in K$ then, by (2.4),

$$|u(x+g_l+S(l))-u(x+g_{l-1}+S(l))|\le C|g_l-g_{l-1}|^p+C|g_l-g_{l-1}||x+g_{l-1}+S(l)|^{p-1}$$

and, similarly, if $x+g_l+S(l)\in K$ then

$$|u(x+g_l+S(l))-u(x+g_{l-1}+S(l))|\le C|g_l-g_{l-1}|^p+C|g_l-g_{l-1}||x+g_l+S(l)|^{p-1}.$$

Hence, if either $x+g_l+S(l)\in K$ or $x+g_{l-1}+S(l)\in K$ then

$$\begin{aligned}&|u(x+g_l+S(l))-u(x+g_{l-1}+S(l))|\\&\quad\le C|g_l-g_{l-1}|^p+C|g_l-g_{l-1}|(|x+g_{l-1}|^{p-1}+|S(l-1)|^{p-1}+|X(l)|^{p-1}).\end{aligned} \tag{3.11}$$

Since $u = 0$ outside of the cone the inequality (3.11) is obvious if both $x + g_l + S(l) \notin K$ and $x + g_{l-1} + S(l-1) \notin K$. Using (3.11), we have

$$\sum_{l=1}^{k} |\mathbf{E}[u(x + g_l + S(l)) - u(x + g_{l-1} + S(l)); \tau_x > l-1]|$$
$$\leq C \sum_{l=1}^{k} \mathbf{P}(\tau_x > l-1) \left(|g_l - g_{l-1}|^p + |g_l - g_{l-1}||g_l|^{p-1}\right)$$
$$+ C \sum_{l=1}^{k} |g_l - g_{l-1}| \mathbf{E}[|x|^{p-1} + |S(l-1)|^{p-1} + |X(l)|^{p-1}; \tau_x > l-1]. \tag{3.12}$$

By (2.14), noting that $|g_l - g_{l-1}| \leq Cl^{-1/2-\gamma}$ we obtain for every $p \geq 1$,

$$\sum_{l=1}^{k} \mathbf{P}(\tau_x > l-1) \left(|g_l - g_{l-1}|^p + |g_l - g_{l-1}||g_l|^{p-1}\right)$$
$$\leq C \sum_{l=1}^{k} l^{p/2-1-p\gamma} \mathbf{P}(\tau_x > l-1)$$
$$\leq C\mathbf{E}[\tau_x^{p/2-p\gamma}] \leq C(1 + |x|^{p-\gamma}). \tag{3.13}$$

Similarly,

$$(|x|^{p-1} + \mathbf{E}[|X(1)|]^{p-1}) \sum_{l=1}^{k} |g_l - g_{l-1}| \mathbf{P}(\tau_x > l)$$
$$\leq C(1 + |x|^{p-1}) \sum_{l=1}^{k} \mathbf{P}(\tau_x > l) l^{-1/2-\gamma}$$
$$\leq C(1 + |x|^{p-1}) \mathbf{E}[\tau_x^{1/2-\gamma}] \leq C(1 + |x|^{p-\gamma}). \tag{3.14}$$

It follows from (3.4) and from Lemma 2.5 that

$$\mathbf{E}\left[|S(l-1)|^{p-1}; \tau_x > l-1\right] \leq l^{(p-1)/2+\varepsilon(p-1)/2} \mathbf{P}(\tau_x > l-1) + C\mathbf{E}[\tau_x] l^{-1/2-\varepsilon/2}$$

in the case $p > 2$. Therefore, for $\varepsilon < \gamma/(p-1)$,

$$\sum_{l=1}^{k} |g_l - g_{l-1}| \mathbf{E}\left[|S(l-1)|^{p-1}; \tau_x > l-1\right]$$
$$\leq C \sum_{l=1}^{k} l^{p/2-1-\gamma/2} \mathbf{P}(\tau_x > l-1) + C\mathbf{E}[\tau_x] \sum_{l=1}^{k} l^{-1-\gamma}$$
$$\leq C\mathbf{E}[\tau_x^{p/2-\gamma/2}] + C\mathbf{E}[\tau_x].$$

Then, taking into account (2.14),

$$\sum_{l=1}^{k} |g_l - g_{l-1}| \mathbf{E}\left[|S(l-1)|^{p-1}; \tau_x > l-1\right] \leq C(1 + |x|^{p-\gamma}). \tag{3.15}$$

Similarly one shows that this relation is true in the case $p \leq 2$. (Here one has to use the assumption $\mathbf{E}|X|^{2+\delta} < \infty$ instead of $\mathbf{E}|X|^p < \infty$.) Plugging (3.13)–(3.15) into (3.12), we infer that (3.10) holds in the case $p \geq 1$.

Assume now that $p < 1$. On the event $\{\tau_x > l-1\}$ one has

$$\mathrm{dist}(x + g_{l-1} + S(l-1), \partial K) \ge (l-1)^{1/2-\gamma}.$$

This implies that

$$x + g_{l-1} + S(l) \in K \qquad |x + g_{l-1} + S(l)| \ge \frac{(l-1)^{1/2-\gamma}}{2}$$

provided that $\tau_x > l-1$ and $|X(l)| \le \frac{(l-1)^{1/2-\gamma}}{2}$. From these observations and from (2.5) we obtain

$$\begin{aligned}&\mathbf{E}[|u(x+g_l+S(l)) - u(x+g_{l-1}+S(l))|; \tau_x > l-1, |X(l)| \le (l-1)^{1/2-\gamma}/2]\\ &\qquad \le C|g_l - g_{l-1}|(l-1)^{(p-1)(1/2-\gamma)}\mathbf{P}(\tau_x > l-1).\end{aligned}$$

Furthermore, by (2.6),

$$\begin{aligned}&\mathbf{E}[|u(x+g_l+S(l)) - u(x+g_{l-1}+S(l))|; \tau_x > l-1, |X(l)| > (l-1)^{1/2-\gamma}/2]\\ &\le C|g_l - g_{l-1}|^p\mathbf{P}(\tau_x > l-1)\mathbf{P}(|X(l)| > (l-1)^{1/2-\gamma}/2).\end{aligned}$$

Recalling that $\mathbf{E}|X(1)|^{2+\delta} < \infty$ and that $|g_l - g_{l-1}| \le C(l-1)^{-1/2-\gamma}$ for $l > 2$, one gets easily

$$\begin{aligned}&\mathbf{E}[|u(x+g_l+S(l)) - u(x+g_{l-1}+S(l))|; \tau_x > l-1]\\ &\qquad \le C(l-1)^{p/2-1-p\gamma}\mathbf{P}(\tau_x > l-1),\ l \ge 2.\end{aligned}$$

Summing over l and using (2.14) we complete the proof. □

Lemma 3.5. *For every $x \in K$,*

$$\mathbf{E}\left[\sum_{l=0}^{\infty} |f(x+g_l+S(l))|\mathbb{I}\{\tau > l\}\right] \le C\left(1 + |x|^{p-\gamma} + \frac{|x|^p}{(\mathrm{dist}(x,\partial K))^\gamma}\right) \tag{3.16}$$

and

$$\mathbf{E}[W_k^{(3)}(x)] \to \mathbf{E}\left[\sum_{l=0}^{\infty} f(x+g_l+S(l))\mathbb{I}\{\tau > l\}\right]. \tag{3.17}$$

Proof. Recalling the definition of the function f, we have

$$\mathbf{E}[W_k^{(3)}(x)] = \mathbf{E}\left[\sum_{l=0}^{k-1} f(x+g_l+S(l))\mathbb{I}\{\tau > l\}\right].$$

This equality yields that (3.17) is a simple consequence of (3.16).

