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Abstract. We investigate possible nontrivial phases of a two-subband quantum wire. It is found that inter- and intra-subband interactions may drive the electron system of the wire into a gapped state. If the nominal electron densities in the two subbands are sufficiently close to each other, then the leading instability is the inter-subband charge-density wave (CDW). For larger difference in the densities, the interaction in the inter-subband Cooper channel may lead to a superconducting instability. The total charge density mode, responsible for the conductance of an ideal wire, always remains gapless, which enforces the two-terminal conductance to be at the universal value of $2e^2/h$ per occupied subband. On the contrary, the tunneling density of states (DOS) in the bulk of the wire acquires a hard gap, above which the DOS has a non-universal singularity. This singularity is weaker than the square-root divergency characteristic for non-interacting quasiparticles near a gap edge due to the "dressing" of massive modes by a gapless total charge density mode. The DOS for tunneling into the end of a wire in a CDW-gapped state, however, preserves the power-law behavior due to the frustration the edge introduces into the CDW order. This work is related to the vast literature on coupled 1D systems, and most of all, on two-leg Hubbard ladders. Whenever possible, we give derivations of the important results by other authors, adopted for the context of our study.

1 Introduction

From a theorist's point of view, electrons in quantum wires should provide a simplest realization of a Luttinger liquid [1,2]. Indeed, as the motion is confined in the direction transverse to the axis of a wire, the system is effectively one-dimensional; the electron-electron interaction is strong enough (typically, of the order of the Fermi energy) for the interaction effects not be washed by the temperature; and the state-of-the-art wires (at least the semiconductor version of them) are clean enough for disorder effects to be sufficiently weak. A clear experimental proof of the existence of the Luttinger-liquid state in quantum wires would be provided by tunneling into a wire, either in the middle or into the end. As is well-known, the tunneling density of states (DOS) of a Luttinger

liquid reveals a pseudogap behavior, i.e, it is suppressed at energies close to the Fermi-energy, which should result in a sublinear (power-law) bias dependence of the tunneling current, and in a power-law temperature dependence of the Ohmic conductance. This finite-bias and finite-temperature measurements has already been performed on a very special realization of quantum wires – carbon nanotubes, and observed non-linear current-voltage dependences were interpreted in terms of the Luttinger-liquid theory [3,4]. Features of resonant tunneling, characteristic for a Luttinger liquid, have recently been observed on GaAs quantum wires [5] prepared by cleaved edge overgrowth technique. Also, a Luttinger-liquid behavior has been reported in tunneling into InSb wires naturally grown in a porous material (asbestos) [6]. Tunneling pseudogap of a quantum wire has been described in terms of a Luttinger-liquid model both for a single- [7] and multi-subband wire [8], the latter system exhibiting a smooth healing of the pseudogap as the number of the occupied channels increases. In anticipation of more and better controlled tunneling experiments on quantum wires, and also from a general point of view, we would like to ask if there are any processes which could open a true gap, but not a pseudogap, in the electron spectrum of a wire, and if yes, what are the properties of the corresponding gapped phases.

To this end, we consider in this paper a two-subband quantum wire, having in mind semiconductor nanostructures studied recently in, e.g., Refs. [9–11]. To some approximation, this system is similar to other two important classes of 1D two-band systems studied extensively over the last few years, i.e., two-leg Hubbard ladders [12–18] and (single-wall) carbon nanotubes [19–23] (for an account of earlier results on coupled 1D systems, see, e.g., [24]). Studies of two-leg Hubbard ladders provided the classification of scattering processes in 1D two-subband systems and identification of processes capable of driving the system into a gapped state. Phase diagrams of a generic ladder, containing a multitude of gapped states, were constructed in Refs. [14–16]. A similarly formulated problem, with applications to a 1D system with electron and hole bands (valence-fluctuation problem), was investigated some time ago in [25].

The goal of the present paper is two-fold. First of all, we would like to understand which of the gapped phases, found in Hubbard ladders, have a chance to occur in quantum wires. The main difference between these two systems is that the Coulomb interaction in wires is (i) (supposed to be) purely repulsive; (ii) relatively long-ranged (even in the presence of a metallic gate); (iii) relatively well-known at distances larger than the lattice spacing (which is the range relevant for quantum wires); this imposes constraints on the choice of coupling constants for the Hubbard model. Also, because the electron wavelength in wires is larger than the lattice spacing, Umklapp scattering is unimportant. All these constraints reduce the variety of possible gapped states to (a) inter-subband charge-density wave (CDW), and (b) superconducting state. [We will come back to a more detailed description of these states shortly.] The main difference between carbon nanotubes and quantum wires is that the former, because of its special crystal structure, has two conducting subbands with *commensurate* Fermi-momenta. Although, as we will show, a quantum wire with nominally dif-

ferent subband Fermi-momenta may be driven into the commensurate state by inter-subband backscattering, this process occurs in a competition with other processes and requires special analysis.

Having determined which gapped phase can in principle occur in a quantum wire, we focus on the calculation of measurable quantities in each of these phases, which is the second goal of the present paper. Our main emphasis is on the tunneling density of states, which turns out to exhibit an unusual threshold behavior due to coexistence of gapped and gapless modes and also be sensitive to the presence of open boundaries. In addition, we consider the two-terminal conductance both in the absence and in the presence of impurities.

Although this paper is not supposed to be a review, we present, when possible, derivations of important results by other authors, e.g., Refs. [22,26–28] adopted to the context of our study. Hopefully, this would help a reader, who is not an expert in this field, to understand connections between different approaches.

Having formulated the goals and scope of this paper, we now return to a generic two-subband quantum wire with *incommensurate* Fermi-momenta k_{1F} and k_{2F} in subbands 1 and 2, respectively. In the basis of occupied transverse states, this becomes the problem of two Luttinger liquids coupled by inter-subband interactions. To understand possible phases of such a system, one should consider all possible scattering processes involving electrons from different subbands, i.e., forward scattering, backscattering, and "Cooper scattering" (cf. Figures 1,2,3). Forward scattering simply renormalizes the parameters of Luttinger liquids formed by electrons of each subbands but does not result in new phases, although it does change the conditions for occurrence of new phases. (To be more precise, forward scattering between Luttinger liquids in different subbands is responsible for the crossover into the Fermi-liquid state, but this crossover occurs smoothly as the number of channels increases).

The momentum transfer in a backscattering event involving electrons of different subbands is equal to $k_{1F} \pm k_{2F}$. If $|k_{1F} - k_{2F}| \gg T/\min\{v_{1F}, v_{2F}\}$, then there are no final states available for electrons involved in such a process (here v_{1F} and v_{2F} are the Fermi velocities in the two subbands). Thus, if the temperature T is low enough, interchannel backscattering is forbidden. However, it may become energetically favorable for a system to equalize the charge densities, and hence the Fermi momenta of different subbands. This may occur if the Fermi-momenta difference is small enough and the amplitude of backscattering is large enough. After the densities are adjusted, backscattering becomes possible. As a result, inter-subband charge-density wave (CDW) phase may be formed, in which charge densities of the subbands form a staggered pattern, see Fig. 4. Similarly to classical charge-density waves, this phase is very sensitive to a random potential, resulting in pinning of the CDW and strong suppression of conductance with disorder.

The "Cooper" scattering event, on the other hand, always conserves momentum and energy. In this process, two electrons starting in, *e.g.*, subband 1 with momenta k_{1F} and $-k_{1F}$, scatter on each other and end up in the other sub-

band, also with opposite momenta k_{2F} and $-k_{2F}$. "Cooper scattering" can be considered as formation of a fluctuational Cooper pair in one of the subband followed by its tunneling into the other one. When kinetic energy gain due to such tunneling overcomes Coulomb repulsion between electrons forming pairs, the wire is in a Cooper (or superconducting) phase. This phase is characterized by locking of fluctuating charge currents in different subbands to each other as well as by spin gaps in each of the subbands. The Cooper phase is favored when Fermi-momenta imbalance is largest, i.e. when the second subband just starts to fill up. Disorder has a less pronounced effect on Cooper phase than on CDW one, similarly to what happens in higher dimensions.

It is important to emphasize here that inter-subband backscattering and Cooper scattering block only modes describing relative charge- and spin- density and current excitations, but leave the center-of-mass charge mode free. As a result, conductance of a clean CDW and Cooper phases remains at the universal value of $2e^2/h$ per occupied subband.

Despite being ideal conductors, both the CDW and Cooper phases are characterized by the truly gapped behavior of the tunneling density of states at energies below corresponding gaps. This is so because a 1D electron decomposes into charge and spin collective density excitations, and if some of these elementary collective excitations acquire a gap, the entire electron acquires it as well.

Somewhat surprisingly, we find that tunneling into the end of the CDW wire is quite different. A tunnel barrier at the end of the wire distorts charge-density wave profile and creates a static semi-soliton. This allows tunneling into the end to occur even at energies below the bulk CDW gap.
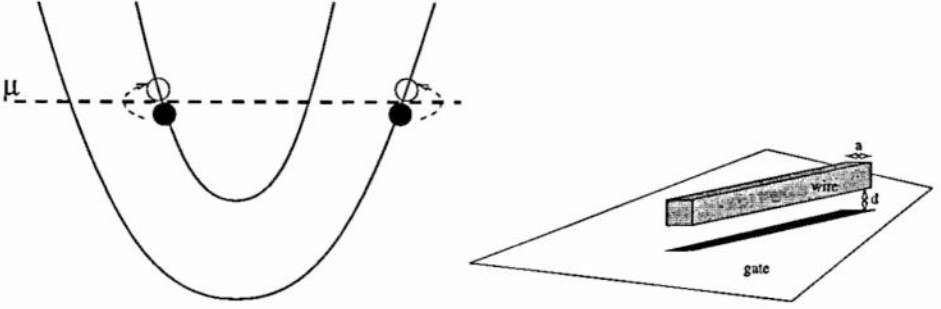
In Section 6.2 we consider, for illustrative purposes, the density of states of the "Mott" phase, which occurs in a single- or multi-subband wire subject to an external periodic potential [29,30]. In the case of a semiconductor wire, this potential may be provided by an additional electrostatic gate of a periodic shape [31]. Varying the potential applied to this gate, one can tune electrons of the wire into the half-filling condition (one electron per unit cell of the periodic potential). Unlike the two strong-coupling phases mentioned above, the Mott phase, which is described by the half-filled Hubbard model, does not conduct current because its total charge fluctuations are gapped by the external potential.

2 Hamiltonian of a Two-Subband Quantum Wire

2.1 Classification of Scattering Processes

Electrons in a quantum wire are described by the following Hamiltonian

$$H = \sum_s \int d^3 r \Psi_s^\dagger(\mathbf{r}) \left(-\frac{1}{2m}\nabla_r{}^2 - \mu + V_{conf}(\mathbf{r}_\perp) \right) \Psi_s(\mathbf{r})$$
$$+\frac{1}{2}\sum_{s,s'} \int d^3 r d^3 r' U(\mathbf{r} - \mathbf{r}') \Psi_s^\dagger(\mathbf{r}) \Psi_{s'}^\dagger(\mathbf{r}') \Psi_{s'}(\mathbf{r}') \Psi_s(\mathbf{r}), \qquad (1)$$

Fig. 1. Left: Example of inter-subband forward scattering. Filled (empty) circles denote initial (final) states of electrons. Dashed lines with arrows indicate "direction" of the scattering. Right: Schematic view of a gated wire. The wire of typical transverse size a is separated by distance d from the metallic gate. Distance d determines the range of interaction among electrons inside the wire.

where $V_{conf}(r_\perp)$ is the confining potential in the transverse direction and $U(r)$ is the electron-electron interaction potential. The Fermi-wavelength of electrons in a semiconducting quantum wire is much larger than the lattice spacing of the underlying crystal structure. Because of that, we do not consider umklapp processes, in which electron momentum is transferred to the lattice (except for in Ch.6.2). Hamiltonian (1) is Galilean-invariant, and hence our subsequent calculations have to preserve this invariance as well. We will return to this important point later on in our discussion.

If the chemical potential in the leads is such that only two lowest subbands of transverse quantization are occupied, the electron wavefunction is given by

$$\Psi_s(r) = \sum_{n=1}^{2} \phi_n(r_\perp)\psi_{ns}(x), \tag{2}$$

where $\phi_n(r_\perp)$ are the orthogonal wavefunctions of transverse quantization, which we choose to be real. In this basis, the kinetic part of Hamiltonian (1) becomes

$$H_0 = \sum_{n,s} \int dx \psi_{ns}^\dagger(x)(-\frac{\partial_x^2}{2m} - \mu + \epsilon_{ns})\psi_n(x) \tag{3}$$

where ϵ_n is the energy of the n-th transverse subband.

To describe low-energy excitations in the n-th subband, we expand the longitudinal part of the Ψ-operator, $\psi_s(x)$, in terms of right- and left-moving excitations, residing around $\pm k_{nF}$ Fermi-points of the n-th channel:

$$\psi_{ns}(x) = R_{ns}(x)e^{ik_{nF}x} + L_{ns}(x)e^{-ik_{nF}x}. \tag{4}$$

In this representation, the interaction (four-fermion) part of Hamiltonian (1) reduces to a sum of two terms. The first one, U_{intra}, describes the interaction of electrons within the same subband, and contains usual forward and backward

scattering processes. The second one, U_{inter}, describes the inter-subband inter-action. It splits naturally into *forward* (U^F), *backward* (U^B), and *Cooper* (U^C) parts.

$$U_{inter} = U^F + U^B + U^C. \tag{5}$$

Forward scattering involves no momentum transfer between subbands (cf. Fig.1). This process is also an example of a *direct* process, in a sense that electrons stay in the same subband, as is evident from the explicit expression for U^F

$$U^F = \frac{1}{2} \sum_{n \neq m} \int_{x,x'} M_d^{\{nm\}}(x - x') \sum_{s,s'} [R_{ns}^\dagger(x)R_{ns}(x) + L_{ns}^\dagger(x)L_{ns}(x)]$$
$$\times [R_{ms'}^\dagger(x')R_{ms'}(x') + L_{ms'}^\dagger(x')L_{ms'}(x')], \tag{6}$$

where the *direct* matrix element is given by

$$M_d^{\{nm\}}(x - x') = \int_{\mathbf{r}_\perp, \mathbf{r}_\perp'} U(\mathbf{r} - \mathbf{r}')\phi_n^2(\mathbf{r}_\perp)\phi_m^2(\mathbf{r}_\perp'); \tag{7}$$

and $\int_{z,z'} \equiv \int dz \int dz'$. By "backward scattering", we understand processes with a non-zero momentum transfer $\delta k = k_{1F} \pm k_{2F}$ between subbands. These processes can be divided further into *direct* and *exchange* parts

$$U^B = U_d^B + U_x^B, \tag{8}$$

In an *exchange* process [see Figs.2,3], electrons change subbands. Two parts of backscattering can be written as (for the sake of brevity, we omit here the $x - x'$-dependence of the matrix elements):
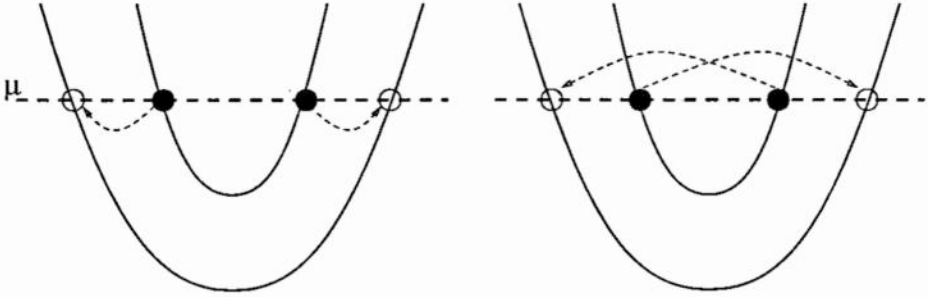
$$U_d^B = \frac{1}{2} \sum_{n \neq m} \sum_{s,s'} \int_{x,x'} M_d^{\{nm\}} [R_{ns}^\dagger(x)L_{ns}(x)L_{ms'}^\dagger(x')R_{ms'}(x')e^{2i(k_{mF}-k_{nF})(x+x')}$$
$$+ (R \Leftrightarrow L)e^{-2i(k_{mF}-k_{nF})(x+x')}] \tag{9}$$

and

$$U_x^B = -\frac{1}{2} \sum_{n \neq m} \sum_{s,s'} \int_{x,x'} M_x^{\{nm\}} [\{R_{ns}^\dagger(x)R_{ns'}(x')R_{ms'}^\dagger(x')R_{ms}(x)e^{i(k_{mF}-k_{nF})(x-x')}$$
$$+ (R \Leftrightarrow L)e^{-i(k_{mF}-k_{nF})(x-x')}\}$$
$$+ \{R_{ns}^\dagger(x)R_{ns'}(x')L_{ms'}^\dagger(x')L_{ms}(x)e^{i(k_{mF}+k_{nF})(x-x')}$$
$$+ (R \Leftrightarrow L)e^{-i(k_{mF}+k_{nF})(x-x')}\}], \tag{10}$$
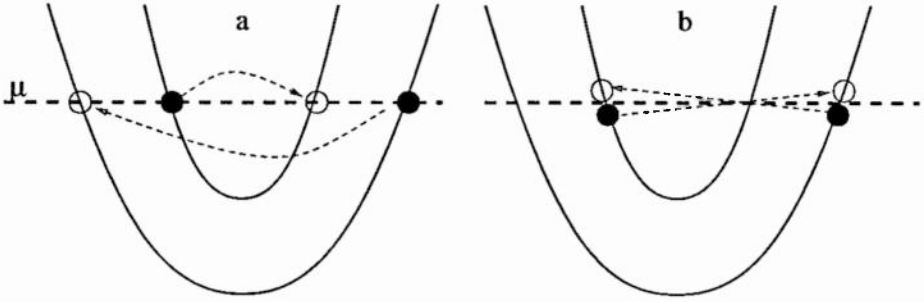
Fig. 2. Example of inter-subband backscattering: (a) direct, (b) exchange. Notations as in Fig.1.

Fig. 3. Examples of inter-subband Cooper scattering. Notations as in Fig.1.

where the exchange matrix element is

$$M_x^{\{nm\}}(x - x') = \int_{\mathbf{r}_\perp, \mathbf{r}'_\perp} U(|\mathbf{r} - \mathbf{r}'|)\phi_n(\mathbf{r}_\perp)\phi_m(\mathbf{r}'_\perp)\phi_n(\mathbf{r}'_\perp)\phi_m(\mathbf{r}_\perp). \qquad (11)$$

Momentum conservation requires that the energy of at least one of the states, involved into direct backscattering, should be far away from the Fermi energy, which forbids this process at not too high temperatures ($T \ll |k_{mF} - k_{nF}| \min\{v_{nF}, v_{mF}\}$). This is reflected in the presence of the exponential factors in front of the fermion operators in Eq.(9), which oscillate rapidly as functions of $(x + x')$. This restriction can be lifted though, if the system prefers to gain energy from backscattering by equalizing the subband densities, so that $k_{mF} = k_{nF}$.

Finally, we call "Cooper scattering" (Fig.3) a process in which two electrons with zero total momentum (a fluctuational Cooper pair) hop from, e.g., channel m, into channel n, so that the total momentum $Q = -k_m + k_m = 0 \to -k_n + k_n = 0$ is conserved. This process is also referred to as "Josephson coupling" [32,24], or as "$g_{00\pi\pi}$ process" [13],[16]. The Hamiltonian of Cooper scat-

tering is given by

$$U^C = \frac{1}{2} \sum_{n \neq m} \sum_{s,s'} \int_{x,x'} M_x^{\{nm\}} [R_{ns}^\dagger(x) L_{ns'}^\dagger(x') e^{ik_{nF}(x'-x)} +$$
$$+ L_{ns}^\dagger(x) R_{ns'}^\dagger(x') e^{-ik_{nF}(x'-x)}] \times$$
$$\times [R_{ms'}(x') L_{ms}(x) e^{ik_{mF}(x'-x)} + L_{ms'}(x') R_{ms}(x) e^{-ik_{mF}(x'-x)}]. \quad (12)$$

By construction, Cooper scattering is of the *exchange* type.

In what follows, we use the following abbreviations: forward scattering $\equiv$ FS, direct backward scattering $\equiv$ dBS, exchange backward scattering $\equiv$ xBS, Cooper scattering $\equiv$ CS.

For a generic situation of $k_n \neq k_m$, the only momentum-conserving inter-subband scattering processes are FS, xBS, and CS. The amplitudes of these processes depend on the ratio a/d, where a is a typical transverse size of the wire (which determines the spatial extension of $\phi_n(\mathbf{r}_\perp)$) and d is the range of interactions. In the limit $a/d \to 0$, the interaction potential can be taken out of integrals (7) and (11), upon which M_d remains finite, whereas M_x vanishes. It can be readily shown that for finite but small ratio a/d the exchange matrix element is small: $M_x \sim (a/d)^2 M_d$. The long-range interaction thus discriminates against exchange processes. If (as it is most often the case) a wire is formed by means of a gate deposited over the 2D heterostructure, d is given by the distance to this gate, which screens the Coulomb interaction in the wire (see Fig. 1). Typically, $a/d = 0.1 - 1$.

2.2 Bosonized Form of the Hamiltonian

We use the conventional bosonization procedure, in which

$$R_{ns}(x) = \frac{1}{\sqrt{2\pi\alpha}} e^{i\sqrt{\pi}(\varphi_{ns} - \theta_{ns})}, \quad (13)$$

$$L_{ns}(x) = \frac{1}{\sqrt{2\pi\alpha}} e^{-i\sqrt{\pi}(\varphi_{ns} + \theta_{ns})}, \quad (14)$$

and short-range cut-off $\alpha \sim k_F^{-1}$. Boson fields φ_{ns} and θ_{ns} with $n = 1, 2$; $s = \pm 1$, are decomposed into charge-(ρ) and spin-(σ) collective modes

$$\varphi_{ns} = \frac{1}{\sqrt{2}}(\varphi_{n\rho} + s\varphi_{n\sigma}),$$

$$\theta_{ns} = \frac{1}{\sqrt{2}}(\theta_{n\rho} + s\theta_{n\sigma}). \quad (15)$$

Parts of the Hamiltonian, describing the free motion and intra-subband interactions $(H_0 + U_{intra})$, take the well-known Luttinger-liquid form:

$$H_{n\rho} = \frac{1}{2} \int dx \{ v_{n\rho} K_{n\rho} (\partial_x \theta_{n\rho})^2 + \frac{v_{n\rho}}{K_{n\rho}} (\partial_x \varphi_{n\rho})^2 \}, \tag{16}$$

$$H_{n\sigma} = \frac{1}{2} \int dx \{ v_{n\sigma} K_{n\sigma} ((\partial_x \theta_{n\sigma})^2 + \frac{v_{n\sigma}}{K_{n\sigma}} (\partial_x \varphi_{n\sigma})^2 \}$$
$$+ \frac{2U(2k_{nF})}{(2\pi\alpha)^2} \int dx \cos[\sqrt{8\pi}\varphi_{n\sigma}], \tag{17}$$

which describes *independent* charge- and spin-density excitations ($H_{n\rho}$ and $H_{n\sigma}$, respectively). The cosine term in Eq.(17) is due to backscattering within a single subband. Explicit expressions for the Luttinger-liquid parameters will be discussed later.

Upon bosonization, the three types of intersubband interactions take the following form :

$$U^F = \frac{2f_0}{\pi} \int dx \partial_x \varphi_{1\rho} \partial_x \varphi_{2\rho}; \tag{18}$$

$$U_d^B = \frac{4f_{bs}}{\pi^2 \alpha^2} \int dx \cos \left[\sqrt{2\pi} (\varphi_{1\rho} - \varphi_{2\rho}) + 2(k_{1F} - k_{2F})x \right]$$
$$\times \cos[\sqrt{2\pi}\varphi_{1\sigma}] \cos[\sqrt{2\pi}\varphi_{2\sigma}]; \tag{19}$$

$$U_x^B = -\frac{1}{2} \int dx \Big(\frac{b_1 + b_2}{\pi} (\partial_x \varphi_{1\rho} \partial_x \varphi_{2\rho} + \partial_x \varphi_{1\sigma} \partial_x \varphi_{2\sigma})$$
$$+ \frac{b_1 - b_2}{\pi} (\partial_x \theta_{1\rho} \partial_x \theta_{2\rho} + \partial_x \theta_{1\sigma} \partial_x \theta_{2\sigma})$$
$$+ \frac{2}{\pi^2 \alpha^2} \cos[\sqrt{2\pi}(\theta_{1\sigma} - \theta_{2\sigma})] \{ (b_1 + b_2) \cos[\sqrt{2\pi}\varphi_{1\sigma}] \cos[\sqrt{2\pi}\varphi_{2\sigma}] -$$
$$(b_1 - b_2) \sin[\sqrt{2\pi}\varphi_{1\sigma}] \sin[\sqrt{2\pi}\varphi_{2\sigma}] \} \Big); \tag{20}$$

$$U^C = \frac{4}{2\pi^2 \alpha^2} \int dx \{ t_{sp} \cos[\sqrt{2\pi}(\theta_{1\rho} - \theta_{2\rho})] \cos[\sqrt{2\pi}\varphi_{1\sigma}] \cos[\sqrt{2\pi}\varphi_{2\sigma}] + \tag{21}$$
$$+ t_{tp} \cos[\sqrt{2\pi}(\theta_{1\rho} - \theta_{2\rho})] \Big(\cos[\sqrt{2\pi}(\theta_{1\sigma} - \theta_{2\sigma})] - \sin[\sqrt{2\pi}\varphi_{1\sigma}] \sin[\sqrt{2\pi}\varphi_{2\sigma}] \Big) \}.$$

The corresponding amplitudes are given by

$$f_0 = \int dx M_d^{\{12\}}(x),$$

$$f_{bs} = \int dx M_d^{\{12\}}(x) \cos[(k_{1F} + k_{2F})x],$$

$$b_{1,2} = \int dx M_x^{\{12\}}(x) \cos[(k_{1F} \mp k_{2F})x],$$

$$t_{sp} = \int dx M_x^{\{12\}}(x) \cos(k_{1F}x) \cos(k_{2F}x),$$

$$t_{tp} = \int dx M_x^{\{12\}}(x) \sin(k_{1F}x) \sin(k_{2F}x). \tag{22}$$

In the last two lines, $t_{sp}(t_{tp})$ are the amplitudes of singlet (triplet) Cooper processes.

The highly non-linear (cosine) terms in Eqs.(19,20,21) signal potential instabilities of the ground state due to interactions. For the dBS process [Eq.(19)], this instability is of the charge-density-wave (CDW) type, quantity $\varphi_{1\rho} - \varphi_{2\rho}$ being the phase of the CDW (particle-hole) condensate. If subbands are equivalent ($k_{1F} = k_{2F}$), the energy is minimized by adjusting the CDW-condensate phase is such a way that the cosine takes its minimum value (-1, for repulsive interactions). For non-equivalent subbands ($k_{1F} \neq k_{2F}$), the global minimization of the energy is impossible due to the position-dependent phase shift, and thus the CDW instability is suppressed. Nevertheless, if the energy gain due to opening of the CDW gap is large enough, the system may choose to adjust the subband densities, which makes the CDW instability possible. Density equilibration is most likely to occur if the cross-section of the wire is approximately symmetric. For example, if it is a perfect square, the ground state is doubly degenerate. Deformation lifts the degeneracy but the energy splitting between the states is small for small deformations. Such states are almost equally occupied, and a small difference in densities is likely to be eliminated by opening the CDW-gap. It seems that cleaved edge quantum wires investigated by Yacoby et al. [11] satisfy this requirement.

In the "two-chain model", the CDW-process of this type is known as "deconfinement" [13]: degeneracy of bonding and antibonding subbands implies that the amplitude of interchain tunneling, $t_\perp$, is renormalized to zero by interactions, and electrons thus remain "confined" to their respective chains.

The xBS process (20) contains both harmonic terms, arising from backscattering of electrons with parallel spins, and cosine terms, arising from backscattering with antiparallel spins. The latter contain only spin fields and thus can lead to the instability only in the spin channel. In the terminology of Ref. [16,14], this instability corresponds to the "orbital antiferromagnet phase" (OAF). The OAF instability occurs only if backscattering is sufficiently strong [16,14]. For a quantum wire, in which all amplitudes are given just by the corresponding Fourier components of the same interaction potential, this conditions means that $U(2k_F) > 2U(0)$ (for identical subbands), which is never the case for any physical $U(\mathbf{r})$. In what follows, we will not therefore consider the OAF phase. (Note that in a "two-chain model", amplitudes of various scattering processes may be determined by entirely different physics, e.g., some of them may result from direct electron-electron interaction and some from exchange of virtual phonons. Hence, the ratios of amplitudes may be arbitrary, and the OAF instability is possible, at least *a priori*.)

Finally, the CS process [Eq.(21)] may lead to a superconducting instability (of both singlet and triplet types), accompanied by opening of spin gaps in each of the subbands, in a analogy with a superconducting transition in higher dimensions. The quantity $\theta_{1\rho} - \theta_{2\rho}$ plays the role of the superconducting condensate phase. Inter-subband forward scattering [Eq. (18)] plays an important role in developing a superconducting instability–it reduces electron repulsion in

the relative charge-density fluctuation channel, making it possible for Cooper scattering to become relevant. The superconducting phase is also known as "C1S0-phase"(meaning: one gapless charge mode and no gapless spin modes) [15] or "d-wave superconductor"(indicating that order parameter is odd upon interchanging subband index of electrons, forming a Cooper pair) [16,14]. This particular instability received much attention recently as one of the models of HTC superconductivity [17]. We also note in passing that the idea of super-conductivity in a two-band system has a long history, starting from the 1968 paper by Frölich [33] (for a review, see Ref. [34]). The idea, employed in earlier work, is that if the masses of electrons in two subbands are significantly different, there always–even in 3D–exists a gapless plasmon excitation (a direct analog of a Langmuir-Tonks ion sound wave in plasma), which serves as a mediator of ef-fective attraction. The superconducting phase in 1D two-band system is already composed of gapless excitations and is not limited by the condition of different masses (although, as we will see shortly, there is no lack of other constraints).

Note also that U_d^B (19) and U^C (21) mix charge and spin modes, and thus spoil the spin-charge separation present in the Hamiltonian of a single subband.

Processes (18 - 21) have been written down in the literature in many different ways, so it is worth to make a connection to previous work here. Identification of our notations with the g-ology ones (used by Schulz in his two important papers [16,14]) is as follows: $t_{sp} = g_{12} + g_{23}$, $t_{tp} = g_{23} - g_{12}$, $b_1 = g_{13}$, and his g_{11}-process (backscattering with opposite spins) should be equated with our $U(2k_{nF})$ in Eq.(17). There is no correspondence to our amplitude b_2, which describes exchange inter-subband backscattering of the type $R_{ns}^\dagger R_{ms'}^\dagger R_{ns'} R_{ms} + (R \to L)$, see Eq.(10). That such a process is absent in Ref. [16] is clear from Eq.(2) of that reference.

We can make one more connection by noting that (somewhat lengthy) Eq.(20) can be also represented compactly as

$$U_x^B = -\int dx\{(b_1 + b_2)(\rho_1\rho_2 + S_1 S_2) + (b_1 - b_2)(j_{c1} j_{c2} + j_{s1} j_{s2})\}, \qquad (23)$$

where ρ_n (S_n) is the charge (spin) density, and j_{cn} (j_{sn}) is the charge (spin) current in the n-th subband, using notations of Emery, Kivelson, and Zachar [17].