Applying Lemma 2.4 and using the elementary bound $\mathrm{dist}(y+g_l, \partial K) \ge l^{1/2-\gamma}$ for

$y \in K$, following from the choice of x_0 and R_0 in the definition of g_l, we have

$$\begin{aligned}
&\sum_{l=0}^{k-1} \mathbf{E}[|f(x+g_{l-1}+S(l-1))|;\tau_x > l-1] \\
&\qquad \le C\frac{|x|^p}{(\mathrm{dist}(x,\partial K))^{2+\delta}} + C\sum_{l=1}^{k-1}\mathbf{E}\left[\frac{|x+g_l+S(l)|^p}{\mathrm{dist}(x+g_l+S(l),\partial K)^{2+\delta}};\tau_x>l\right] \\
&\qquad \le C|x|^p\sum_{l=0}^{k-1}\mathbf{E}\left[(\mathrm{dist}(x+g_l+S(l)),\partial K)^{-2-\delta};\tau_x>l\right] \\
&\qquad\qquad + C\sum_{l=1}^{k-1}|g_l|^{p-2-\delta}\mathbf{P}(\tau_x>l) \\
&\qquad\qquad + C\sum_{l=1}^{k-1}|g_l|^{-2-\delta}\mathbf{E}[|S(l)|^p;\tau_x>l].
\end{aligned}$$

Choosing γ sufficiently small, we have

$$\begin{aligned}
\sum_{l=1}^{k-1}|g_l|^{p-2-\delta}\mathbf{P}(\tau_x>l) &\le C\sum_{l=1}^{k-1} l^{p/2-1-\gamma}\mathbf{P}(\tau_x>l) \\
&\le C\mathbf{E}[\tau_x^{p/2-\gamma/2}] \le C(1+|x|^{p-\gamma}). \qquad (3.18)
\end{aligned}$$

We next show that

$$\sum_{l=1}^{k-1}|g_l|^{-2-\delta}\mathbf{E}[|S(l)|^p;\tau_x>l] \le C(1+|x|^{p-\gamma}). \qquad (3.19)$$

Assume first that $p>2$. Applying Lemma 2.5, we have

$$\mathbf{E}[|S(l)|^p;\tau_x>l] \le l^{p(1+\varepsilon)/2}\mathbf{P}(\tau_x>l) + C\mathbf{E}[\tau_x].$$

Combining this with the estimate $|g_l| \ge cl^{1/2-\gamma}$, we obtain, for $\gamma < \frac{\delta}{5+2\delta}$ and $\varepsilon < \frac{\delta-5\gamma-2\gamma\delta}{p}$,

$$\begin{aligned}
&\sum_{l=1}^{k-1}|g_l|^{-2-\delta}\mathbf{E}[|S(l)|^p;\tau_x>l] \\
&\qquad \le C\sum_{l=1}^{k-1} l^{p(1+\varepsilon)/2-(2+\delta)(1/2-\gamma)}\mathbf{P}(\tau_x>l) + C\mathbf{E}[\tau_x]\sum_{l=1}^{k} l^{-(2+\delta)(1/2-\gamma)} \\
&\qquad \le C\sum_{l=1}^{k-1} l^{p/2-\gamma/2-1}\mathbf{P}(\tau_x>l) + C\mathbf{E}[\tau_x] \\
&\qquad \le C\mathbf{E}[\tau_x^{p/2-\gamma/2}].
\end{aligned}$$

Taking into account Lemma 2.6, we get (3.19) for $p>2$.

If $p \le 2$ then the moment of order $2+\delta$ is finite and, consequently,

$$\begin{aligned}
&\mathbf{E}[|S(l)|^p;\tau_x>l] \\
&\qquad \le l^{p(1+\varepsilon)/2}\mathbf{P}(\tau_x>l) + l^{-(2+\delta-p)(1+\varepsilon/2)}\mathbf{E}[|S(l)|^{2+\delta};|S(l)|>l^{(1+\varepsilon)/2},\tau_x>l] \\
&\qquad \le l^{p(1+\varepsilon)/2}\mathbf{P}(\tau_x>l) + Cl^{-(2+\delta-p)(1+\varepsilon/2)}\mathbf{E}[\tau_x\wedge l],
\end{aligned}$$

where in the last step we used Lemma 2.5 once again. Noting that

$$\mathbf{E}[\tau_x\wedge l] \le l^{1-p/2+\gamma/2}\mathbf{E}[\tau_x^{p/2-\gamma/2}]$$

we obtain

$$\begin{aligned}&\mathbf{E}[|S(l)|^p;\tau_x>l]\\&\qquad\le l^{p(1+\varepsilon)/2}\mathbf{P}(\tau_x>l)+Cl^{-(2+\delta-p)(1+\varepsilon/2)+1-p/2+\gamma/2}\mathbf{E}[\tau_x^{p/2-\gamma/2}].\end{aligned}$$

Summing over l we infer that (3.19) holds also for $p\le 2$, provided that γ and ε are sufficiently small.

Using the bound $\mathrm{dist}(y+g_l,\partial K)\ge l^{1/2-\gamma}$ and choosing γ sufficiently small, we conclude that

$$\begin{aligned}\sum_{l=g(x)}^{\infty}&\mathbf{E}\left[(\mathrm{dist}(x+g_l+S(l)),\partial K)^{-2-\delta};\tau_x>l\right]\\&\le\sum_{l=g(x)}^{\infty}l^{-(1/2-\gamma)(2+\delta)}\le Cg^{-\gamma}(x).\end{aligned}$$

Furthermore, if $|S(l)|\le g(x)/2$ then $\mathrm{dist}(x+g_l+S(l),\partial K)>g(x)/2$. Thus, by the Chebyshev inequality,

$$\mathbf{E}\left[(\mathrm{dist}(x+g_l+S(l)),\partial K)^{-2-\delta};\tau_x>l\right]\le Cg(x)^{-2-\delta}+Cl^{-(1/2-\gamma)(2+\delta)}\frac{l}{g^2(x)}.$$

Consequently,

$$\sum_{l=0}^{g(x)}\mathbf{E}\left[(\mathrm{dist}(x+g_l+S(l)),\partial K)^{-2-\delta};\tau_x>l\right]\le Cg^{-\gamma}(x).$$

As a result,

$$\sum_{l=0}^{\infty}\mathbf{E}\left[(\mathrm{dist}(x+g_l+S(l)),\partial K)^{-2-\delta};\tau_x>l\right]\le Cg^{-\gamma}(x).$$

Combining this with (3.18) and (3.19), we arrive at (3.16). □

The claim of Proposition 3.2 is immediate from Lemmas 3.3, 3.4 and 3.5. Furthermore, we have, for a sufficiently small γ, the estimate

$$|V(x)-u(x)|\le C\left(1+|x|^{p-\gamma}+\frac{|x|^p}{(\mathrm{dist}(x,\partial K))^{\gamma}}\right). \tag{3.20}$$

Lemma 3.6. *The function V possesses the following properties.*

(a) For any $\gamma>0,R>0$, uniformly in $x\in D_{R,\gamma}$ we have $V(tx)\sim u(tx)$ as $t\to\infty$.