3   Spinless Electrons

Model In this section we consider a "toy"model of two subbands of spinless electrons, which contains all interesting effects we want to discuss and, at the same time, allows a rather complete analytic treatment. In this model, various

parts of the Hamiltonian reduce to

$$H_0 \to \tilde{H}_0 = \int dx \sum_n \left[\frac{v_n}{2K_n} (\partial_x \varphi_n)^2 + \frac{v_n K_n}{2} (\partial_x \theta_n)^2 \right] \tag{24}$$

$$U^F \to \tilde{U}^F = \frac{f_0}{\pi} \int dx \partial_x \varphi_1 \partial_x \varphi_2 \tag{25}$$

$$U_d^B \to \tilde{U}_d^B = \frac{f_{bs}}{\pi^2 \alpha^2} \int dx \cos[\sqrt{4\pi}(\varphi_1 - \varphi_2) + 2(k_{1F} - k_{2F})x] \tag{26}$$

$$U_x^B \to \tilde{U}_x^B = -\frac{1}{2\pi} \int dx \left[(b_1 + b_2)\partial_x \varphi_1 \partial_x \varphi_2 + (b_1 - b_2)\partial_x \theta_1 \partial_x \theta_2 \right] \tag{27}$$

$$U^C \to \tilde{U}^C = \frac{f_C}{2\pi^2 \alpha^2} \int dx \cos\sqrt{4\pi}(\theta_1 - \theta_2); \tag{28}$$

so that

$$\tilde{H} = \tilde{H}_0 + \tilde{U}^F + \tilde{U}_d^B + \tilde{U}_x^B + \tilde{U}^C. \tag{29}$$

Amplitude f_C plays now the role of t_{sp} for spinless electrons. Analysis of potentially "dangerous"(in a sense of inducing instabilities) intersubband processes reduces to estimating the scaling dimensions of corresponding cosine operators in terms of the parameters of the harmonic part. In their turn, these parameters are related to the Fourier components of the electron-electron interaction potential. As it turns out, the latter relation is not that straightforward, and we will clarify this point in the next Section.

Galilean invariance and Pauli principle: single-subband Luttinger liquid To begin with, we consider the simplest case when there is no intersubband interaction and the Hamiltonian is given by the sum of two single-subband Hamiltonians (24). (As our discussion is referred now to a single subband, we suppress temporarily the subband index.) For a given effective 1D interaction potential $U(x)$, the Luttinger-liquid parameters (K and v) depend on the $q = 0$ and $q = 2k_F$ Fourier components of U, as well on the bare Fermi velocity v_F:

$$K = \mathcal{K}[U(0)/v_F, U(2k_F)/v_F], \tag{30}$$

$$v = v_F \mathcal{V}[U(0)/v_F, U(2k_F)/v_F], \tag{31}$$

where $\mathcal{K}(x, y)$ and $\mathcal{V}(x, y)$ are some dimensionless functions of their arguments. Relations (30),(31) have to satisfy (i) the Pauli principle and (ii) Galilean invariance. The Pauli principle for spinless fermions means that for the case of contact interaction, i.e., when $U(0) = U(2k_F)$, the system should behave as if there is no interaction at all. Accordingly, $K = 1$ and $v = v_F$ for this case, or

$$\mathcal{K}(x, x) = \mathcal{V}(x, x) = 1. \tag{32}$$

Galilean invariance stipulates that $Kv = v_F$, or

$$\mathcal{K}(x, y)\mathcal{V}(x, y) = 1, \forall x, y. \tag{33}$$

Physically, condition (33) comes about either by requiring that the shift of the ground state energy due to the motion of a system as a whole does not depend on the interaction [35], or by requesting that the dc conductivity of a uniform system[0] does not depend on interactions (Peierls theorem) [1,36]. (Also, one can use the interaction-invariance of the persistent current in a ring threaded by the Aharonov-Bohm flux).

Conventional bosonization of the g-ology Hamiltonian (see, e.g., review [1]) leads to

$$K = \sqrt{\frac{2\pi v_F + g_4 - g_2}{2\pi v_F + g_4 + g_2}}, \quad v = v_F \sqrt{(1 + \frac{g_4 - g_2}{2\pi v_F})(1 + \frac{g_4 + g_2}{2\pi v_F})}. \quad (34)$$

In terms of the Fourier components of the interaction potential the g-parameters are expressed as $g_4 = U(0)$ (right-right and left-left amplitude) and $g_2 = U(0) - U(2k_F)$ (right-left amplitude), and Eq.(34) gives

$$K = \left[\frac{1 + \frac{U(2k_F)}{2\pi v_F}}{1 + \frac{2U(0) - U(2k_F)}{2\pi v_F}} \right]^{1/2},$$

$$v = v_F \left[1 + \frac{U(2k_F)}{2\pi v_F} \right]^{1/2} \cdot \left[1 + \frac{2U(0) - U(2k_F)}{2\pi v_F} \right]^{1/2}. \quad (35)$$

One can see that the expressions above do not satisfy conditions (32), (33). Indeed, it follows from Eq. (35) that $v \neq v_F$ for $U(0) = U(2k_F)$ and that $Kv \neq v_F$ as long as $U(2k_F) \neq 0$. Usually, the spinless Luttinger liquid model does not include backscattering explicitly. The rationale for such a simplification is that for spinless particles in 1D this process is indistinguishable from forward scattering, see, e.g., Ref. [37]. We do not find this approach satisfactory, as it is clear that the behavior of the system should be determined both by forward and backward amplitudes. Also, correct expressions for K and v should include both $U(0)$ and $U(2k_F)$, otherwise the Pauli principle cannot be satisfied. This argument can also be re-phrased in terms of direct and exchange contributions to the self-energy [38].

What did we do wrong to arrive at the Luttinger-liquid model which does not satisfy two basic physical principles? As one can show by using the Ward indentities (conservation laws) for the system of interacting electrons with *linear* spectrum [37], the problem occurs already at the level of fermions and is thus not inflicted by some subtleties of bosonization. Rather, it is a manifestation of an *anomaly*, i.e., a violation of the conservation law caused by regularization, which one is forced to used in a model with linear and unbound spectrum [37].

One way to deal with this problem is to replace Eqs.(35) by expressions which do not follow directly from the original fermion Hamiltonian with linear spectrum, but do satisfy all necessary criteria. This is an accord with the point of view [40] that one should consider K and v as phenomenological parameters,

---

[0] Here we consider a uniform Luttinger liquid. The role of reservoirs, to which the wire is attached to, will be discussed in Sec.5.

which are renormalized from their bare values by irrelevant or marginal operators neglected in the course of linearization. We will be able to find *exact* expressions for K and v, satisfying the minimal set of requirements.

First, we notice that the Pauli principle requires $\mathcal{K}(x, y)$ to be a function of either $x - y$ or x/y. The latter choice contradicts to the requirement that $\mathcal{K}$ must have Taylor expansions both around $x = 0$ and $y = 0$. Therefore,

$$\mathcal{K}(x, y) = \kappa(x - y), \tag{36}$$

$$\kappa(0) \quad = 1. \tag{37}$$

Then we notice that the model with forward scattering only, i.e., the original Luttinger model [39], does respect Galilean invariance. Therefore one can take the Luttinger model expression for K as the correct one, which means that

$$\mathcal{K}(x, 0) = 1/\sqrt{1 + x}. \tag{38}$$

Combining Eq. (37) with Eq. (38), we see that

$$\mathcal{K}(x, y) = 1/\sqrt{1 + x - y}, \tag{39}$$

$$\mathcal{V}(x, y) = \sqrt{1 + x - y}, \tag{40}$$

or,

$$K = \left[1 + \frac{U(0) - U(2k_F)}{\pi v_F} \right]^{-1/2}, \tag{41}$$

$$v = v_F \left[1 + \frac{U(0) - U(2k_F)}{\pi v_F} \right]^{1/2}. \tag{42}$$

The physical meaning of Eq.(41),(42) is obvious: the effective interaction is equal to backscattering minus forward scattering. One can check that a Luttinger-liquid model with parameters given by Eq.(41),(42) reproduces correctly results for a 1D electron system, obtained without linearization but in the limit of weak interactions. For instance, the (inverse) compressibility of a Luttinger liquid, parametrized by K and v from Eq.(41),(42), is given by

$$\frac{1}{\chi} = \frac{\pi v}{K} = \pi v_F + U(0) - U(2k_F). \tag{43}$$

As one can check, Eq.(43) coincides with the inverse compressibility of electrons with a *quadratic* spectrum obtained in the Hartree-Fock approximation. A perturbative (linear in U) form of Eq.(42) has recently been derived in [41]. It can also be read of from the tunneling exponent of a 1D system with a *quadratic* dispersion [42].

Galilean invariance and Pauli principle: coupled subbands Now we allow for harmonic coupling between subbands, i.e., take into account intersubband

forward [Eq.(25)] and exchange backscattering [Eq.(27)]. For the contact interaction, the amplitudes of these two processes coincide: $b_1 = b_2 = f_0$ and, as a result, inter-subband interaction drops out. The Pauli principle is thus satisfied. Intersubband exchange backscattering does violate Galilean invariance, and the correction procedure, similar to that for a single subband, is necessary. We will not do it here however, because for a long-range interaction ($a/d \ll 1$), the violation is "weak": the deviation from the Galilean-invariant result is proportional to the *exchange* amplitudes, which are small compared to the *direct* ones.

3.1 Nearly Equivalent Subbands

First, we discuss the CDW-instability, which may occur if the density equilibration between subbands is energetically favorable. To simplify the discussion, we consider the case of a long range interaction ($a/d \ll 1$), when amplitudes of all exchange processes are small. In the leading order, $f_C = b_{1,2} = 0$ and the only "dangerous" process to be considered is direct intersubband backscattering. Furthermore, we assume that subbands are nearly equivalent and put $v_{1F} = v_{2F}$ and $K_1 = K_2$ but keep $\delta k_F = k_{1F} - k_{2F}$ in Eq. (26) finite. It is convenient to introduce symmetric and antisymmetric combinations of boson fields

$$\varphi_\pm = \frac{\varphi_1 \pm \varphi_2}{\sqrt{2}}; \; \theta_\pm = \frac{\theta_1 \pm \theta_2}{\sqrt{2}}, \tag{44}$$

which correspond to fluctuations of total (+) and relative (-) subband charge and current. In terms of these fields,

$$
\begin{aligned}
H &= H_+ + H_-, \\
H_+ &= \frac{1}{2} \int dx \left\{ \frac{v_+}{K_+}(\partial_x\varphi_+)^2 + v_+ K_+(\partial_x\theta_+)^2 \right\}, \\
H_- &= \frac{1}{2} \int dx \left\{ \frac{v_-}{K_-}(\partial_x\varphi_-)^2 + v_- K_-(\partial_x\theta_-)^2 + \frac{f_{bs}}{\pi^2\alpha^2}\cos[\sqrt{8\pi}\varphi_- + 2\delta k_F x] \right\}
\end{aligned}
\tag{45}
\tag{46}
$$

where

$$K_+ = \left[1 + \frac{2U(0) - U(2k_F)}{\pi v_F}\right]^{-1/2} \quad \text{and} \quad K_- = \left[1 - \frac{U(2k_F)}{\pi v_F}\right]^{-1/2}, \tag{47}$$

and $v_\pm = v_F/K_\pm$. Note that $K_- > 1$ for $U(2k_F) > 0$, which signals effective attraction in the ($-$) channel.

Collective adjustment of densities as a commensurate-incommensurate transition To understand how the CDW-instability works, we consider first a model situation when $K_- < 1$, so that operator $\cos[\sqrt{8\pi}\phi_-]$ is relevant in the RG sense. Finite δk_F stops the RG-flow at scale $\ell \sim 1/\ln|\delta k_F|\alpha$, thus precluding the system from reaching its strong-coupling limit. However, this consideration
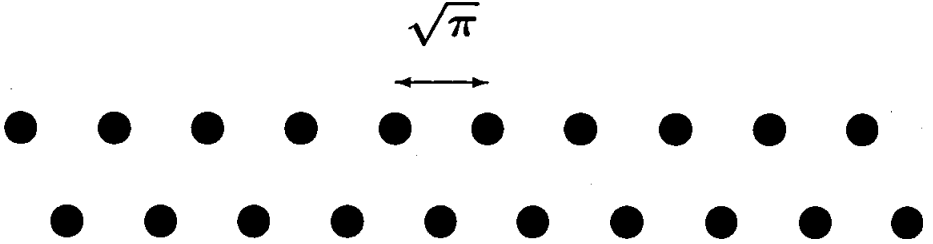
$$\sqrt{\pi}$$

Fig. 4. An illustration of the charge density wave in two coupled subbands. A staggered configuration lowers the energy due to short-range repulsion, if the densities are commensurate.

does not take into account the possibility of a collective density readjustment between subbands. Such a readjustment may occur, if the kinetic energy loss $\Omega = v_F \delta k_F$ is compensated by the gain in the potential energy due to opening of the gap in the (-) channel. In other words, when the difference in electron densities is sufficiently small, the total energy is minimized by equating the densities and opening the charge gap. Obviously, such process cannot be considered at the level of single-particle description of transverse quantization. Instead, one should now treat eigenstates and eigenenergies of the wire as being determined by a self-consistent procedure, involving both single-particle and many-body effects.

The mechanism described above can be considered as a commensurate-incommensurate transition. The incommensurability, defined as $\mathcal{I} = L^{-1}\langle \int dx \partial_x \varphi_- \rangle$, where L is the length of the wire, is known to have a threshold behavior [43,44]: $\mathcal{I} \sim \sqrt{\Omega^2 - \Omega_c^2}\Theta(\Omega - \Omega_c)$, where $\Omega_c = \sqrt{2\pi K_-}\Delta_{CDW}$ and the gap follows from mapping on exactly solvable Heisenberg spin chain [45], $\Delta_{CDW} \sim (f_{bs})^{1/2(1-K_-)}$. As follows from the definition of the incommensurability, $\mathcal{I} = 0$ implies $\delta k_F = 0$, i.e., equal subband densities. Therefore, the re-adjustment takes place if $\Omega < \Omega_c$. Backscattering is then enabled and relevant (for $K_- < 1$), even for a non-zero initial value of δk_F.

What is the physical meaning of this instability? A simple picture can be obtained in the limit of strong (both inter- and intra-subband) interactions, when the potential energy dominates over quantum fluctuations. In this case, electrons of each of the subbands form a regular lattice (Wigner crystal). Boson fields φ_n also have periodic structures with period equal to $\sqrt{\pi}$ (recall that a shift of $\sqrt{\pi}$ corresponds to adding one electron into the system). For $f_{bs} > 0$, the energy of intrasubband repulsion

$$f_{bs}\cos[\sqrt{8\pi}\varphi_-] = -f_{bs}\cos[\sqrt{4\pi}(\varphi_1 - \varphi_2 + \sqrt{\pi}/2)] \tag{48}$$

is minimized by a relative phase shift of $\sqrt{\pi}/2$ between the subbands, which corresponds to a shift of electron lattices by half-a-period. This is an inter-subband charge-density wave (CDW).

Competition between CDW and Cooper channels Let us now suppose that the density re-adjustment did occur, i.e., $\delta k_F = 0$, but Cooper scattering is also present, so that the Hamiltonian of the (-)-channel is

$$H_- = \frac{1}{2} \int dx \left\{ \frac{v_-}{K_-}(\partial_x \varphi_-)^2 + v_- K_-(\partial_x \theta_-)^2 \right.$$
$$\left. + \frac{f_{bs}}{\pi^2 \alpha^2} \cos \sqrt{8\pi}\varphi_- + \frac{f_C}{2\pi^2 \alpha^2} \cos \sqrt{8\pi}\theta_- \right\}. \tag{49}$$

Which of the two instabilities–CDW or superconductivity–wins? The situation of this type, when cosines of both mutually conjugated fields (φ_- and θ_-) are present, was analyzed by Schulz and Giamarchi [46]. They found that the result is very sensitive not only to the value of K_-, which determines the scaling dimensions of the fields, but also to the ratio of amplitudes, f_C/f_{bs}. As $K_- > 1$ for repulsive $U(r)$, it may seem that superconductivity is favored over CDW. The situation is not that straightforward, however. For example, consider the situation of weak and long-range interactions, i.e., assume that $U(0), U(2k_F) \ll v_F$ and $a \ll d$. Because the interaction is weak, both processes are almost marginal, CDW being on the slightly irrelevant and superconductivity on the slightly relevant side. For long-range interactions, $f_{bs} \sim U(2k_F)$ and $f_C \sim (a/d)^2 U(0)$. Modeling $U(x)$ by

$$U(x) = \begin{cases} e^2/\epsilon x, & \text{for } x < d; \\ 0, & \text{for } x > d, \end{cases}$$

we get $U(2k_F)/U(0) \sim \ln(k_F a)/\ln(d/a)$. Thus

$$\frac{f_C}{f_{bs}} \sim \left(\frac{a}{d}\right)^2 \frac{\ln d/a}{\ln k_F a} \ll 1. \tag{50}$$

The RG-equation for K_- [46]

$$\frac{d}{dl} K_- = f_C^2 - f_{bs}^2 \tag{51}$$

shows that K_- decreases, if $|f_C| < |f_{bs}|$. Even if initially $K_-(0) > 1$, the situation with $K_-(l) < 1$, when CDW is relevant, will be reached in the process of renormalization. For weak and long-ranged interactions, CDW thus wins over superconductivity.

If interactions are not sufficiently weak and/or long-ranged, only a full RG solution can determine the leading instability. We will not analyze the general case here.

3.2 Non-Equivalent Subbands: Renormalization Group

Now we consider a generic situation of non-equivalent subbands, when δk_F is not small enough for the density re-adjustment to occur. We find that a strong

imbalance between Fermi-velocities of occupied subbands actually helps super-conducting instability to develop (see *case B* below), despite the fact that a naive scaling dimension estimate does not show this. This effect follows from the next-to-leading order perturbative RG calculations, which we present here.

Because $\delta k_F \neq 0$, we neglect the dBS process [Eq. (26)] from the outset but keep the Cooper one [Eq. (28)]. For long-range interaction, one can also neglect the xBS process, Eq.(27), whose amplitude is small for this case: $b_i \propto (a/d)^2$. Its inclusion is straightforward (U_x^B is quadratic) but does not lead to any qualitatively new results, while complicating the analysis significantly. The Hamiltonian then reads

$$\tilde{H} = \tilde{H}_0 + \tilde{U}^F + \tilde{U}^C. \tag{52}$$

Because $\tilde{U}^C$ contains θ-fields, it is convenient to switch from the Hamiltonian to the Lagrangian approach and to integrate out the φ-fields. The quadratic part of the resulting action is diagonalized by the following transformation

$$\begin{pmatrix} \bar{\theta}_1 \\ \bar{\theta}_2 \end{pmatrix} = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \sqrt{v_1 K_1}\theta_1 \\ \sqrt{v_2 K_2}\theta_2 \end{pmatrix}, \tag{53}$$

where

$$\tan 2\beta = \frac{u_0^2}{v_1^2 - v_2^2}, \quad u_0 = \left[2f_0\sqrt{v_1 K_1 v_2 K_2}/\pi \right]^{1/2}, \tag{54}$$

and

$$\mu_1 = \frac{\cos\beta}{\sqrt{v_1 K_1}} - \frac{\sin\beta}{\sqrt{v_2 K_2}},$$
$$\mu_2 = \frac{\sin\beta}{\sqrt{v_1 K_1}} + \frac{\cos\beta}{\sqrt{v_2 K_2}}. \tag{55}$$

In terms of new fields, the action is given by

$$S = \frac{1}{2} \int dx d\tau \left[\sum_n R_n \{ \frac{1}{u_n}(\partial_\tau \bar{\theta}_n)^2 + u_n(\partial_x \bar{\theta}_n)^2 \} + \frac{f_C}{\pi^2 \alpha^2} \cos\sqrt{4\pi}(\bar{\theta}_1 - \bar{\theta}_2) \right], \tag{56}$$

where

$$u_n^2 = \frac{1}{2}\left(v_1^2 + v_2^2 \pm \sqrt{(v_1^2 - v_2^2)^2 + u_0^4} \right) \tag{57}$$

are the velocities of new collective modes and $R_n = 1/(u_n \mu_n^2)$ are the new stiffness coefficients.

We are now ready to perform the momentum-shell RG, i.e., to develop perturbatively in coupling constant f_C and integrate out high-energy fluctuations with 2-momentum k within a thin strip $\Lambda - d\Lambda \leq k \leq \Lambda$ ($d\Lambda/\Lambda \ll 1$). The first-order contributions renormalize f_C, whereas the second-order one renormalize

stiffnesses R_n. The main difference from the conventional RG-treatment of the sine-Gordon action (see, e.g., Ref. [47]) is that the f_C^2-contribution produces (among others) mixed gradient terms of the type $\partial_\nu \bar\theta_1 \partial_\nu \bar\theta_2$ ($\nu = \tau, x$), which are absent in bare action (56). To eliminate these terms, we transform fields one more time :

$$\begin{pmatrix} \tilde\theta_1 \\ \tilde\theta_2 \end{pmatrix} = \left(1 + \frac{\beta'}{t'}\right) \begin{pmatrix} \cosh\beta' & t'\sinh\beta' \\ \frac{1}{t'}\sinh\beta' & \cosh\beta' \end{pmatrix} \begin{pmatrix} \bar\theta_1 \\ \bar\theta_2 \end{pmatrix}, \tag{58}$$

where

$$t' = \frac{u_2 R_2}{u_1 R_1} \text{ and } \beta' = \frac{d\Lambda}{\Lambda}\left(\frac{f_C}{\pi}\right)^2 \frac{1}{u_1 u_2 R_1 R_2} \tag{59}$$

are chosen in such a way that the coefficients in front of mixed gradient terms vanish. When written in terms of $\tilde\theta_n$, the action is brought into its original form but with renormalized parameters. The resulting RG equations read

$$\frac{d}{dl}\frac{1}{R_1} = -\bar{f}^2 \frac{1}{R_1}\left(\frac{4\gamma^2}{R_2^2(1+\gamma^2)} + \frac{2}{R_1^2(1+\gamma^2)} + \frac{\gamma}{R_1 R_2}\right), \tag{60}$$

$$\frac{d}{dl}\frac{1}{R_2} = -\bar{f}^2 \frac{1}{R_2}\left(\frac{4}{R_1^2(1+\gamma^2)} + \frac{2\gamma^2}{R_2^2(1+\gamma^2)} + \frac{1}{R_1 R_2 \gamma}\right), \tag{61}$$

$$\frac{d}{dl}\gamma = \bar{f}^2 \frac{1-\gamma^2}{R_1 R_2}, \tag{62}$$

$$\frac{d}{dl}\bar{f} = \left(2 - \frac{1}{R_1} - \frac{1}{R_2}\right)\bar{f}, \tag{63}$$

where $\bar{f} = (1/\pi)f_C\sqrt{u_1^{-2} + u_2^{-2}}$ is the dimensionless coupling constant and $\gamma = u_1/u_2$. To the $\bar{f}^2$-accuracy, all terms multiplying $\bar{f}^2$ on the right-hand-side of the first three equations above have to be treated as a constants determined by the initial conditions. The system of RG-equations has an obvious integral of motion

$$C = \frac{x^2}{c_1} + \frac{y^2}{c_2} - \bar{f}^2, \tag{64}$$

where $x = 1 - 1/R_1$, $y = 1 - 1/R_2$ and $c_{1,2}$ are the coefficients in front of $\bar{f}^2$ in Eqs. (60,61), respectively. Note also that $x = (c_1/c_2)y + p$ (we denote $x = x(l)$, $y = y(l)$ whereas initial conditions are denoted by sub-index 0, i.e., $x(0) = x_0$, etc.). Constants of motion C and p are determined by initial conditions.

The flow described by (60-63) is quite similar to that of a canonical Kosterlitz-Thouless system: x and y increase with $\bar{f}$ regardless of its sign. If x_0, $y_0 > 0$, $\bar{f}$ grows unrestrictedly, flowing into the strong-coupling regime with a gap in the $\theta_1 - \theta_2$ channel. Such initial conditions correspond to $R_{1,2} > 1$, i.e., to the attractive interaction in the $\bar\theta_n$-channels. It is worth emphasizing here that due to the presence of inter-subband forward scattering, such effective attraction may arise

in a purely repulsive system, as we shall demonstrate shortly (*case A* below). Another relevant limit is represented by the "repulsive" case (*case B*), where initially $x_0 < 0$, $y_0 < 0$. For a strong repulsion ($R_{1,2} \ll 1$), $\bar{f}$ quickly renormalizes to zero and the resulting phase is a two-subband Luttinger liquid. However, there is a region of anomalously small $x_0, y_0 \sim \bar{f}_0$ (which requires strong inter-subband scattering), where Cooper scattering may still be important. One finds that if

$$(c_1 + c_2)\bar{f}_0^2 > (x_0 + y_0)^2, \tag{65}$$

the Cooper process wins over repulsion and initially negative variables x, y change sign during renormalization. In this case, the Cooper scattering amplitude initially decreases with l, but then passes through the minimum and finally flows into strong-coupling regime $\bar{f} \geq 1$. Equation (65) is a condition for the development of superconducting fluctuations in the system with purely repulsive interactions.

Now we again apply our analysis to a wire with weak and long-range interactions. Two limits are possible.

Case A: $\Delta_{CDW}/k_F \ll v_{1F} - v_{2F} \ll U_0$.

The first inequality allows one to neglect direct backscattering, which leads to inter-subband CDW, whereas the second one allows to consider subbands as "nearly equivalent". Denoting $v_F \equiv v_{1F}$, $\delta v_F \equiv v_{1F} - v_{2F}$, and the $2k_{nF}$-component of electron interaction potential in the n-th subband by $U_{2k_F}^{(n)}$, one finds

$$\frac{1}{R_1} + \frac{1}{R_2} = 2 - \frac{U_{2k_F}^{(1)} + U_{2k_F}^{(2)}}{2\pi v_F} - \frac{\delta v_F}{v_F}\frac{U_{2k_F}^{(1)}}{2\pi v_F} < 2, \tag{66}$$

which corresponds to an effective attraction. Thus, thanks to inter-subband forward scattering, the Cooper process is relevant in the system with purely repulsive interaction. Note that a small velocity imbalance $\delta v_F > 0$ enhances the relevance of Cooper scattering . (As our second subband is chosen to have a higher energy, δv_F is always positive.)

Case B: $v_{1F} - v_{2F} \gg U(0)$. *Van Hove singularity.*

In this limit an analytic solution is also possible. Generally, one finds that $1/R_1 + 1/R_2 > 2$, which corresponds to effective repulsion. Neither CDW nor superconducting instability can develop, and the resulting phase is a two-subband Luttinger liquid. This is not true, however, in the limit of a strong velocity imbalance, when $v_{2F}/v_{1F} \ll 1$, which corresponds to opening of the second subband for conduction. Then $u_n \approx v_n$ and $\gamma = u_1/u_2 \sim v_{1F}/v_{2F} \gg 1$. Hence $c_1 \sim \gamma$ and it follows from (65) that Cooper process wins over repulsion, if $f_C > U(0)/\sqrt{\gamma}$. For long-range interactions, this inequality reduces to

$$\frac{v_{2F}}{v_{1F}} \ll \left(\frac{a}{d}\right)^4. \tag{67}$$

The physics of this scenario is well known - interactions are enhanced due to the large value of the density of states ($\propto 1/v_{2F}$) in the upper subband (Van Hove singularity). We should warn here that our calculations do not describe the very onset of conduction in the upper subband, because its proper description requires accounting for the nonlinearity of the electron spectrum, which is beyond our bosonization analysis. However, such a calculation was performed by Balents and Fisher [15], who analyzed the case of a contact interaction. They found that superconducting fluctuations are indeed enhanced in this limit.

We thus see that superconducting fluctuations do have a good chance to overcome the electron-electron repulsion and drive the system into a strong-coupling phase with the gap in the spectrum of relative current fluctuations, $\bar{\theta}_1 - \bar{\theta}_2$.

There is another important feature of the RG-flow described by Eqs. (60-63): the interaction tries to equilibrate densities in the subbands. This is seen from equation (62): $d\gamma/dl$ is proportional to $1 - \gamma^2$, which makes $\gamma = 1$ a stable fixed point. γ tends to increase, if initially $\gamma_0 < 1$, and it tends to decrease, if $\gamma_0 > 1$.

4 Electrons with Spins

Guided by the results of the previous Section, we now comment briefly on what happens if spin is included. As should be clear from the complexity of Eqs.(16-21), this question has no simple answer. For a quantum wire with $0 < U(2k_F) < U(0)$, possible phases are (i) Luttinger liquid, (ii) inter-subband CDW, and (iii) Cooper phase (superconductor). As with spinless fermions, subbands must be nearly equivalent in order for the CDW phase to occur, whereas the Cooper phase needs effective attraction in the relative charge-density excitation channel. When neither of these conditions is met, a two-subband Luttinger liquid is realized. At different degrees of generality, renormalization group analysis of the model defined by Eqs.(16- 21) has been performed in the past and we refer to papers [25,15,17] for a detailed description.

Long-range interactions If the interaction is long-range and weak, a considerable simplification occurs. In this case, amplitudes of forward intra- and inter-subband processes are the same [see discussion after Eq.(12)] and a simple perturbative estimate of the scaling dimension δ_C of the Cooper process (21) is possible.

For weak interactions, $K_{n\rho} = 1 - (2U(0) - U_{2k_F}^{(n)})/(2\pi v_{nF})$, and $f_0 = U(0)$. The $SU(2)$-invariance requires that $K_{n\sigma} = 1$. One finds

$$\delta_C = 2 - \frac{U_{2k_F}^{(1)} + U_{2k_F}^{(2)}}{4\pi v_F} - \frac{\delta v_F}{v_F}\frac{U_{2k_F}^{(1)}}{4\pi v_F} < 2. \tag{68}$$

Observe that this result coincides with Eq.(66) upon replacing $U(0) \rightarrow 2U(0)$. Thus, Cooper scattering is relevant for repulsive long-range interactions (and assuming also that $0 < \delta v_F \ll U_0$).

However, if $K_{\rho-}$ is sufficiently close to its non-interacting value, i.e., to unity, backscattering is strong $(f_{bs} \gg t_{sp}, t_{tp})$, and δk_F is small, the CDW-channel can take over the Cooper one, similarly to what happened in Subsection 3.1.