(b) For all $x\in K$ we have $V(x)\le C(1+|x|^p)$.

(c) The function V is harmonic for the killed random walk, that is

$$V(x)=\mathbf{E}\left[V(x+S(n_0)),\tau_x>n_0\right],\quad x\in K,n_0\ge 1.$$

(d) The function V is strictly positive on K_+.

(e) If $x\in K$, then $V(x)\le V(x+x_0)$, for all x_0 such that $x_0+K\subset K$.

The proof is identical with that of Lemma 13 in [9], for the proof of (c) one has to notice that (3.20) implies that $V(x)=u(x)+O(|x|^{p-\gamma})$ for $x\in D_{R,\gamma}$.

4 Proof of Corollary 1.2

As we have mentioned in the introduction, the proofs of the claims in the corollaries are quite close to that in [9]. We demonstrate the needed changes by deriving the tail asymptotics for τ_x. Proofs of other results can similarly be adapted to the present setting.

Let

$$K_{n,\varepsilon} := \left\{ x \in K \,:\, \mathrm{dist}(x, \partial K) \geq n^{1/2-\varepsilon} \right\},$$

where $\varepsilon < \gamma/(1+p)$. Similarly to the proof of Lemma 20 in [9] one can show that

$$u(y+g_k) = (1+o(1))u(y), \quad y \in K_{n,\varepsilon}, |y| \leq \sqrt{n} \tag{4.1}$$

uniformly in $k \leq n^{1-\varepsilon}$. Indeed, by (2.5),

$$|u(y+g_k) - u(y)| \leq C|g_k||y|^{p-1} = C\frac{|g_k|}{|y|}|y|^p \leq Cn^{\varepsilon-\gamma}n^{p/2}.$$

Also, by the lower bound in (2.2)

$$|u(y)| \geq Cn^{p/2-p\varepsilon}.$$

Hence,

$$\frac{|u(y+g_k)-u(y)|}{|u(y)|} \leq Cn^{(1+p)\varepsilon-\gamma},$$

which proves (4.1), as $(1+p)\varepsilon < \gamma$.

Then, it follows from (4.1) that for any sequence $\theta_n \to 0$,

$$\begin{aligned}&\mathbf{E}\left[u(x+S(\nu_n)); \tau_x > \nu_n, \nu_n \leq n^{1-\varepsilon}, |x+S(\nu_n)| \leq \theta_n\sqrt{n}\right] \\ &\quad \sim \mathbf{E}\left[u(x+g_{\nu_n}+S(\nu_n)); \tau_x > \nu_n, \nu_n \leq n^{1-\varepsilon}, |x+S(\nu_n)| \leq \theta_n\sqrt{n}\right],\end{aligned}$$

where

$$\nu_n := \inf\{n \geq 0 : x + S(n) \in K_{n,\varepsilon}\}.$$

Applying this to (50) in [9], we obtain

$$\begin{aligned}&\mathbf{P}(\tau_x > n, \nu_n \leq n^{1-\varepsilon}) \\ &\quad = \frac{\varkappa + o(1)}{n^{p/2}}\mathbf{E}\left[u(x+g_{\nu_n}+S(\nu_n)); \tau_x > \nu_n, \nu_n \leq n^{1-\varepsilon}\right] \\ &\qquad + O\left(\frac{1}{n^{p/2}}\mathbf{E}\left[|x+S(\nu_n)|^p; \tau_x > \nu_n, \nu_n \leq n^{1-\varepsilon}, |x+S(\nu_n)| > \theta_n\sqrt{n}\right]\right).\end{aligned}$$

In view of Lemma 24 in [9], O-term is $o(n^{-p/2})$. Therefore, the proof will be completed if we show that

$$\mathbf{E}\left[u(x+g_{\nu_n}+S(\nu_n)); \tau_x > \nu_n, \nu_n \leq n^{1-\varepsilon}\right] \to V(x). \tag{4.2}$$

According to (3.7),

$$\begin{aligned}&\mathbf{E}\left[u(x+g_{\nu_n}+S(\nu_n)); \tau_x > \nu_n, \nu_n \leq n^{1-\varepsilon}\right] \\ &\qquad = \mathbf{E}[u(x) - W^{(1)}_{\nu_n}(x) + W^{(2)}_{\nu_n}(x) + W^{(3)}_{\nu_n}(x); \nu_n \leq n^{1-\varepsilon}].\end{aligned}$$

Now note that $\mathbf{P}(\nu_n \leq n^{1-\varepsilon}) \to 1$ by the functional central limit theorem. Therefore, since $\nu_n \to \infty$, as $n \to \infty$, by Lemmas 3.3 and 3.4 and the dominated convergence theorem we obtain, as $n \to \infty$,

$$\begin{aligned}&\mathbf{E}[u(x) - W^{(1)}_{\nu_n}(x) + W^{(2)}_{\nu_n}(x); \nu_n \leq n^{1-\varepsilon}] \\ &\qquad \to u(x) - \mathbf{E}u(x+g_{\tau_x}+S(\tau_x)) + \mathbf{E}[W^{(2)}(x)].\end{aligned}$$

Thus, (4.2) will follow from the convergence

$$\mathbf{E}[W^{(3)}_{\nu_n}(x);\nu_n\le n^{1-\varepsilon}]\to\mathbf{E}\left[\sum_{l=0}^{\infty}f(x+g_l+S(l))\mathbb{I}\{\tau>l\}\right]. \tag{4.3}$$

It is immediate from the definition of $f(x)$ that the sequence

$$Y_k:=W^{(3)}_k(x)-\sum_{l=0}^{k-1}f(x+g_l+S(l))\mathbb{I}\{\tau_x>l\}$$

is a martingale. Upper bound (3.16) together with the dominated convergence theorem imply that

$$\begin{aligned}\mathbf{E}&\left[\sum_{l=0}^{\nu_n-1}f(x+g_l+S(l))\mathbb{I}\{\tau_x>l\};\nu_n\le n^{1-\varepsilon}\right]\\&\qquad\to\mathbf{E}\left[\sum_{l=0}^{\infty}f(x+g_l+S(l))\mathbb{I}\{\tau_x>l\}\right].\end{aligned} \tag{4.4}$$

Furthermore, by the optional stopping theorem,

$$\begin{aligned}&\mathbf{E}[Y_{\nu_n};\nu_n\le n^{1-\varepsilon}]=\mathbf{E}[Y_{\nu_n\wedge n^{1-\varepsilon}};\nu_n\le n^{1-\varepsilon}]=\mathbf{E}[Y_{\nu_n\wedge n^{1-\varepsilon}}]-\mathbf{E}[Y_{\nu_n\wedge n^{1-\varepsilon}};\nu_n>n^{1-\varepsilon}]\\&=-\mathbf{E}[Y_{n^{1-\varepsilon}};\nu_n>n^{1-\varepsilon}]\\&=-\mathbf{E}[W^{(3)}_{n^{1-\varepsilon}}(x);\nu_n>n^{1-\varepsilon}]+\mathbf{E}\left[\sum_{l=0}^{n^{1-\varepsilon}-1}f(x+g_l+S(l))\mathbb{I}\{\tau_x>l\};\nu_n>n^{1-\varepsilon}\right].\end{aligned}$$

Using (3.16), the dominated convergence theorem and the fact that $\mathbf{P}(\nu_n>n^{1-\varepsilon})$ converges to 0 once again, we infer that

$$\mathbf{E}[Y_{\nu_n};\nu_n\le n^{1-\varepsilon}]=-\mathbf{E}[W^{(3)}_{n^{1-\varepsilon}}(x);\nu_n>n^{1-\varepsilon}]+o(1).$$

Assume first that $p<1$. Recalling the definition of $W^{(3)}_k$ and using (2.6), we have

$$|W^{(3)}_k|\le C\sum_{l=1}^{k}|X(l)|^p\mathbb{I}\{\tau_x>l-1\}.$$

Then, for every fixed $N\ge1$,

$$\begin{aligned}&\mathbf{E}[|W^{(3)}_{n^{1-\varepsilon}}(x)|;\nu_n>n^{1-\varepsilon}]\\&\quad\le C\mathbf{E}\left[\sum_{l=1}^{N}|X(l)|^p;\nu_n>n^{1-\varepsilon}\right]+C\sum_{l=N+1}^{n^{1-\varepsilon}}\mathbf{E}|X(1)|^p\mathbf{P}(\tau_x>l-1,\nu_n>l-1).\end{aligned}$$

The first summand on the right hand side converges to zero due to the fact that $\mathbf{P}(\nu_n>n^{1-\varepsilon})\to0$. Furthermore, by Lemma 14 in [9],

$$\mathbf{P}(\nu_n>n^{1-\varepsilon},\tau_x>n^{1-\varepsilon})\le\exp\{-Cn^{\varepsilon}\}. \tag{4.5}$$

Therefore, we obtain

$$\begin{aligned}\sum_{l=N+1}^{n^{1-\varepsilon}}\mathbf{P}(\tau_x>l-1,\nu_n>l-1)&\le\sum_{l=N+1}^{n^{1-\varepsilon}}\mathbf{P}(\tau_x>l-1,\nu_l>l-1)\\&\le\sum_{l=N+1}^{\infty}e^{-Cl^{\varepsilon}}.\end{aligned}$$

Letting here $N \to \infty$, we conclude that

$$\mathbf{E}[|W^{(3)}_{n^{1-\varepsilon}}(x)|; \nu_n > n^{1-\varepsilon}] \to 0.$$

It remains to prove this relation for $p \ge 1$. In this case, using (2.4), we have

$$|W_k^{(3)}| \le C \sum_{l=1}^{k} \left(|X(l)|^p + |X(l)||x + g_{l-1} + S(l-1)|^{p-1}\right) \mathbb{I}\{\tau_x > l-1\}.$$

The summands with $|X(l)|^p$ have been already considered. For the remaining summands we have for $N \ge 1$,

$$\begin{aligned}
&\mathbf{E}\left[\sum_{l=1}^{n^{1-\varepsilon}} |X(l)||x + g_{l-1} + S(l-1)|^{p-1}\mathbb{I}\{\tau_x > l-1\}; \nu_n > n^{1-\varepsilon}\right] \\
&\le \mathbf{E}\left[\sum_{l=1}^{N} |X(l)||x + g_{l-1} + S(l-1)|^{p-1}; \nu_n > n^{1-\varepsilon}\right] \\
&\qquad + \mathbf{E}|X(1)| \sum_{l=N+1}^{n^{1-\varepsilon}} \mathbf{E}\left[|x + g_{l-1} + S(l-1)|^{p-1}; \tau_x > l-1, \nu_n > l-1\right].
\end{aligned}$$

The first summand converges again to zero since $\mathbf{P}(\nu_n > n^{1-\varepsilon}) \to 0$. For the second summand applying the Cauchy-Schwarz inequality and then using (4.5), we have

$$\begin{aligned}
&\mathbf{E}\left[|x + g_{l-1} + S(l-1)|^{p-1}; \tau_x > l-1, \nu_n > l-1\right] \\
&\quad \le \left(\mathbf{E}\left[|x + g_{l-1} + S(l-1)|^{p}\right]\right)^{(p-1)/p} \mathbf{P}^{1/p}(\tau_x > l-1, \nu_n > l-1) \\
&\quad \le C l^{p/2} e^{-Cl^{\varepsilon}}.
\end{aligned}$$

Therefore, letting $N \to \infty$, we complete the proof.

5 Second proof of Theorem 1.1

For every $\varepsilon > 0$ define

$$\widetilde{K}_{n,\varepsilon} := \left\{x \in K : \operatorname{dist}(x, \partial K) \ge \frac{1}{2}\left(n^{1/2-\varepsilon} + \frac{|x|}{n^{2\varepsilon}}\right)\right\}.$$

5.1 Preliminary estimates

The next statement is the most important step in this proof of Theorem 1.1.

Proposition 5.1. *Assume that the conditions of Theorem 1.1 are valid. Then, for every sufficiently small $\varepsilon > 0$ there exists $q > 0$ such that*

$$\max_{k \le n} \left|\mathbf{E}[u(x + S(k)); \tau_x > k] - u(x)\right| \le \frac{C}{n^q} u(x), \quad x \in \widetilde{K}_{n,\varepsilon}.$$

Lemma 5.2. *For sufficiently small ε there exists $q > 0$ such that*

$$\max_{k \in [\sqrt{n}, n]} |\mathbf{E}[u(x + S(k)), \tau_x > k] - u(x)| \le \frac{C}{n^q}, \quad x \in \widetilde{K}_{n,\varepsilon}.$$

Proof. For every $x \in K$ define

$$x_k^+ = x + g_k = x + k^{1/2-\gamma} R_0 x_0.$$

Clearly,

$$\begin{aligned}&\mathbf{E}[u(x+S(k));\tau_x>k]\\&\quad=\mathbf{E}[u(x+S(k))-u(x_k^++S(k));\tau_x>k]+\mathbf{E}[u(x_k^++S(k));\tau_x>k].\end{aligned}$$