Electron ladder There is an interesting qualitative question where an RG consideration is very helpful. Suppose that condition (68) is satisfied and thus tunneling of fluctuational Cooper pairs is relevant. What happens to spin excitations? To answer this question, we relax the $SU(2)$-invariance condition and perform the RG calculation for two nearly equivalent subbands so that $v_{n\nu} \equiv v_{\nu}, K_{n\nu} \equiv K_{\nu}$, where $\nu = \rho, \sigma$. Nevertheless, we assume that δk_F is still finite and neglect direct backscattering (19), similarly to Subsection 3.1. The problem then becomes identical to that of two coupled equivalent chains ("electron ladder"). Also, for the sake of simplicity, we consider only the singlet channel of Cooper scattering. Due to enhanced symmetry, the total current fluctuation mode $\theta_{\rho+} = (\theta_{1\rho} + \theta_{2\rho})/\sqrt{2}$ decouples from the rest and is described by a harmonic action with $K_{\rho+}^{-2} = K_{\rho}^{-2} + 2f_0/(\pi v_F)$. Relative current fluctuations $\theta_{\rho-} = (\theta_{1\rho} - \theta_{2\rho})/\sqrt{2}$ are described by the sine-Gordon theory [t_{sp} term in (21)] with $K_{\rho-}^{-2} = K_{\rho}^{-2} - 2f_0/(\pi v_F)$ and $v_{\rho-} = v_{\rho}[1 - 2f_0 K_{\rho}/(\pi v_F)]^{1/2}$. In addition, we have to keep track of spin-density sector, which contain the cosine term corresponding to intra-subband backscattering (17). For convenience, we denote the amplitude of this term by g_{σ}, its initial value being $g_{\sigma}(0) = U(2k_F)$. After tedious but straightforward calculations we arrive at the following system of RG equations:

$$\frac{d}{d\ell}\bar{g} = 2(1 - K_{\sigma})\bar{g} - \bar{t}^2, \tag{69}$$

$$\frac{d}{d\ell}\bar{t} = (2 - K_{\sigma} - \frac{1}{K_{\rho-}} - \bar{g})\bar{t}, \tag{70}$$

$$\frac{d}{d\ell}(1 - K_{\sigma}) = \frac{1}{2}(\bar{g}^2 + \bar{t}^2), \tag{71}$$

$$\frac{d}{d\ell}(1 - \frac{1}{K_{\rho-}}) = \bar{t}^2, \tag{72}$$

where $\bar{g} = g_{\sigma}/(\pi v_F)$ and $\bar{t} = t_{sp}/(\pi v_F)$. Let us recall what happens in the absence of Cooper tunneling first and set $t_{sp} = 0$ everywhere in this system. In the weak-coupling limit, $K_{\sigma} = (1 - \bar{g})^{-1/2} \approx 1 + \bar{g}/2$ and (69) becomes $d\bar{g}/d\ell = -\bar{g}^2$, which gives $\bar{g}_{\ell} = \bar{g}_0/(1 + \bar{g}_0\ell)$. For repulsive interactions $(\bar{g}_0 > 0)$, $\bar{g} \propto \ell^{-1} \to 0$ as $\ell \to \infty$: intra-subband backscattering is marginally irrelevant. Observe now that when the Cooper process is present and relevant, i.e., when $\bar{t}$ increases, the flow of $\bar{g}$ is modified: the $\bar{t}^2$-term on the right-hand-side of Eq. (69) changes the sign of $d\bar{g}/d\ell$. Hence $\bar{g}_{\ell}$ is bound to become negative in the process of renormalization and grows unrestrictedly in its absolute value. Intra-subband spin backscattering is thus driven relevant by singlet-pair tunneling, which results in pinning of φ_{σ} in Eq.(17) and opening of the *spin gap*. Thus, similar to the true superconducting state in higher dimensions, the Cooper phase is characterized by gaps in both

charge- and spin-channels. The only massless excitations are those of the total charge channel. This phenomenon is not restricted to the degenerate electron ladder but rather is a generic feature of the system of coupled subbands and/or chains, see, e.g., [15,17].

5 Conductance

Having realized the importance of inter-subband interactions, we now proceed with the analysis of its effect on observable properties of quantum wires. The first property we consider is conductance G.

5.1 No Disorder

Our results for the conductance of a clean wire can be understood from the following simple considerations. The dc conductance of a single-subband wire is equal to $2e^2/h$ regardless of the interactions in the wire [48–50]. Consider now a wire with several subbands occupied. Those interband interactions, which do not open gaps, lead only to a renormalization of Luttinger-liquid parameters. As these parameters do not enter the final result for G, the conductance remains quantized in units of $2e^2/h$ per occupied subband. Other processes, such as direct backscattering and Cooper scattering, open gaps in channels of *relative* charge fluctuations as well as in spin channels. Neither of these gaps, however, affects the motion of a *center-of-mass* of the electron fluid through the wire, thus G remains unrenormalized by this type of interactions as well.

Now we demonstrate the proof of the statements made above. Consider the case of a superconducting instability, when the cosine term of Cooper scattering in Eq. (12) is relevant and the $\theta_{\rho-}$-field is thus gapped. Gaussian fluctuations of the gapped field can be described by expanding the relevant cosine term around its minimum value:

$$\left(4f_C/\pi^2\alpha^2\right)\cos[\sqrt{2\pi}(\theta_{1\rho} - \theta_{2\rho})] \approx \text{const} + m^2(\theta_{1\rho} - \theta_{2\rho})^2, \qquad (73)$$

where m is the mass of the field. The strong-coupling (superconducting) phase corresponds to $m \neq 0$, whereas in the Luttinger-liquid phase $m = 0$. Expanding $\cos[\sqrt{2\pi}\varphi_{n\sigma}]$ around their minima as well, we find that at the Gaussian level charge and spin modes decouple again. Note though that now these are *massive* modes.

As spin excitations do not affect charge transport, we concentrate on the charge sector of the theory, whose Hamiltonian is given by the sum of Eqs. (16,18) and (73). In order to simplify notations, we suppress index ρ in this section, so

that $\theta_{\rho-} \to \theta_-$, etc. Using $v_n K_n = v_{nF}$, we write the charge Hamiltonian as

$$
\begin{aligned}
H_\rho = \frac{1}{2} \int dx \Big\{ & \frac{v_{1F} + v_{2F}}{2} \left[(\partial_x \theta_+)^2 + (\partial_x \theta_-)^2 \right] \\
& + \frac{1}{2} \left(\frac{v_1}{K_1} + \frac{v_2}{K_2} \right) \left[(\partial_x \varphi_+)^2 + (\partial_x \varphi_-)^2 \right] \\
& + \frac{2f_0}{\pi} \left[(\partial_x \varphi_+)^2 - (\partial_x \varphi_-)^2 \right] + (v_{1F} - v_{2F}) \partial_x \theta_+ \partial_x \theta_- \\
& + \left(\frac{v_1}{K_1} - \frac{v_2}{K_2} \right) \partial_x \varphi_+ \partial_x \varphi_- + m^2 \theta_-^2 \Big\}.
\end{aligned}
\tag{74}
$$

The total charge current is given by $j = e\sqrt{2/\pi} \sum_n \partial_t \varphi_n = e(2/\sqrt{\pi})\partial_t \varphi_+$. In order to find the current-current correlation function, one needs to know the retarded Green's function $G_{++}(xt) = -i\Theta(t)\langle [\varphi_+(xt), \varphi_+(00)] \rangle$, which is coupled to another Green's function $G_{-+}(xt) = -i\Theta(t)\langle [\varphi_-(xt), \varphi_+(00)] \rangle$ by the following equations of motion

$$
\begin{aligned}
(-i\partial_t)^2 G_{++} = & \frac{v_1 + v_2}{2} \delta(x)\delta(t) - \frac{1}{2}\partial_x \left\{ \frac{v_1^2}{K_1} + \frac{v_2^2}{K_2} + \frac{2f_0}{\pi}(v_1 + v_2) \right\} \partial_x G_{++} \\
& - \frac{1}{2}\partial_x \left\{ \frac{v_1^2}{K_1} - \frac{v_2^2}{K_2} - \frac{2f_0}{\pi}(v_1 - v_2) \right\} \partial_x G_{-+};
\end{aligned}
$$

$$
\begin{aligned}
(-i\partial_t)^2 G_{-+} = & \frac{v_1 - v_2}{2} \delta(x)\delta(t) - \frac{1}{2}\partial_x \left\{ \frac{v_1^2}{K_1} + \frac{v_2^2}{K_2} - \frac{2f_0}{\pi}(v_1 + v_2) \right\} \partial_x G_{-+} \\
& - \frac{1}{2}\partial_x \left\{ \frac{v_1^2}{K_1} - \frac{v_2^2}{K_2} + \frac{2f_0}{\pi}(v_1 - v_2) \right\} \partial_x G_{++} \\
& - m^2 \left\{ \frac{1}{2}\left(\frac{v_1}{K_1} + \frac{v_2}{K_2} \right) - \frac{2f_0}{\pi} \right\} G_{-+} - m^2 \frac{1}{2}\left(\frac{v_1}{K_1} - \frac{v_2}{K_2} \right) G_{++}.
\end{aligned}
\tag{75}
$$

In the massless limit ($m = 0$), this system of equations is solved readily. In order to model the effect of non-interacting electron reservoirs, which the wire is attached to, we assume that K_n, v_n vary with x adiabatically and approach their non-interacting values $K_n = 1, v_n = v_F$ for $x \to \pm\infty$ [48–50]. In the zero-frequency limit, the solution is particularly simple: $G_{++}(x, \omega \to 0) = 1/2i\omega$, $G_{-+}(x, \omega = 0) = 0$. As a result, the conductivity $\sigma(x, y; \omega \to 0) = (e^2/h)(2/\sqrt{\pi})^2 i\omega/2i\omega = 4e^2/h$ is x-independent, and the conductance is simply $G = 2 \times 2e^2/h$.

In order to see the effect of the gap, we consider first the case of equivalent subbands ("electron ladder"), introduced in Sec. 4. One observes immediately that conditions $v_1 = v_2$, $K_1 = K_2$ lead to complete decoupling of the equations for G_{++} and G_{-+}. As a result, the total charge mode φ_+ is not affected by the gap. Taking the boundary condition for K and v into account gives again the universal result $G = 4e^2/h$. The result for the "electron ladder" thus gives us a hint that G remains at its universal value despite the presence of the gap in the relative charge channel. In order to prove this statement in the general case, we neglect for the moment the boundary conditions for K_n and v_n, i.e.,

consider a uniform wire with two coupled subbands. System (75) is then solved by Fourier transformation. The key feature of the result for $G_{++}(q,\omega)$ is that it still has a pole corresponding to a massless mode $\omega \propto q$, despite the presence of the massive term. The conductivity becomes

$$\sigma(q,\omega) \sim \frac{\omega F(\omega,q)}{(\omega^2 - \bar{v}^2 q^2)(\omega^2 + \omega_m^2 - \bar{u}^2 q^2)}, \tag{76}$$

where $\bar{v}, \bar{u}$ are some renormalized velocities, ω_m is some energy proportional to m^2, and $F(\omega,q)$ is a smooth function of its arguments. As a result, $\sigma(q,\omega) = \bar{G}\delta(q)$ in the limit $\omega \to 0$, where $\bar{G}$ has a meaning of the conductance. Because we neglected the boundary conditions corresponding to presence of non-interacting leads in this calculation, $\bar{G}$ depends on all interaction parameters – v_n, K_n, and m^2–and is of course different from $4e^2/h$. It is clear though that once the boundary conditions are restored, this non-universal value is replaced by the universal factor of $4e^2/h$. The only other possibility is $G = 0$, which, however, is ruled out by the fact that $G_{++}(q,\omega)$ has a massless pole.

We thus conclude that the conductance of a clean wire remains at the universal quantized value irrespective of whether the relative charge mode is gapped or not. The case of a CDW instability can be treated in a similar manner.

5.2 Disordered Wire

A disordered two-band system in the presence of interaction-induced instabilities was considered by Orignac and Giamarchi [28] and by Egger and Gogolin [22]. Our discussion of a disordered two-subband wire follows largely these two papers.

Results of the subsequent analysis can be summarized as follows. If Cooper scattering opens a gap, the system does not become a real superconductor: at $T \to 0$ a single weak impurity splits the wire into two disconnected pieces at low enough energies and weak random potential leads to localization of electrons, similar to the case of a gapless Luttinger liquid. Nevertheless, effects of disorder are less pronounced than for a gapless Luttinger liquid. On the contrary, the CDW-state is more sensitive to disorder than a gapless Luttinger liquid.

Spinless electrons We begin by considering a single impurity described as a potential perturbation $w(x, \mathbf{r}_\perp)$. The impurity causes backscattering of electrons within the occupied subbands, as well as inter-subband backscattering. The amplitudes of the corresponding processes are given by

$$W_n(2k_{nF}) = \int dx dr_\perp w(x, \mathbf{r}_\perp) \phi_n^2(\mathbf{r}_\perp) \cos(2k_{nF}x), \quad n = 1,2; \tag{77}$$

$$W_{\text{inter}} = \int dx dr_\perp w(x, \mathbf{r}_\perp) \phi_1(\mathbf{r}_\perp) \phi_2(\mathbf{r}_\perp) \cos[(k_{1F} + k_{2F})x]. \tag{78}$$

If w varies slowly across the wire, then $W_{\text{inter}} \ll W_n$ due to the orthogonality of transverse wavefunctions, and we consider intra-subband backscattering first.

The bosonized form of intra-subband backscattering is

$$W_{\text{intra}}^n = \frac{W_n(2k_{nF})}{\pi\alpha}\cos[\sqrt{4\pi}\varphi_n] = \frac{W_n(2k_{nF})}{\pi\alpha}\cos[\sqrt{2\pi}(\varphi_+ \pm \varphi_-)], \ n = 1, 2; \tag{79}$$

so the total backscattering operator is given by

$$W_{\text{intra}} = \sum_{n=1,2} W_{\text{intra}}^n = \frac{W_2(2k_{2F}) - W_1(2k_{1F})}{\pi\alpha}\sin[\sqrt{2\pi}\varphi_+]\sin[\sqrt{2\pi}\varphi_-]$$
$$+ \frac{W_2(2k_{2F}) + W_1(2k_{1F})}{\pi\alpha}\cos[\sqrt{2\pi}\varphi_+]\cos[\sqrt{2\pi}\varphi_-]. \tag{80}$$

Note that W_{intra} is local in space and thus cannot change the RG-flows of bulk parameters of the wire. Depending on these bulk parameters, however, W_{intra} will either grow, splitting eventually the wire into two disconnected pieces, or decay, in which case the impurity effectively disappears.

(i) *Cooper phase.*
In the Cooper phase, θ_- is gapped, hence φ_- fluctuates strongly. [This follows from the fact that θ_- and φ_- are canonically conjugated fields, see Sec. 6.] On the first sight, it may seem that these strong fluctuations render W_{intra} to zero. To see that it is not so, consider a second-order impurity contribution, e.g.,

$$\left(\frac{W_1(2k_{1F})}{\pi\alpha}\right)^2 \int d\tau \int d\tau' \left(\langle e^{i\sqrt{2\pi}\varphi_-(\tau)}e^{-i\sqrt{2\pi}\varphi_-(\tau')}\rangle\right.$$
$$\times \left. \cos[\sqrt{2\pi}\varphi_+(\tau)]\cos[\sqrt{2\pi}\varphi_+(\tau')]\right) \tag{81}$$

As we will be explained in more details in Sec. 6, the correlator of φ_--fields in the Cooper phase decays exponentially, i.e., $\langle e^{i\sqrt{2\pi}\varphi_-(\tau)}e^{-i\sqrt{2\pi}\varphi_-(\tau')}\rangle \sim e^{-\Delta_{SC}|\tau-\tau'|}$, where Δ_{SC} is the Cooper gap in the θ_--channel. As a result, the double integration over τ, τ' in Eq. (81) is effectively contracted into a single one, which gives $\Delta_{SC}^{-1}\left(W(2k_{1F})/\pi\alpha\right)^2 \int d\tau \cos[\sqrt{8\pi}\varphi_+(\tau)]$. The mechanism of generating higher order impurity backscattering was discovered in [28,22]. ($\sqrt{8\pi}$ under the cosine indicates that this is a two-particle backscattering process.) Following the RG-calculations of Kane and Fisher [7], one finds that impurity backscattering, generated in this way, becomes relevant for $K_+ < 1/2$. Note that this requires rather strong electron repulsion. For weaker repulsion, impurity scales to zero and the wire retains the universal conductance of $2e^2/h$. Without superconducting correlations, i.e., when θ_- is not gapped, an impurity is effective for $K < 1$. Thus the Cooper phase weakens but does not eliminate impurity scattering.

(ii) *CDW phase.*
In this phase, $\delta k_F = 0$ and φ_- is pinned by the bulk nonlinear term $\cos[\sqrt{8\pi}\varphi_-]$, so that φ_- acquires average value $\langle\varphi_-\rangle = \sqrt{\pi/8}$. Allowing for fluctuations

around the average value, we substitute $\varphi_- = \langle \varphi_- \rangle + \delta\varphi_-$ into (80)

$$W_{\text{intra}} = \frac{W_2(2k_F) - W_1(2k_F)}{\pi\alpha} \sin[\sqrt{2\pi}\varphi_+] \cos[\sqrt{2\pi}\delta\varphi_-]$$
$$- \frac{W_2(2k_F) + W_1(2k_F)}{\pi\alpha} \cos[\sqrt{2\pi}\varphi_+] \sin[\sqrt{2\pi}\delta\varphi_-] \qquad (82)$$

Observing that for small fluctuations one can replace $\cos[\sqrt{2\pi}\delta\varphi_-] \approx 1$, we see that the first term in (82) gives the strongest contribution to backscattering, which is relevant already for $K_+ < 2$. The second term requires more work. To the second order in the amplitude of this term, an expression similar to (81) is generated, but now it involves the following average

$$\langle \sin[\sqrt{2\pi}\delta\varphi_-(\tau)] \sin[\sqrt{2\pi}\delta\varphi_-(\tau')] \rangle \sim \frac{\Delta_{CDW}}{\Delta_0} \sinh[K_0(\Delta_{CDW}|\tau - \tau'|)], \quad (83)$$

where Δ_{CDW} is the CDW gap, Δ_0 is the ultraviolet energy cutoff, and $K_0(x)$ is the modified Bessel function, $[K_0(x) \sim e^{-x}/\sqrt{x}$ for $x \gg 1]$. This result is due to the fact that in the *massive* phase

$$\langle e^{i\sqrt{2\pi}\delta\varphi_-(\tau)} e^{\pm i\sqrt{2\pi}\delta\varphi_-(\tau')} \rangle \sim \exp\left[-K_0\left(\Delta_{CDW}/\Delta_0\right) \mp K_0(\Delta_{CDW}|\tau - \tau'|)\right]. \qquad (84)$$

As a result, correlation functions of *sines* and *cosines* of massive fields are different: the first ones decay exponentially with distance, whereas the second ones reach constant values. Thus the double integral over τ, τ' in (81) can be reduced to the single one again, and, similarly to the Cooper-phase case, two-particle impurity backscattering, relevant for $K_+ < 1/2$, is generated.

It is also straightforward to analyze the effect of inter-subband impurity scattering

$$W_{\text{inter}} = \frac{2W_{\text{inter}}(k_{1F} + k_{2F})}{\pi\alpha} \cos[\sqrt{2\pi}\varphi_+] \cos[\sqrt{2\pi}\theta_-]. \qquad (85)$$

Similarly to Eq.(81), we have to average over the strongly fluctuating θ_--field, which generates again the two-particle backscattering term $\sim \cos[\sqrt{8\pi}\varphi_+]$, relevant for $K_+ < 1/2$.

Hence, the perturbative correction to the conductance of a CDW wire behaves as

$$-\delta G_{CDW} \propto w^2 \epsilon^{K_+ - 2} + \left(w^2/\Delta_{CDW}\right)^2 \epsilon^{4K_+ - 2}, \qquad (86)$$

where $\epsilon = \max\{T, \text{bias}\}$ and where we have also indicated the order of the impurity potential. Please note that the exponent of the weak-link counterpart of (86), derived in Eq.(134) of Section 7, is not related to the leading exponent $K_+ - 2$ here by the conventional duality relation [7]. We conjecture that this violation of the duality signals phase transition separating regimes of weak and strong tunneling.

For the Cooper-phase case, $\cos[\sqrt{2\pi}\theta_-]$ is replaced by $\sin[\sqrt{2\pi}\delta\theta_-]$. Hence, W_{inter} also generates the effective two-particle term in the second order of perturbation theory (cf. with our analysis of the second term in (82)). The correction to the conductance is given by

$$-\delta G_{SC} \propto \left(w^2/\Delta_{SC}\right)^2 \epsilon^{4K_+-2}. \tag{87}$$

To summarize, the Cooper phase is insensitive to a single impurity as long as $K_+ > 1/2$, whereas the CDW one is stable only for $K_+ > 2$.

We now turn to the case of weak random potential produced by many impurities. To establish the boundary between delocalized and localized regimes for the case of weak disorder, it suffices to replace $\epsilon \to 1/L$ in Eqs. (86,87) and to multiply δG by the total number of impurities [51], proportional to L. Depending on whether δG increases or decreases with L, the wire is in the localized or delocalized phase. By doing so, one concludes that a wire is localized for $K_+ < 3$, if it is in the CDW phase, and for $K_+ < 3/4$, if it is in the Cooper phase. For comparison, a (spinless) single-subband Luttinger liquid is localized for $K < 3/2$.

Electrons with spins The bosonized form of impurity backscattering is given by

$$W_{\text{intra}}^{1(2)} = \frac{4W_{1(2)}(2k_{F1,2})}{2\pi\alpha} \cos[\sqrt{\pi}(\varphi_{\rho+} \pm \varphi_{\rho-})] \cos[\sqrt{\pi}(\varphi_{\sigma+} \pm \varphi_{\sigma-})]; \tag{88}$$

$$W_{\text{inter}} = \frac{4W_{\text{inter}}(k_{1F}+k_{2F})}{2\pi\alpha} \cos[\sqrt{\pi}(\varphi_{\rho+} + \theta_{\rho-})] \cos[\sqrt{\pi}(\theta_{\sigma-} + \varphi_{\sigma+})], \tag{89}$$

where $+(-)$ in the argument of cosines refers to the 1st (2nd) subband and all operators are evaluated at the position of the impurity.

In the Cooper phase, the $\theta_{\rho-}-$ and $\varphi_{\sigma\pm}$-modes are gapped, whereas the conjugated modes, i.e., $\varphi_{\rho-}$ and $\theta_{\sigma\pm}$, exhibit strong fluctuations. Integrating out $\varphi_{\rho-}$ and $\varphi_{\sigma\pm}$, we find: $W_{\text{intra}} \sim \Delta_{SC}^{-1} [W(2k_{1,2\,F})]^2 \cos[\sqrt{4\pi}\varphi_{\rho+}]$, which is relevant for $K_{\rho+} < 1$. The same is true for W_{inter}, where strong fluctuations of $\theta_{\sigma-}$ produce a similar operator. The correction to the conductance behaves as

$$-\delta G_{SC} \propto \left(w/\Delta_{SC}\right)^2 \epsilon^{2K_{\rho+}-2}, \tag{90}$$

i.e., as if we were dealing with a single channel gapless Luttinger liquid, characterized by parameter $K_{\rho+}$, subject to an effectively reduced impurity potential. Weak random potential leads to localization for $K_{\rho+} < 3/2$.

In the CDW-state, the situation is different. In this case, the $\varphi_{\rho-}-$ and $\varphi_{\sigma-}$-modes are gapped [14,28], whereas the $\theta_{\rho-}-$ and $\theta_{\sigma\pm}-$ modes fluctuate strongly. As a result, intersubband backscattering is renormalized into $\cos[\sqrt{4\pi}\varphi_{\rho+}]$, as in the Cooper phase, but intra-subband one remains unchanged and is determined by the dynamics of the only gapless $\varphi_{\rho+}$ mode:

$$W_{\text{intra}} \propto \cos[\sqrt{\pi}\varphi_{\rho+}]. \tag{91}$$

Pinning of $\varphi_{\rho+}$ at the impurity site leads to the suppression of the conductance. W_{intra} is relevant, i.e., an impurity eventually splits the wire into two disconnected wires for $K_{\rho+} < 4$. The correction to the conductance behaves as $-\delta G \propto w^2 \epsilon^{\frac{K_{\rho\pm}}{2}-2}$. This is to be contrasted with the case of a gapless Luttinger liquid, when the impurity is relevant only for repulsive interactions ($K < 1$). This reflects the fact that a real (gapped) charge-density-wave is pinned stronger than the fluctuating one (Luttinger liquid). Finally,a weak random potential localizes the CDW-wire for $K_{\rho+} < 6$. It is worth pointing out here that such large values of critical $K_{\rho+}$, separating localized from pure behavior, imply strong effective attraction between charge fluctuations in the $\rho+$ channel. It might well be that for such an attraction instead of CDW instability charge segregation will take place [52].

Thus, for electrons with spin, the Cooper phase is more stable to impurities than the CDW one, similar to a spinless case, but both are unstable in the physically relevant region of $K_{\rho+} < 1$.

6 Single-Particle Density of States

We now turn to the discussion of tunneling into a quantum wire. To the leading order in barrier transparency $\mathcal{T}$, the differential tunneling conductance is

$$G(V) = \frac{dI}{dV} = |\mathcal{T}|^2 \rho_c \rho(eV), \tag{92}$$

where ρ_c is the density of states (DOS) in the contact (which we assume to be energy independent), $\rho(\epsilon)$ is the DOS of the wire, and V is the applied voltage. When the wire is in the gapless Luttinger-liquid phase, $\rho(\epsilon) \propto |\epsilon|^{\beta}$. This behavior has recently been observed in tunneling into carbon nanotubes [3]. Tunneling into the edge of a fractional quantum Hall system also exhibits a power-law current-voltage dependence [53], which might be an indication of a chiral Luttinger-liquid state at the edge. Here, however, the situation is not that straightforward, and other explanations, different from a chiral Luttinger liquid, have also been suggested [54,55].

Suppose now that a two-subband quantum wire is in one of the possible gapped phases, i.e., CDW or Cooper phase. The goal of this Section is to analyze what would a tunneling experiment show in this case. The answer turns out to depend crucially on the geometry of the experiment. If the tunneling contact probes the interior of the wire, the gapped behavior is predicted: $G(V) = 0$ for $eV < \Delta$, where Δ is the appropriate energy gap. For $eV > \Delta$ the behavior is non-universal: the threshold behavior of ρ is determined by gapless charge and spin modes. More surprisingly, tunneling into the end of the CDW-wire exhibit a gapless behavior, similar to the Luttinger-liquid case. The tunneling exponent though is different from that for the gapless phase.

6.1 Tunneling Preliminaries

The local single-particle (or tunneling) density of states is given by

$$\rho(\omega, x) = -\frac{1}{\pi} Im\{G_{ret}(\omega, x)\}, \tag{93}$$

where $G_{ret}(\omega, x)$ is a Fourier transform of retarded Green's function $G_{ret}(t, x) = -i\Theta(t)\Sigma_s\langle\{\Psi_s(t, x), \Psi_s^\dagger(0, x)\}\rangle$. Representing the electron of the n-th subband as a sum of right- and left-movers and accounting for the orthogonality of transverse wavefunctions, the Green's function becomes $G_{ret}(t, x) = \sum_n [G_{ret}^{R_n}(t, x) + G_{ret}^{L_n}(t, x)]$, where the summation is over all occupied subbands and $G_{ret}^{N_n}(t, x) = -i\Theta(t)\ \Sigma_s\langle\{N_{ns}(t, x), N_{ns}^\dagger(0, x)\}\rangle$, $N = L, R$, see (13,14). The contribution of the off-diagonal terms $\sim \langle R_n L_n^\dagger\rangle$ is less singular and is thus neglected. $\rho(\omega, x)$ is a sum of contributions from right and left movers of all occupied subbands. To find $\rho(\omega, x)$, it is convenient to calculate first the Matsubara Green's function

$$G^R(\tau, x) = -\langle T_\tau R_s(\tau, x) R_s^\dagger(0, 0)\rangle, \tag{94}$$

and then to make the analytic continuation to real frequencies. Left- and right-moving fermions give identical contributions to ρ, thus the result obtained from (94) is simply multiplied by a factor of two at the end.

The key feature of gapped phases in multisubband 1D systems is the co-existence of gapped and gapless modes, which also makes the calculations to be slightly less trivial. The single-particle Green's function under these circumstances has recently been considered in Ref. [27,26], and our analysis follows largely these two papers.

6.2 Warm-Up: DOS of a Half-Filled Hubbard Chain

To warm up, we consider the simplest system in which gapped and gapless modes co-exist – a single-band Hubbard chain at half-filling. In the context of nanostructure physics, such a system is produced by imposing an artificial periodic potential of period a_0 over a quantum wire [31]. At half-filling, the Fermi-momentum $k_F = \pi/2a_0$ is commensurate with the reciprocal lattice spacing, which gives rise to Umklapp scattering. An Umklapp process occurs as simultaneous backscattering of two right- or left-moving electrons, the total momentum transferred to the lattice being $\pm 4 \times \pi/2a_0 = \pm 2\pi/a_0$. This process is responsible for opening of the (Mott-Hubbard) gap in the charge excitations spectrum. On the other hand, spin excitations remain gapless, and are described by the $SU(2)$-invariant Luttinger-liquid Hamiltonian ($K_\sigma = 1$).

The corresponding Hamiltonian of the charge sector is

$$H = \frac{1}{2} \int dx \{v_\rho K_\rho (\partial_x \theta_\rho)^2 + \frac{v_\rho}{K_\rho} (\partial_x \varphi_\rho)^2 + g \cos[\sqrt{8\pi}\varphi_\rho]\}, \tag{95}$$

where the last (cosine) term represents Umklapp scattering. Substituting Eq. (13) into Eq. (94), one finds that G^R factorizes into a product of spin and charge parts

$$G^R(\tau, x) = -\frac{sgn(\tau)}{2\pi\alpha}F_\sigma(\tau, x)F_\rho(\tau, x);$$

$$F_\nu = \langle \exp[i\sqrt{\frac{\pi}{2}}(\varphi_\nu(1) - \varphi_\nu(0))]\exp[-i\sqrt{\frac{\pi}{2}}(\theta_\nu(1) - \theta_\nu(0))]\rangle_\nu; \nu = \rho, \sigma \quad (96)$$

where shorthand notations $1 \equiv (\tau, x)$, $0 \equiv (0, 0)$ have been used. F_σ is gapless, whereas F_ρ contains massive fields.