If $p\ge 1$ then, using (2.4), we get

$$\begin{aligned}&\mathbf{E}[|u(x+S(k))-u(x_k^++S(k))|;\tau_x>k]\\&\qquad\le Ck^{1/2-\gamma}\mathbf{E}\left[|x|^{p-1}+|S(k)|^{p-1}+k^{(1/2-\gamma)(p-1)}\right]\\&\qquad\le Ck^{1/2-\gamma}\left(|x|^{p-1}+k^{(p-1)/2}\right).\end{aligned}$$

In the case $p<1$ we use (2.6) to obtain

$$\mathbf{E}[|u(x+S(k))-u(x_k^++S(k))|;\tau_x>k]\le C|g_k|^p\le Ck^{p(1/2-\gamma)}.$$

Combining these two cases, we have

$$\mathbf{E}[|u(x+S(k))-u(x_k^++S(k))|;\tau_x>k]\le Ck^{1/2-\gamma}\left(|x|^{p-1}\mathbf{1}\{p\ge 1\}+k^{(p-1)/2}\right).$$

Next,

$$\begin{aligned}&\mathbf{E}[u(x_k^++S(k));\tau_x>k]\\&\quad=u(x_k^+)+\sum_{l=1}^{k}\big(\mathbf{E}[u(x_k^++S(l));\tau_x>l]-\mathbf{E}[u(x_k^++S(l-1));\tau_x>l-1]\big)\\&\quad=u(x_k^+)+\sum_{l=1}^{k}\mathbf{E}[u(x_k^++S(l))-u(x_k^++S(l-1);\tau_x>l-1]\\&\qquad\quad-\sum_{l=1}^{k}\mathbf{E}[u(x_k^++S(l));\tau_x=l]\\&\quad=u(x_k^+)+\sum_{l=1}^{k}\mathbf{E}[f(x_k^++S(l-1));\tau_x>l-1]-\mathbf{E}[u(x_k^++S(\tau_x));\tau_x\le k].\end{aligned}$$

Using (2.4) and (2.6) once again, we have

$$|u(x_k^+)-u(x)|\le Ck^{1/2-\gamma}\left(|x|^{p-1}\mathbf{1}\{p\ge 1\}+k^{(1/2-\gamma)(p-1)}\right).\qquad(5.1)$$

By Lemma 2.4,

$$\left|\sum_{l=1}^{k}\mathbf{E}[f(x_k^++S(l-1));\tau_x>l-1]\right|\le C\sum_{l=0}^{k-1}\mathbf{E}\left[\frac{|x_k^++S(l)|^p}{\mathrm{dist}(x_k^++S(l),\partial K)^{2+\delta}};\tau_x>l\right].$$

Now note that on the event $\{\tau_x>l\}$ the random variable $x+S(l)\in K$. Hence $\mathrm{dist}(x_k^++S(l),\partial K)\ge Ck^{1/2-\gamma}$. Therefore,

$$\begin{aligned}\left|\sum_{l=1}^{k}\mathbf{E}[f(x_k^++S(l-1);\tau_x>l-1]\right|&\le C\sum_{l=0}^{k-1}\mathbf{E}\left[\frac{|x_k^++S(l)|^p}{k^{(1/2-\gamma)(2+\delta)}};\tau_x>l\right]\\&\le C\frac{k|x_k^+|^p+kk^{p/2}}{k^{(1/2-\gamma)(2+\delta)}}\le C\frac{|x|^p+k^{p(1/2-\gamma)}+k^{p/2}}{k^{(1/2-\gamma)(2+\delta)-1}}\\&\le C\frac{|x|^p+k^{p/2}}{k^{(1/2-\gamma)(2+\delta)-1}}.\end{aligned}\qquad(5.2)$$

If $p \geq 1$ then, using (2.2) and the fact that $u(x) = 0$ for $x \notin K$, we obtain

$$\begin{aligned}\mathbf{E}[u(x_k^+ + S(\tau_x)); \tau_x \leq k] &\leq \mathbf{E}[|x_k^+ + S(\tau_x)|^{p-1}\mathrm{dist}(x_k^+ + S(\tau_x), \partial K), \tau_x \leq k] \\ &\leq Ck^{1/2-\gamma}\mathbf{E}[|x_k^+ + S(\tau_x)|^{p-1}; \tau_x \leq k] \\ &\leq Ck^{p(1/2-\gamma)} + Ck^{1/2-\gamma}\mathbf{E}[|x + S(\tau_x)|^{p-1}; \tau_x \leq k].\end{aligned}$$

To bound the second term we use the Burkholder inequality,

$$\begin{aligned}\mathbf{E}[|x + S(\tau_x)|^{p-1}; \tau_x \leq k] &\leq C|x|^{p-1} + C\mathbf{E}[\max_{l \leq k} |S(l)|^{p-1}] \\ &\leq C|x|^{p-1} + Ck^{(p-1)/2}.\end{aligned}$$

Then,

$$\mathbf{E}[u(x_k^+ + S(\tau_x)); \tau_x \leq k] \leq Ck^{p(1/2-\gamma)} + Ck^{p/2-\gamma} + Ck^{1/2-\gamma}|x|^{p-1}. \tag{5.3}$$

If $p < 1$ then, applying (2.6), we obtain

$$\mathbf{E}[u(x_k^+ + S(\tau_x)); \tau_x \leq k] \leq Ck^{p/2-p\gamma}.$$

In other words, (5.3) holds also for $p < 1$.

Combining now (5.1), (5.2) and (5.3), we obtain

$$\begin{aligned}&|\mathbf{E}[u(x + S(k)); \tau_x > k] - u(x)| \\ &\quad \leq C\left(k^{1/2-\gamma}|x|^{p-1} + k^{p(1/2-\gamma)} + k^{p/2-\gamma} + \frac{|x|^p + k^{p/2}}{k^{(1/2-\gamma)(2+\delta)-1}}\right).\end{aligned}$$

We can assume that $\gamma < 1/2$ is sufficiently small to ensure that $\gamma < p/2$ and

$$p/2 > (1/2 - \gamma)(2 + \delta) - 1 = \delta/2 - 2\gamma - \gamma\delta > 0.$$

Then,

$$\begin{aligned}&\max_{\sqrt{n} \leq k \leq n} |\mathbf{E}[u(x + S(k)); \tau_x > k] - u(x)| \\ &\quad \leq C\left(n^{1/2-\gamma}|x|^{p-1} + n^{p(1/2-\gamma)} + n^{p/2-\gamma} + n^{p/2-(\delta/2-2\gamma-\gamma\delta)} + \frac{|x|^p}{n^{\delta/4-\gamma-\gamma\delta/2}}\right).\end{aligned}$$

For every $x \in \widetilde{K}_{n,\varepsilon}$ one has

$$|x| \leq 2n^{2\varepsilon}\mathrm{dist}(x, \partial K) \quad \text{and} \quad \mathrm{dist}(x, \partial K) \geq \frac{1}{2}n^{1/2-\varepsilon}.$$