Bosonic calculation To calculate F_ρ in the bosonic language, we adopt the semiclassical approximation, i.e., expand the cosine in (95) around its minimum to the second order in fluctuations. This is equivalent to replacing $g\cos[\sqrt{8\pi}\varphi_\rho] \to m^2\varphi_\rho^2$, which defines the mass $m^2 \equiv 4\pi g$. Now the averaging is straightforward:

$$F_\rho(\tau, x) = \quad (97)$$
$$\exp\left\{-\frac{\pi}{2}\int\frac{d^2k}{(2\pi)^2}(1 - \cos[k \cdot z])\Big(G_\theta(k) + G_\varphi(k) + 2iG_\theta(k)\frac{k_0k_1}{vKk_1^2 + m^2}\Big)\right\},$$

where $k \equiv (k_0, k_1) = (\omega_n, q)$, $z \equiv (\tau, x)$ and

$$G_\varphi(\omega_n, q) \equiv \langle T_\tau\varphi_\rho\varphi_\rho\rangle_{\omega_n q} = \frac{vK}{v^2q^2 + \omega_n^2 + m^2vK}. \quad (98)$$

The Green's function of θ_ρ–fields can be written as

$$G_\theta(\omega_n, q) \equiv \langle T_\tau\theta_\rho\theta_\rho\rangle_{\omega_n q} = \frac{v}{K}\Big(\frac{1}{v^2q^2 + \omega_n^2} + \frac{\omega_n^2}{v^2q^2}\frac{\bar{m}^2}{(v^2q^2 + \omega_n^2)(v^2q^2 + \omega_n^2 + \bar{m}^2)}\Big)$$
$$= G_\theta^{(1)}(\omega_n, q) + G_\theta^{(2)}(\omega_n, q), \quad (99)$$

where $\bar{m}^2 \equiv m^2vK$. The first term in Eq. (99) is just a free Green's function, whereas the second one is present only in strong-coupling phase and contains a strong infrared divergence at $q \to 0$. This divergence is often explained by the "uncertainty principle": in a gapped phase, a position-like field (φ_ρ) acquires an average value, hence its canonical conjugate, momentum-like field (θ_ρ) fluctuates strongly, hence its correlation function diverges. Let us analyze the Fourier transform of G_θ in more details, and define

$$I(x, \tau) = \int\frac{d^2k}{(2\pi)^2}(1 - \cos[k \cdot z])G_\theta^{(2)}(\omega, q). \quad (100)$$

Changing to polar coordinates $\omega_n = k\cos\phi, q = k\sin\phi, \tau = z\cos\vartheta, x = z\sin\vartheta$, we get

$$I(z, \vartheta) = \frac{\bar{m}^2z^2}{(2\pi)^2}\int_0^{2\pi}d\phi\frac{\cos^2\phi}{\sin^2\phi}\cos^2(\phi - \vartheta)\int_0^\infty\frac{dy}{y}\frac{1 - \cos y}{y^2 + \bar{m}^2z^2\cos^2(\phi - \vartheta)}. \quad (101)$$

The integral over ϕ diverges at $\phi = 0$, which, if taken literally, means that $I = \infty$ and thus $F_\rho(\tau, x = 0) = 0$ for any finite τ. However, this divergence is absent at $\vartheta = \pi/2$, which corresponds to $\tau = 0, z = x$. Let us therefore continue the calculation at this special point. Despite the cancellation of the infrared divergence, the integral is still controlled by the region of small ϕ: $\phi \sim 1/(\bar{m}x) \ll 1$. Expanding $\sin\phi \sim \phi$ and extending the limits of angular integration to $\pm\infty$, we find that $I(z, \vartheta = \pi/2) = \bar{m}x/4 + O(1/(\bar{m}x))$ for $\bar{m}x \gg 1$. Collecting regular contributions from G_φ and $G_\theta^{(1)}$, we find that the equal-time exponential correlator of θ_ρ–fields is given by

$$\langle e^{ia\theta_\rho(x,0)} \, e^{-ia\theta_\rho(0,0)} \rangle = \exp\left[-\frac{a^2}{4\pi K}\ln(x^2/\alpha^2) - \frac{a^2}{4K}\bar{m}x\right]. \qquad (102)$$

An important feature here is that the expected exponential decay of this correlator is modified by the power-law prefactor, given by the usual Luttinger-liquid correlator. Symbolically, $\langle e^{ia\theta_\rho(x,0)} \, e^{-ia\theta_\rho(0,0)} \rangle_{\bar{m}\neq 0} = \langle e^{ia\theta_\rho(x,0)} \, e^{-ia\theta_\rho(0,0)} \rangle_{\bar{m}=0}$ $\times \, e^{-\bar{m}x}$. One should be careful in using Eq.(102): the Luttinger-liquid parameter K, which appears here, should in fact be understood as the strong-coupling fixed-point value, K^*, which is often unknown.

Fortunately, the fixed-point value of K is known for a half-filled Hubbard chain: $K^* = 1/2$. For $\bar{m}x \gg 1$ then

$$\langle e^{i\sqrt{\frac{\pi}{2}}\varphi_\rho(x)} \, e^{-i\sqrt{\frac{\pi}{2}}\varphi_\rho(0)} \rangle \to const,$$
$$\langle e^{i\sqrt{\frac{\pi}{2}}\theta_\rho(x)} \, e^{-i\sqrt{\frac{\pi}{2}}\theta_\rho(0)} \rangle \sim \exp[-\pi\bar{m}x/4]/\sqrt{x}, \qquad (103)$$

and the full Green's function behaves as

$$G^R(0, x) \propto \frac{\exp[-\pi\bar{m}x/4]}{\sqrt{x}} \times \frac{1}{\sqrt{x}}, \qquad (104)$$

in agreement with Ref. [57]. The second $x^{-1/2}$-factor in Eq. (104) is due to gapless spin excitations.

So far all calculations have been straightforward. Now we would like to argue that the infrared divergence of $I(z, \vartheta \neq 0)$ is an artifact of the semiclassical approximation, which ignores degeneracy of $\cos[\sqrt{8\pi}\varphi]$ with respect to a uniform shift $\varphi \to \varphi + \sqrt{\frac{\pi}{2}}N$ with integer N. The proper theory of both massive and massless phases should be Lorentz-invariant. We thus propose that the correct result, valid for any $z = \sqrt{x^2 + v^2\tau^2}$, is given by Eq.(102) where x is replaced by Euclidian distance z: $x \to z$. Therefore,

$$\langle e^{ia\theta(x,\tau)} \, e^{-ia\theta(0,0)} \rangle = \exp\left[-\frac{a^2}{4\pi K}\ln(\frac{x^2 + v^2\tau^2}{\alpha^2}) - \frac{a^2}{4K}\bar{m}\sqrt{x^2 + v^2\tau^2}\right]. \quad (105)$$

Similar arguments in favor of such replacement were given by Voit [26].

We now use Eq. (105) to evaluate Eq. (97), and find [compare with (103)]

$$F_\rho(\tau, 0) \sim \sqrt{\frac{\alpha}{v|\tau|}} \exp[-\pi\bar{m}v|\tau|/4]. \qquad (106)$$

The spin sector average is non-zero and universal (thanks to $K_\sigma = 1$), therefore $F_\sigma(\tau, 0) \sim \sqrt{\alpha/v|\tau|}$. The corresponding DOS will be calculated later, see Eq.(117). The correctness of the procedure described above is verified in the next Section.

Re-fermionization. To check that our proposition makes sense, we now switch gears and derive Eq. (106) in a completely different way. To this end, we use the Luther-Emery refermionization procedure [58], which works for $K = 1/2$, that is at the fixed-point of a half-filled Hubbard chain. This procedure begins with an innocuous looking transformation $\varphi_\rho = \varphi/\sqrt{2}$, $\theta_\rho = \sqrt{2}\theta$, which changes Umklapp scattering in Eq.(95) into backscattering of some auxiliary particles (solitons): $\cos(\sqrt{8\pi}\varphi_\rho) = \cos(\sqrt{4\pi}\varphi)$. Right- and left-going solitons are defined by

$$\psi_\pm = \frac{1}{\sqrt{2\pi\alpha}} \exp\left\{\pm i\sqrt{\pi}(\varphi \mp \theta)\right\}. \tag{107}$$

In terms of new bosons, the original fermion operator (13) becomes

$$R_s = \frac{e^{-i\pi/8}}{\sqrt{2\pi\alpha}} \exp[is\sqrt{\frac{\pi}{2}}(\varphi_\sigma - \theta_\sigma)] \exp(i\frac{\sqrt{\pi}}{2}\varphi)\exp(-i\sqrt{\pi}\theta). \tag{108}$$

It can also be written in terms of solitons

$$R_s = e^{i\pi/8} \exp[is\sqrt{\frac{\pi}{2}}(\varphi_\sigma - \theta_\sigma)]\exp(-i\frac{\sqrt{\pi}}{2}\varphi)\ \psi_+, \tag{109}$$

where φ is expressed in terms of soliton density fluctuations as $(1/\sqrt{\pi})\partial_x\varphi =: \psi_+^\dagger\psi_+ + \psi_-^\dagger\psi_-$:. The usefulness of these formal manipulations is based on the fact that Hamiltonian (95) is quadratic in *massive* solitons $\psi_\pm$, and the mass (or the gap Δ) in their spectrum is determined by coupling constant g: $\Delta = g/(2\pi\alpha)$. Due to the presense of the gap, charge density fluctuations are suppressed, which results in the suppression of fluctuations of φ. Therefore, at energies below the gap the phase factor $\exp(-i\sqrt{\pi}\varphi/2)$ in Eq. (109) can be replaced by its average value. The charge part of the Green's function (F_ρ) coincides with the Green's function of massive fermions

$$F_\rho(\tau, x) \sim \int \frac{d\omega_n dq}{(2\pi)^2} e^{i\omega_n\tau + iqx}\left(-\frac{i\omega_n + vq}{\omega_n^2 + \Delta^2 + v^2q^2}\right). \tag{110}$$

At $x = 0$,

$$F_\rho(\tau, 0) \sim -\frac{1}{v}\int d\omega_n \frac{i\omega_n e^{i\omega\tau}}{\sqrt{\omega_n^2 + \Delta^2}} = -\frac{1}{v}\partial_\tau \int d\omega_n \frac{\cos(\omega_n\tau)}{\sqrt{\omega_n^2 + \Delta^2}}$$

$$= \frac{\Delta}{v}sgn(\tau)K_1(\Delta |\tau|). \tag{111}$$

Asymptotically, $F_\rho(\tau, 0) \sim (\Delta/v\sqrt{\Delta|\tau|})e^{-\Delta|\tau|}$, in agreement with our earlier proposition (106).

It follows from Eqs.(110)-(111) that upon continuing to real frequencies $F_\rho^{ret}(\omega = i\omega_n) = -(\omega/2v)\left(\Delta^2 - \omega^2\right)^{-1/2}$. Hence, the DOS of massive fermions is given by

$$\rho_\rho(\omega, x = 0) = \frac{1}{2\pi v}\Theta(\omega - \Delta)\frac{\omega}{\sqrt{\omega^2 - \Delta^2}}. \tag{112}$$

Up to a factor of $1/2$, which is due to the fact that this is the contribution of right-movers only, the obtained result is just the DOS of free massive particles with dispersion $\epsilon(q) = \sqrt{v^2 q^2 + \Delta^2}$. Since there are no particles above energy Δ at zero temperature, there are no interaction corrections to the density of states as well [59]. Another way of deriving this result consists in using the Ising-model representation of φ, θ fields (see Ref. [60] for details). In this representation,

$$\begin{aligned}\exp(-i\sqrt{\pi}\theta) &= \sigma_1\mu_2 - i\mu_1\sigma_2,\\ \exp(i\sqrt{\pi}\varphi) &= \mu_1\mu_2 + i\sigma_1\sigma_2,\end{aligned} \tag{113}$$

where μ_i (σ_i) are order (disorder) fields of the $d = 2$ Ising model, whose correlators are known. At long times, i.e., when $\Delta\tau \gg 1$,

$$\begin{aligned}\langle\mu_i(\tau)\mu_j(0)\rangle &\sim \delta_{ij}\langle\mu\rangle^2,\\ \langle\sigma_i(\tau)\sigma_j(0)\rangle &\sim \delta_{ij}K_0(\Delta|\tau|).\end{aligned} \tag{114}$$

As a result, $F_\rho(\tau) \sim sgn(\tau)K_0(\Delta|\tau|)$. Because of the condition $\Delta\tau \gg 1$, there is no discrepancy between Eqs. (114) and (111), since the leading asymptotic term of $K_\nu(x)$ is ν-independent. Hence one again finds a square-root singularity in $F_\rho^{ret}(\omega)$ for $\omega - \Delta \ll \Delta$. The correspondence with the Ising model allows one to estimate neglected terms as $e^{-3\Delta\tau}$.

A more illuminating way to understand the square-root singularity is provided by the real-time calculation. Starting from Eq.(111), it can be shown that [61]

$$\begin{aligned}Im[F_\rho^{ret}(\omega)] &= \frac{\Delta}{4\pi v}\int_{-\infty}^{\infty}dt\,e^{i\omega t}\left(K_1(-i\Delta t) - K_1(i\Delta t)\right)\\ &= \frac{\Delta}{2v}\int_0^{\infty}dt\,\sin(\omega t)Y_1(\Delta t) = \Theta(\omega - \Delta)\frac{\omega}{2v\sqrt{\omega^2 - \Delta^2}},\end{aligned} \tag{115}$$

where $Y_1(x)$ is the Bessel function of the second kind. Since asymptotically $Y_1(x) \sim \sin(x)/\sqrt{x}$, the origin of the singularity at $\omega = \Delta$ can be easily understood. For $\omega - \Delta \ll \Delta$, the integrand of (115) oscillates very slowly, with period $t_0 = 2\pi/(\omega - \Delta)$. The integral is thus determined by long times, $t \approx t_0$, and can be estimated as $\int_0^{t_0} 1/\sqrt{t} \sim \sqrt{t_0}$. We see that the threshold behavior of the DOS is determined by times much longer than $1/\Delta$, which justifies our use of the long-time asymptotics of Bessel functions to evaluate the DOS at $\omega \approx \Delta$.

DOS of a physical electron To find the density of states of a physical electron, we have to convolute Eq.(112) with the contribution of the gapless spin mode:

$$\rho(\omega, x = 0) = \int \frac{d\epsilon_n}{2\pi} F_\rho(\epsilon_n, x = 0) F_\sigma(\omega_n - \epsilon_n, x = 0)|_{\omega_n = -i\omega}$$

$$= \frac{2}{\pi} \int_0^\omega d\epsilon \, Im[F_\rho^{ret}(\epsilon)] Im[F_\sigma^{ret}(\omega - \epsilon)]. \tag{116}$$

Since $F_\sigma(\epsilon_n) \sim \sqrt{\alpha/iv\epsilon_n}$, we find

$$\rho(\omega, x = 0) = \frac{2}{\pi v}\sqrt{\frac{\alpha}{v}}\Theta(\omega - \Delta) \int_\Delta^\omega d\epsilon \frac{\epsilon}{\sqrt{\epsilon^2 - \Delta^2}}\frac{1}{\sqrt{\omega - \epsilon}}. \tag{117}$$

For $\omega - \Delta \ll \Delta$, where our derivation is valid,

$$\rho(\omega, x = 0) = \frac{\pi}{v}\sqrt{\frac{g}{v}}\,\Theta(\omega - \Delta). \tag{118}$$

Instead of a square-root singularity Eq. (112), the DOS of a physical electron exhibits a regular behavior approaching a finite value at the threshold. This modification is due to dressing of the gapped charge mode by gapless spin excitations. At energies much above the gap DOS increases, $\rho(\omega, x = 0) \sim \sqrt{\omega}$, which means that the spectral weight is shifted to higher energies. The energy-independent electron DOS near the threshold was obtained in [26,27]. Parenthetically, the functional form (118) remains valid when the spin channel is gapped as well. In this case the density of states of spin excitations is given by Eq.(112) with $\Delta \to \Delta_\sigma$. Doing the integral (116), we again find behavior (118) near the threshold $\Delta + \Delta_\sigma$. In other words, the gap in the electron's DOS is given by the sum of charge- and spin-sector gaps.