Combining these estimates with the lower bound in (2.2), we obtain

$$|x|^p \leq \frac{2^p n^{2p\varepsilon}}{C_1}u(x), \tag{5.4}$$

$$|x|^{p-1} \leq \frac{2^{p-1}n^{2(p-1)\varepsilon}}{C_1}\frac{u(x)}{\mathrm{dist}(x, \partial K)} \leq \frac{2^p}{C_1}n^{(2p-1)\varepsilon}\frac{u(x)}{n^{1/2}} \tag{5.5}$$

and

$$n^{p/2} \leq 2^p n^{p\varepsilon}\left(\mathrm{dist}(x, \partial K)\right)^p \leq \frac{2^p}{C_1}u(x)n^{p\varepsilon}. \tag{5.6}$$

Taking into account (5.4), (5.5) and (5.6), we arrive at the bound

$$\begin{aligned}&\max_{\sqrt{n} \leq k \leq n} |\mathbf{E}[u(x + S(k)); \tau_x > k] - u(x)| \\ &\leq Cu(x)\left(n^{(2p-1)\varepsilon-\gamma} + n^{p(\varepsilon-\gamma)} + n^{p\varepsilon-\gamma} + n^{p\varepsilon-(\delta/2-2\gamma-\gamma\delta)} + n^{2p\varepsilon-(\delta/4-\gamma-\gamma\delta/2)}\right).\end{aligned}$$

Clearly, we can pick sufficiently small $\varepsilon > 0$ in such a way that all exponents on the right hand side of the previous inequality are negative. This completes the proof of the lemma. □

Proof of Proposition 5.1. If $k \in [\sqrt{n}, n]$ then the desired estimate is immediate from Lemma 5.2. Thus, it remains to consider the case $k < \sqrt{n}$. Clearly,

$$\begin{aligned}\mathbf{E}[u(x+S(k));\tau_x>k]-u(x)\\ =\mathbf{E}[u(x+S(k))-u(x);\tau_x>k]-u(x)\mathbf{P}(\tau_x\le k)\end{aligned}\tag{5.7}$$

By the Doob inequality, for every $x \in \widetilde{K}_{n,\varepsilon}$,

$$\mathbf{P}(\tau_x\le k)\le \mathbf{P}\left(\max_{j\le k}|S(j)|^2\ge n^{1-2\varepsilon}\right)\le C\frac{k}{n^{1-2\varepsilon}}.\tag{5.8}$$

Using (2.4) and (2.6), we conclude that, for all $k \le \sqrt{n}$,

$$\begin{aligned}\mathbf{E}\left[|u(x+S(k))-u(x)|;\tau_x>n\right]&\le C\mathbf{E}\left[|S(k)||x|^{p-1}\mathbf{1}\{p\ge 1\}+|S(k)|^p\right]\\ &\le C\left(n^{1/4}|x|^{p-1}+n^{p/4}\right).\end{aligned}$$

Taking into account (5.5) and (5.6), we obtain

$$\max_{k\le\sqrt{n}}\mathbf{E}\left[|u(x+S(k))-u(x)|;\tau_x>n\right]\le\frac{C}{n^q}u(x),\quad x\in\widetilde{K}_{n,\varepsilon}.\tag{5.9}$$

Combining (5.7)–(5.9) completes the proof of the proposition. □

Define

$$\nu_n:=\inf\left\{n\ge 0: x+S(n)\in K_{n,\varepsilon}\right\}$$

and

$$\widetilde{\nu}_n:=\inf\left\{n\ge 0: x+S(n)\in \widetilde{K}_{n,\varepsilon}\right\}.$$

Lemma 5.3. *There exists $\gamma > 0$ such that, for every $x \in K$,*

$$\max_{k\in[n^{1-\varepsilon},n]}\mathbf{E}\left[u(x+S(k));\tau_x>k,\widetilde{\nu}_n>[n^{1-\varepsilon}]\right]\le\frac{C(1+|x|^{p-\gamma})}{n^q}.$$

Proof. Set

$$M(k):=\max_{j\le k}|x+S(j)|$$

and split the expectation into two parts:

$$\begin{aligned}&\mathbf{E}\left[u(x+S(k));\tau_x>k,\widetilde{\nu}_n>[n^{1-\varepsilon}]\right]\\ &\quad=\mathbf{E}\left[u(x+S(k));\tau_x>k,\widetilde{\nu}_n>[n^{1-\varepsilon}],M([n^{1-\varepsilon}])\le n^{1/2+\varepsilon/2}\right]\\ &\qquad+\mathbf{E}\left[u(x+S(k));\tau_x>k,\widetilde{\nu}_n>[n^{1-\varepsilon}],M([n^{1-\varepsilon}])>n^{1/2+\varepsilon/2}\right].\end{aligned}\tag{5.10}$$

Set, for brevity, $m = [n^{1-\varepsilon}]$. Now, using the relation,

$$\begin{aligned}&\left\{\widetilde{\nu}_n>m,M(m)\le n^{1/2+\varepsilon/2}\right\}\\ &\subset\left\{\mathrm{dist}(x+S(j),\partial K)\le\frac{1}{2}\left(n^{1/2-\varepsilon}+\frac{|x+S(j)|}{n^{2\varepsilon}}\right),|x+S(j)|\le n^{1/2+\varepsilon/2},j\le m\right\}\\ &\subset\left\{\mathrm{dist}(x+S(j),\partial K)\le n^{1/2-\varepsilon},j\le m\right\}=\{\nu_n>m\},\end{aligned}$$

we obtain

$$\begin{aligned}&\mathbf{E}\left[u(x+S(k));\tau_x>k,\widetilde{\nu}_n>n^{1-\varepsilon},M(m)\le n^{1/2+\varepsilon/2}\right]\\ &\quad\le\mathbf{E}\left[u(x+S(k));\tau_x>k,\nu_n>m,M(m)\le n^{1/2+\varepsilon/2}\right].\end{aligned}$$

Now, if $p \ge 1$ then by (2.4),

$$u(x+S(k)) \le u(x+S(m)) + C|S(k)-S(m)|^p + C|S(k)-S(m)||x+S(m)|^{p-1}.$$

If $p<1$ we make use of (2.6) to obtain

$$u(x+S(k)) \le u(x+S(m)) + C|S(k)-S(m)|^p$$

As a result,

$$\begin{aligned} u(x+S(k)) \le u(x+S(m)) &+ C|S(k)-S(m)|^p \\ &+ C|S(k)-S(m)||x+S(m)|^{p-1}\mathbf{1}\{p\ge 1\}. \end{aligned} \tag{5.11}$$

Hence,

$$\begin{aligned} &\mathbf{E}\left[u(x+S(k)); \tau_x>k, \nu_n>m, M(m)\le n^{1/2+\varepsilon/2}\right] \\ &\le \mathbf{E}\left[u(x+S(m)); \tau_x>k, \nu_n>m, M(m)\le n^{1/2+\varepsilon/2}\right] \\ &+ C\mathbf{E}\left[|S(k)-S(m)|^p; \tau_x>k, \nu_n>m, M(m)\le n^{1/2+\varepsilon/2}\right] \\ &+ C\mathbf{1}\{p\ge1\}\mathbf{E}\left[|S(k)-S(m)||x+S(m)|^{p-1}; \tau_x>k, \nu_n>m, M(m)\le n^{1/2+\varepsilon/2}\right]. \end{aligned}$$

First, since $u(y)\le C|y|^p$, by (4.5), we conclude that

$$\begin{aligned} \max_{k\in[n^{1-\varepsilon},n]} &\mathbf{E}\left[u(x+S(m)); \tau_x>k, \nu_n>m, M(m)\le n^{1/2+\varepsilon/2}\right] \\ &\le Cn^{p/2+\varepsilon/2}\mathbf{P}\left(\tau_x>m, \nu_n>m\right) \le Cn^{p/2+\varepsilon/2}e^{-cn^{\varepsilon}} \end{aligned} \tag{5.12}$$

and

$$\begin{aligned} \max_{k\in[n^{1-\varepsilon},n]} &\mathbf{E}\left[|S(k)-S(m)|^p; \tau_x>k, \nu_n>m, M(m)\le n^{1/2+\varepsilon/2}\right] \\ &\le \max_{k\in[n^{1-\varepsilon},n]} \mathbf{E}\left[|S(k)-S(m)|^p\right]\mathbf{P}\left(\tau_x>m, \nu_n>m\right) \\ &\le Cn^{p/2}e^{-cn^{\varepsilon}}. \end{aligned} \tag{5.13}$$

Second, for $p\ge 1$ one has by the same argument,

$$\begin{aligned} \max_{k\in[n^{1-\varepsilon},n]} &\mathbf{E}\left[|S(k)-S(m)||x+S(m)|^{p-1}; \tau_x>k, \nu_n>m, M(m)\le n^{1/2+\varepsilon/2}\right] \\ &\le Cn^{1/2}n^{(p-1)/2+\varepsilon/2}e^{-cn^{\varepsilon}}. \end{aligned} \tag{5.14}$$

Therefore, combining (5.12), (5.13) and (5.14) we obtain, that the first expectation on the right hand side of (5.10) can be estimated as follows,

$$\begin{aligned} \max_{k\in[n^{1-\varepsilon},n]} &\mathbf{E}\left[u(x+S(k)); \tau_x>k, \widetilde{\nu}_n>[n^{1-\varepsilon}], M([n^{1-\varepsilon}])\le n^{1/2+\varepsilon/2}\right] \\ &\le Cn^{p/2+\varepsilon/2}e^{-cn^{\varepsilon}}. \end{aligned} \tag{5.15}$$

Using (5.11), we have for the second expectation on the right hand side of (5.10),

$$\begin{aligned} &\mathbf{E}\left[u(x+S(k)); \tau_x>k, \widetilde{\nu}_n>m, M(m)>n^{1/2+\varepsilon/2}\right] \\ &\quad\le \mathbf{E}\left[u(x+S(m)); \tau_x>m, \widetilde{\nu}_n>m, M(m)>n^{1/2+\varepsilon/2}\right] \\ &\qquad+ C\mathbf{E}\left[|S(k)-S(m)|^p; \tau_x>m, M(m)>n^{1/2+\varepsilon/2}\right] \\ &\qquad+ C\mathbf{1}\{p\ge1\}\mathbf{E}\left[|S(k)-S(m)||x+S(m)|^{p-1}; \tau_x>m, M(m)>n^{1/2+\varepsilon/2}\right] \\ &\quad:= E_1+E_2+E_3. \end{aligned}$$

To estimate E_1 we apply the upper bound from (2.2), and use the fact that on the event $\{\widetilde{\nu}_n > m\}$,

$$\mathrm{dist}(x+S(m),\partial K) \le \frac{1}{2}\left(n^{1/2-\varepsilon} + \frac{|x+S(m)|}{n^{2\varepsilon}}\right).$$

Then,

$$E_1 \le \frac{1}{2} n^{1/2-\varepsilon}\mathbf{E}\left[(x+S(m))^{p-1};\tau_x > m, M(m) > n^{1/2+\varepsilon/2}\right] + \frac{1}{2n^{2\varepsilon}}\mathbf{E}\left[(x+S(m))^{p};\tau_x > m, M(m) > n^{1/2+\varepsilon/2}\right].$$

Using independence of increments we obtain

$$E_2 \le Cn^{p/2}\mathbf{P}\left(\tau_x > m, M(m) > n^{1/2+\varepsilon/2}\right)$$

and

$$E_3 \le Cn^{1/2}\mathbf{E}\left[(x+S(m))^{p-1};\tau_x > m, M(m) > n^{1/2+\varepsilon/2}\right].$$

Combining these estimates and using the Markov inequality, we obtain

$$E_1+E_2+E_3 \le \frac{C}{n^{\min(p\varepsilon/2,\varepsilon/2)}}\mathbf{E}\left[(M(m))^{p};\tau_x > m, M(m) > n^{1/2+\varepsilon/2}\right].$$

Now note that by Lemma 2.5,

$$\mathbf{E}\left[(M(m))^{p};\tau_x > m, M(m) > n^{1/2+\varepsilon/2}\right] \le C\mathbf{E}[\tau_x \wedge n]. \tag{5.16}$$

Note that for $p > 2$ the desired statement immediately follows from (2.14). If $p \le 2$ then, using (2.14),

$$\mathbf{E}[\tau_x \wedge n] \le n^{1-p/2+\delta/2}\mathbf{E}[\tau_x^{p/2-\delta/2}] \le C(1+|x|^{p-\delta})n^{1-p/2+\delta/2}.$$

By the assumption $\mathbf{E}|X|^{2+\delta} < \infty$,

$$\mathbf{E}\left[|X|^p;|X| > an^{1/2+\varepsilon/2}\right] \le Cn^{-(1/2+\varepsilon/2)(2+\delta-p)}.$$

Then using directly the last inequality in the proof of Lemma 2.5 we can see that (5.16) remains valid for $p \le 2$. The proof is complete. □

Remark 5.4. The only place we need to use the results of [16] is the end of the last Lemma. To make the proof self-contained we can use a different estimate in (5.16). Namely, we can directly use the estimate (2.13) with $t = p$ and then apply estimates $\mathbf{E}[\tau_x \wedge n] \le n$ and further assuming that $\mathbf{E}|X|^{p+2} < \infty$ the Markov inequality to probability. This would give the desired estimate in (5.16). Thus we can avoid using the results of [16] by imposing 2 additional moments.