6.3 DOS of the Electron Ladder

We consider now the tunneling density of states of a two-channel wire. Application of Eq.(102) requires knowledge of strong-coupling fixed point values of K_ν, which are not known for a general case of two non-equivalent channels coupled by the Coulomb interaction. Just to illustrate what kind of behavior one might expect in this case, we consider an electron ladder with $K_{1\nu} = K_{2\nu}$ ($\nu = \rho, \sigma$) in the Cooper phase. The right-moving fermion is represented by

$$R_{n=1,s} = \frac{e^{ik_F x}}{\sqrt{2\pi\alpha}} e^{is\sqrt{\pi}(\varphi_{\sigma-} - \theta_{\sigma-})/2} e^{is\sqrt{\pi}(\varphi_{\sigma+} - \theta_{\sigma+})/2}$$

$$\times \, e^{i\sqrt{\pi}(\varphi_{\rho-} - \theta_{\rho-})/2} e^{i\sqrt{\pi}(\varphi_{\rho+} - \theta_{\rho+})/2}. \tag{119}$$

Now we apply Eq.(102) to the correlator $F(\tau) = -\langle T_\tau R_{1,s}(\tau, 0) R_{1,s}^\dagger(0,0)\rangle$. In the Cooper phase, $\theta_{\rho-}$ and $\varphi_{\sigma\pm}$ are gapped, hence their conjugates are exponentially suppressed. As a consequence, e.g.,

$$\langle e^{i\sqrt{\pi}(\varphi_{\sigma-}(\tau) - \theta_{\sigma-}(\tau))/2} e^{-i\sqrt{\pi}(\varphi_{\sigma-}(0) - \theta_{\sigma-}(0))/2}\rangle = C_\sigma\left(\frac{\alpha}{v_\sigma|\tau|}\right)^{1/8}\exp\left[-\frac{\pi m_\sigma v_\sigma|\tau|}{16}\right]. \tag{120}$$

On the other hand $\varphi_{\rho+}$ and $\theta_{\rho+}$ remain critical. As a result,

$$F(\tau) \propto sgn(\tau) \, |\tau|^{-\kappa} \exp\left[-\frac{\pi}{16}(2m_\sigma v_\sigma + m_{\rho-}v_{\rho-})|\tau|\right], \ \kappa = \frac{1}{8}(3 + K_{\rho+} + 1/K_{\rho+}).$$

(121)

Hence the DOS behaves as

$$\rho(\omega) \propto \frac{\Theta(\omega - \Delta_{SC})}{(\omega - \Delta_{SC})^\gamma}, \ \gamma = 1 - \kappa = \frac{1}{8}(5 - K_{\rho+} - 1/K_{\rho+}),$$

(122)

where $\Delta_{SC} = (\pi/16)(2m_\sigma v_\sigma + m_{\rho-}v_{\rho-})$. Note that $\gamma \leq 3/8 < 1/2$ for $K_{\rho+} \leq 1$.

Exactly at half-filling, when the $\rho+$ mode is also gapped due to Umklapp scattering and there are no more gapless modes, our procedure gives $F(\tau) \sim e^{-\Delta|\tau|}/\sqrt{|\tau|}$, in agreement with recent exact result [62]. The corresponding DOS is that of a free massive particle, $\rho_{hf}(\omega) \sim \Theta(\omega - \Delta)/\sqrt{\omega - \Delta}$.

Comparison of (122) with (118) shows that softening of the square-root singularity is less pronounced for the electron ladder than for the Hubbard chain, because there are three gapped and only one gapless mode now. As repulsion in the $\rho+$ channel becomes stronger, i.e., as $K_{\rho+}$ decreases, the singularity becomes weaker and disappears at $K_{\rho+} = (5 - \sqrt{21})/2 \approx 0.2$. For even smaller $K_{\rho+}$, we have $\rho(\omega = \Delta_{SC}) = 0$. This behavior, though, is not very realistic as it requires very strong repulsion. For weak repulsion, i.e., when $K_{\rho+} \approx 1$,

$$\rho(\omega) \propto \Theta(\omega - \Delta_{SC})/(\omega - \Delta_{SC})^{3/8},$$

(123)

and the threshold singularity is still present albeit softened compared to the free massive particle case. We note that $\rho(\omega)$ (122) is similar to the DOS of high unoccupied subbands of the wire, considered recently by Balents [63].

One would expect that the long-range order of the Cooper phase affect tunneling. Indeed one finds that *pair correlations* are determined by the Luttinger parameter of the total charge fluctuations only [18],

$$\langle R_{1,s}(\tau)L_{1,-s}(\tau) \, L_{1,-s}^\dagger(0)R_{1,s}^\dagger(0)\rangle \sim \langle e^{-i\sqrt{\pi}\theta_{\rho+}(\tau)}e^{i\sqrt{\pi}\theta_{\rho+}(0)}\rangle \sim \tau^{-1/(2K_{\rho+})},$$

(124)

whereas all other two-particle combinations decay exponentially. Thus, although the single-particle density of states is strictly zero, the two-particle one is not. In principle, this effect can be checked experimentally by tunneling into a two-channel wire from the superconducting tip - one should observe then a nonzero tunneling current of Cooper pairs. Its magnitude, however, will be much smaller than the current in a system of a normal tip and gapless wire, because the probability of two-particle (Copper pair) tunneling ($|\mathcal{T}|^4$) is exponentially smaller than that of single-particle tunneling ($|\mathcal{T}|^2$).

7 Tunneling into the End of a Gapped Wire

Tunneling into the end of a Luttinger liquid is different from tunneling into the bulk. The reason for this difference is the open boundary condition $\psi = 0$

for the electron wavefunction. For the boson modes describing charge and spin displacements, this condition means pinning at the boundary. The difference between the edge and bulk tunneling was considered first theoretically by Kane and Fisher [7], and has been recently observed in experiments with the tuneling into a carbon nanotube [3]. A rigorous treatment of a Luttinger liquid with open boundary conditions, which involves re-formulation of the bosonization procedure, can be found in Refs. [64–66].

Suppose now that a two-subband wire is driven into a CDW state by direct backscattering processes accompanied by density adjustment, as described in Sec. 3.1. The relative mode of charge excitations is described by Hamiltonian (46), in which we put $\delta k_F = 0$. From the equivalence of Eqs. (95) and (46) (with $\delta k_F = 0$), we expect the fixed point-value of K_- in the CDW-phase to be the same as for a half-filled Hubbard chain, i.e., $K_-^* = 1/2$. The total charge mode (φ_+) remains gapless and plays the same role as the spin mode of the Hubbard chain [see Eqs. (118) and (122)]: it softens the threshold singularity of the DOS. For $\omega < \Delta_{CDW}$, the density of states is equal to zero. Thus if the tunneling contact probes the interior of the wire, a gapped behavior is observed.

However, the DOS at the end of a wire exhibits *gapless* behavior, i.e, $\rho_{end} \propto |\omega|^{\alpha_d}$, as we will demonstrate in the rest of this Section. Consequently, $I(V) \propto |V|^{\alpha_d+1}$ for tunneling from a macroscopic (Fermi-liquid contact) into the end of the wire, and $I(V) \propto |V|^{2\alpha_d+1}$ for tunneling through a barrier located somewhere within the wire.

This very different behavior of the DOS in the bulk and at the end of the wire can be understood physically for tunneling through a barrier located within the wire (cf. Fig. 4). Without the barrier, the CDW is free to slide and the conductance is the same as in the absence of any interactions. Squeezing one electron into the middle of the wire leads to creation of a soliton-like compression in one of the two modes, and to accompanying it "stretch" in the other mode, which requires an energy of the order of the charge gap Δ_{CDW}. However, such an excitation needs not be created when the barrier distorts the uniform profile of the CDW. Indeed, the boundary condition imposed by the barrier pins the mode φ_- at $x = 0$ to a value which is different from the one in the bulk, $\varphi_-(x \to \infty) = \sqrt{\pi/8}$ (the latter follows from the minimization of the CDW energy, as illustrated in Fig.5). Therefore, the regular order of CDW is already frustrated near the barrier: there is a built-in compression in one of the modes, and depression in the other. The electron that tunnels through the barrier arrives into the "stretched"mode. Upon the proper shift of both modes, the system arrives into a state with the same energy but with a switched "polarity"of frustration. This consideration is true if the barrier is strong enough to destroy the CDW order in its vicinity, i.e., the barrier height is larger than Δ_{CDW}.

We now give a derivation for the current through the barrier. To model the boundary conditions corresponding to a wire cut into two semi-infinite pieces, we choose the potential barrier in the form $w(x, \mathbf{r}_\perp) = \pi \alpha W \delta(x)$, so that $W_{n=1,2} = \pi \alpha W$ and $W_{inter} = 0$ [cf. Eqs. (77,78)]. To find the current in, e.g., the 1st
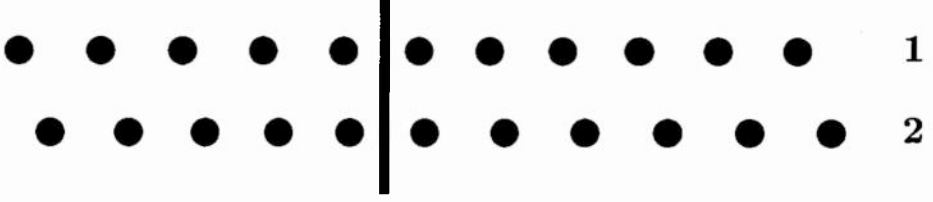
Fig. 5. Illustration of the CDW in the vicinity of a barrier : while the bulk value for φ_- is $\sqrt{\pi/8}$, the barrier forces φ_- to vanish at $x = 0$.

subband, through the barrier at $x = 0$

$$I \propto \lim_{t \to \infty} \langle \partial_t \varphi_1(x = 0, t) \rangle / e, \tag{125}$$

we need to calculate the rate at which $\varphi_1(x = 0, t)$ increases in the stationary limit in the presence of a potential drop $\sim -eV \int dx \ (\partial_x \varphi_+/2\sqrt{\pi})\mathrm{sgn}(x)$, proportional to the applied voltage V (due to their equivalence, it does not matter which of the two modes we use for measuring the current). Eq. (125) reduces the transport problem to that of the dynamics of a quantum particle $q(t) \equiv \varphi_1(x = 0, t)$ subject to "damping" by all of the remaining bulk degrees of freedom, including those of the second mode. Therefore, we can employ methods of dissipative quantum mechanics [67,7] to solve this problem.

The effective action of the boundary mode S_{eff} is obtained by tracing over these remaining degrees of freedom

$$e^{-S_{eff}[q]} = \int \mathcal{D}[\varphi_1]\mathcal{D}[\varphi_2] \, \delta[q(\tau) - \varphi_1(x = 0, \tau)] \, \delta[\varphi_2(x = 0, \tau)] \, e^{-S}, \tag{126}$$

where

$$S = \int dx \int d\tau \ \Big[\frac{1}{2v_F}(\partial_\tau \varphi_+)^2 + \frac{v_F}{2K_+^2}(\partial_x \varphi_+)^2 \tag{127}$$

$$+ \frac{1}{2v_F}(\partial_\tau \varphi_-)^2 + \frac{v_F}{2K_-^2}(\partial_x \varphi_-)^2 + \frac{4f_{bs}}{\pi\alpha^2}\varphi_-^2 \Big] , \tag{128}$$

where $\pm$ combinations are defined in (44). Performing the integration, we get

$$S_{\mathrm{eff}}[q] = \frac{1}{2} \int d\tau \int d\tau' \ q(\tau)\hat{\mathcal{K}}(\tau - \tau')q(\tau'), \tag{129}$$

which accounts for the influence of the bulk modes exactly. In (126) we assume a large barrier so that incoherent single electron passages through subband 1 determine the current, while subband 2 is fixed by the barrier, cf. (126). Simultaneous contributions from subband 2 represent coherence effects, and are of higher order in the barrier transmission coefficient. The dynamics of q is governed by

$$S_0 = \int d\tau \left(W \cos 2\sqrt{\pi}q - \frac{eV}{\sqrt{2\pi}}q \right) + S_{\mathrm{eff}}, \tag{130}$$

which includes the applied voltage. Eq. (130) describes a damped quantum particle on a tilted washboard potential. Tunneling between adjacent minima of the potential corresponds to single electron transfers.

For large W, Eq. (130) maps onto a tight binding model with nearest neighbor hopping $\mathcal{T}$, whose value can in principle be deduced from W, assuming a δ-barrier : it is renormalized by damping compared to the bare value [68]. In this analogy, (125) corresponds to the particle's mobility, studied in Ref. [69]. In leading order $\sim |\mathcal{T}|^2$, the result is

$$I(V) = e|\mathcal{T}|^2 \int_0^\infty dt \, \sin(eVt) \operatorname{Im} e^{-w(t)}, \tag{131}$$

where

$$w(t) = \int_0^\infty \frac{J(\omega)}{\omega^2}(1 - e^{-i\omega t}) \,. \tag{132}$$

The spectral function $J(\omega)$ is related to the Fourier transform $\mathcal{K}(\omega_n)$ of the kernel [67] appearing in (129),

$$J(\omega) = -\lim_{\eta \to 0} \operatorname{Im} \mathcal{K}(-i\omega + \eta) = \frac{\omega}{K_+} + \frac{\sqrt{\omega^2 - \omega_0^2}}{K_-} \Theta(\omega - \omega_0), \tag{133}$$

with $\omega_0^2 = 8f_{bs}v_F/\pi\alpha^2$. This yields

$$I(V) \sim V^{1/K_+ - 1}, \tag{134}$$

which gives $\alpha_d = 1/2K_+ - 1$ at voltages $V < \omega_0/e$. Above the gap, i.e, for $V \gg \omega_0/e$, gapless behavior of a two-subband Luttinger liquid is restored $I(V \gg \omega_0/e) \sim V^{1/K_+ + 1/K_- - 1}$.

The gapless behavior of fermion's DOS at the end of the wire can be interpreted in terms of a midgap state in the $(-)$ channel localized near the boundary. This is a bound state with zero energy formed in the potential well created by the static distortion of the φ_- field subject to an open boundary condition $\varphi_-(x = 0) = 0$, see Appendix D of Fabrizio and Gogolin paper [64]. Qualitatively, this is the same state which appears in a doped two-leg spin ladder, and which represents free $S = 1/2$ spin induced by a charged impurity, see, e.g., [70]. As a result, *local* density of states of the $(-)$ channel at the end of the wire takes the form $\rho_-(\epsilon) \sim \lambda\delta(\epsilon) + \rho_-^{reg}$, where ρ_-^{reg} stands for contribution of massive modes with energies *above* the CDW gap. On the other hand the $(+)$ mode remains gapless, and its end-chain DOS is given by $\rho_+(x = 0, \omega) \sim \omega^{\frac{1}{2K_+} - 1}$, [7,64,66]. The factor of 1/2 in the exponent is due to "factorization" of the electron operator into $(\pm)$ modes. Another consequence of such factorization is that DOS of a physical electron is a convolution of the DOS of the $(\pm)$ channels [cf. Eq. (116)]

$$\rho_{phys}(x = 0, \omega) \sim \int_0^\omega d\epsilon \rho_+(x = 0, \epsilon)\rho_-(x = 0, \omega - \epsilon). \tag{135}$$

Hence,

$$\rho_{phys}(x=0,\omega) \sim \rho_+(x=0,\omega) = \omega^{\frac{1}{2K_+}-1}, \tag{136}$$

which implies $\alpha_d = 1/(2K_+) - 1$, in agreement with the result of the explicit calculation of $I(V)$ presented above, (134).

8 Experimental Consequencies and Conclusions

Apart from the rather well-known Luttinger-liquid phase, a two-subband quantum wire may also exhibit either a CDW or a (Cooper) superconducting phase. In both of these phases, certain modes of inter-subband charge- and spin-excitations are gapped, whereas the center-of-mass charge mode remains gapless. As a result, the conductance remains at the universal value of $2e^2/h$ per occupied subband, irrespective of whether the wire is in a gapless or gapped phase. However, the single-particle density of states in the middle of the wire has a hard gap. Above the gap, the DOS exhibits a non-universal threshold behavior $\rho(\omega) \sim \Theta(\omega-\Delta)(\omega-\Delta)^{-\beta}$, where $\beta \leq 1/2$. Softening of the threshold singularity is due "dressing" of the gapped modes by the remaining gapless one.

We find that the DOS for tunneling into the end of a wire in the CDW phase remains gapless, with the exponent determined by the center-of-mass mode only. This effect is due to frustration introduced into the CDW order by an open boundary (strong barrier).

Where should one look for such exotic phases of a quantum wire? We believe that quantum wires [11] prepared by cleaved edge overgrowth technique may be well-suited for observing the CDW phase. Indeed, the cross-section of such a wire is close to a square, which implies that the lowest states of transverse quantization should be close in energy. Hence, one-dimensional subbands can have close Fermi-momenta, which is a necessary condition for the formation of the CDW state. The Cooper phase, on the other hand, requires effective attraction in the relative charge channel, and has the best chance to occur when the second (upper) subband just opens for conduction. i.e., near the Van Hove singularity.

Acknowledgements

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