5.2 Proof of Theorem 1.1

Fix a large integer $n_0 > 0$ and put, for $m \ge 1$,

$$n_m = [n_0^{((1-\varepsilon)^{-m})}],$$

where $[r]$ denotes the integer part of r. Let n be any integer. There exists unique m such that $n \in (n_m, n_{m+1}]$. We first split the expectation into two parts,

$$\begin{aligned}\mathbf{E}[u(x+S(n));\tau_x>n] &= E_1(x)+E_2(x)\\ &:= \mathbf{E}\left[u(x+S(n));\tau_x>n,\widetilde{\nu}_n\le n_m\right]+\mathbf{E}\left[u(x+S(n));\tau_x>n,\widetilde{\nu}_n>n_m\right].\end{aligned}$$

By Lemma 5.3, since $n_m \ge n^{1-\varepsilon}$, the second term on the right hand side is bounded by

$$E_2(x)\le \frac{C(x)}{n_m^q},$$

where

$$C(x)=C(1+|x|^{p-\gamma}).$$

For the first term we have

$$E_1(x)=\sum_{i=1}^{n_m}\int_{\widetilde{K}_{n,\varepsilon}}\mathbf{P}\{\widetilde{\nu}_n=i,\tau_x>i,x+S(i)\in dy\}\mathbf{E}[u(y+S(n-i));\tau_y>n-i].$$

Then, by Proposition 5.1,

$$\begin{aligned}E_1(x) &\le \left(1+\frac{C}{n^q}\right)\sum_{i=1}^{n_m}\int_{\widetilde{K}_{n,\varepsilon}}\mathbf{P}\{\widetilde{\nu}_n=i,\tau_x>i,x+S(i)\in dy\}u(y)\\ &\le \frac{\left(1+\frac{C}{n^q}\right)}{\left(1-\frac{C}{n_m^q}\right)}\sum_{i=1}^{n_m}\int_{\widetilde{K}_{n,\varepsilon}}\mathbf{P}\{\widetilde{\nu}_n=i,\tau_x>i,x+S(i)\in dy\}\\ &\qquad\qquad\times\mathbf{E}[u(y+S(n_m-i));\tau_y>n_m-i]\\ &=\frac{\left(1+\frac{C}{n_m^q}\right)}{\left(1-\frac{C}{n_m^q}\right)}\mathbf{E}[u(x+S(n_m));\tau_x>n_m,\widetilde{\nu}_n\le n_m].\end{aligned}$$

As a result we have

$$\mathbf{E}[u(x+S(n));\tau_x>n]\le\frac{\left(1+\frac{C}{n_m^q}\right)}{\left(1-\frac{C}{n_m^q}\right)}\mathbf{E}[u(x+S(n_m));\tau_x>n_m]+\frac{C(x)}{n_m^q}. \tag{5.17}$$

Iterating this procedure m times, we obtain

$$\begin{aligned}&\max_{n\in(n_m,n_{m+1}]}\mathbf{E}[u(x+S(n));\tau_x>n]\\ &\qquad\le\prod_{j=0}^{m}\frac{\left(1+\frac{C}{n_j^q}\right)}{\left(1-\frac{C}{n_j^q}\right)}\left(\mathbf{E}[u(x+S(n_0));\tau_x>n_0]+C(x)\sum_{j=0}^{m}n_j^{-q}\right).\end{aligned} \tag{5.18}$$

Since n_m grows exponentially fast, we infer that

$$\sup_n\mathbf{E}[u(x+S(n));\tau_x>n]\le C(x)<\infty. \tag{5.19}$$

An identical procedure gives a lower bound

$$\begin{aligned}
&\mathbf{E}[u(x+S(n));\tau_x>n] \ge E_1(x)\\
&\ge \frac{\left(1-\frac{C}{n_m^q}\right)}{\left(1+\frac{C}{n_m^q}\right)}\mathbf{E}[u(x+S(n_m));\tau_x>n_m,\widetilde{\nu}_n\le n_m]\\
&\ge \frac{\left(1-\frac{C}{n_m^q}\right)}{\left(1+\frac{C}{n_m^q}\right)}\Big(\mathbf{E}[u(x+S(n_m));\tau_x>n_m]-\mathbf{E}[u(x+S(n_m));\tau_x>n_m,\widetilde{\nu}_n>n_m]\Big)\\
&\ge \frac{\left(1-\frac{C}{n_m^q}\right)}{\left(1+\frac{C}{n_m^q}\right)}\Big(\mathbf{E}[u(x+S(n_m));\tau_x>n_m]-C(x)n_m^{-q}\Big)\\
&\ge \prod_{j=0}^m \frac{\left(1-\frac{C}{n_j^q}\right)}{\left(1+\frac{C}{n_j^q}\right)}\mathbf{E}[u(x+S(n_0));\tau_x>n_0]-C(x)\sum_{j=0}^m n_j^{-q}. \qquad (5.20)
\end{aligned}$$

For every positive δ we can choose $n_0=n_0(\delta)$ such that for all $m\ge 1$,

$$\left|\prod_{j=0}^m \frac{\left(1-\frac{C}{n_j^q}\right)}{\left(1+\frac{C}{n_m^q}\right)}-1\right|\le\delta \quad\text{and}\quad \sum_{j=0}^m n_j^{-q}\le\delta.$$

Then, for this value of n_0 and all $x\in K$,

$$\sup_{n>n_0}\mathbf{E}[u(x+S(n));\tau_x>n]\le(1+\delta)\mathbf{E}[u(x+S(n_0));\tau_x>n_0]+C(x)\delta$$

and

$$\inf_{n>n_0}\mathbf{E}[u(x+S(n));\tau_x>n]\ge(1-\delta)\mathbf{E}[u(x+S(n));\tau_x>n_0]-C(x)\delta. \qquad (5.21)$$

Consequently,

$$\begin{aligned}
\sup_{n>n_0}\mathbf{E}[u(x+S(n));\tau_x>n]&-\inf_{n>n_0}\mathbf{E}[u(x+S(n));\tau_x>n]\\
&\le 2\delta\mathbf{E}[u(x+S(n_0));\tau_x>n_0]+2C(x)\delta.
\end{aligned}$$

Taking into account (5.19) and that δ can be made arbitrarily small we conclude that the limit

$$V(x):=\lim_{n\to\infty}\mathbf{E}[u(x+S(n));\tau_x>n]$$

exists for every $x\in K$.

For positivity of V note that by (2.4),

$$\mathbf{E}[u(tx+S_{n_0});\tau_{tx}>n_0]\ge u(tx)\mathbf{P}(\tau_{tx}>n_0)-\mathbf{E}[|S_{n_0}|^p]-(t|x|)^{p-1}\mathbf{E}[|S_{n_0}|]$$

when $p\ge 1$ and by (2.6),

$$\mathbf{E}[u(tx+S_{n_0});\tau_{tx}>n_0]\ge u(tx)\mathbf{P}(\tau_{tx}>n_0)-\mathbf{E}[|S_{n_0}|^p]$$

when $p<1$. Also, $C(tx)\le t^{p-\gamma}|x|^{p-\gamma}$. Hence, it follows from (5.21) that there exists R such that $V(x)$ is positive for $x\in D_{R,\gamma}$. The rest of the proof follows the corresponding part of Lemma 13 of [9]. The proof is complete.

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