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Charge and Spin Dynamics of Interacting Fermions in a One-Dimensional Harmonic Trap

Lars Kecke,¹ Hermann Grabert,¹ and Wolfgang Häusler²

¹*Physikalisches Institut, Universität Freiburg, Hermann-Herder-Str. 3, D-79104 Freiburg, Germany*

²*Institut für Theoretische Physik, Universität Erlangen, Staudtstr. 7, D-91058 Erlangen, Germany*

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We study an atomic Fermi gas interacting through repulsive contact forces in a one-dimensional harmonic trap. Bethe-ansatz solutions lead to an inhomogeneous Tomonaga-Luttinger model for the low energy excitations. The equations of motion for charge and spin density waves are analyzed both near the trap center and near the trap edges. While the center shows conventional spin-charge separation, the edges cause a giant increase of the separation between these modes.

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Recent advances in cooling technology have allowed us to reach the degenerate regime of Fermionic quantum gases. Different hyperfine states effectively correspond to spin polarizations of spin- $\frac{1}{2}$ particles of respective densities n_\uparrow and n_\downarrow [1]. Furthermore, the spatial dimensionality of the gases can be reduced using either a hollow beam setup [2] or arrays of microtraps [3] so that one-dimensional (1D) gases can be studied, where interaction effects are known to be most pronounced. Away from p -wave resonances [4], one has pure s -wave scattering [5], implying short-ranged interactions [6] acting between particles of opposite spins only. Also, the strength of the forces can, in principle, be varied over wide ranges [7] by tuning the Feshbach resonance [8]. An optical lattice along the trap can further enhance correlations by reducing the bandwidth and thus the kinetic energy [9,10].

As opposed to higher dimensions, total charge ($\rho = n_\uparrow + n_\downarrow$) and relative spin ($\sigma = n_\uparrow - n_\downarrow$) density waves comprise the only low energy excitations in 1D; no fermionic quasiparticles can be excited in this regime. It is generally believed that the low energy physics of homogeneous, gapless 1D systems is described by the Tomonaga-Luttinger liquid (TLL) theory [11,12] entailing that charge and spin modes move at different velocities when interactions are present. This spin-charge separation is considered as a hallmark of TLL behavior. A parabolic trap potential, however, causes the particle density to vary along the trap. To treat such an inhomogeneous gas cloud, new Boson representations of the Fermi operators have been introduced [13]. Here, we follow the other route put forward recently by Recati *et al.* [9] and consider an inhomogeneous TLL [14] with x -dependent parameters, assuming a trap potential which is slowly varying on the scale of the Fermi wavelength.

In the bulk of the gas cloud, away from the trap edges, interactions are sufficiently weak to justify perturbative estimates to the (local) TLL model parameters. In this regime, the picture of charge and spin waves moving at different velocities is recovered [9,15]. Near the trap edges, however, the gas density decreases and interactions become very strong. In this regime we employ the Bethe-

ansatz solution for spin- $\frac{1}{2}$ particles with contact forces [16–18] to compute the TLL parameters exactly [19,20]. To leading order in the inverse interaction strength we can access the dynamics analytically. Charge modes are found to propagate in a similar way as in the bulk of the gas cloud. At the edges they are reflected and thus keep oscillating in the trap until damping processes become significant. Remarkably, spin density waves show an exponential slowing down of their velocity and, ultimately, accumulation at an edge without reflection. Thus, fermionic 1D quantum gases in shallow confinements should exhibit a giant increase of spin-charge separation, much more pronounced than electrons in quantum wires [21]. The effect of spin accumulation should be detectable: one way would be to observe (e.g., by fluorescence measurements) the time evolution of an initial spin-up density peak containing charge and spin modes of equal amounts, $\rho_\uparrow = (\rho + \sigma)/2$.

Without parabolic confinement the quantum gas is described by the Hamiltonian ($\hbar = 1$)

$$H_{\text{hom}} = \frac{1}{2m} \left[\sum_{i=1}^N -\frac{\partial^2}{\partial x_i^2} + 2c \sum_{i < j} \delta(x_i - x_j) \right]. \quad (1)$$

Effectively, by virtue of the Pauli principle, only fermions of opposite spins are interacting. As long as the 3D scattering length a is much smaller than the transversal width d of the gas cloud, the interaction strength $c = 2a/d^2$ [5].

For given particle density n , the ground state energy density $\epsilon_{\text{hom}}(n)$ is obtained exactly from the Hamiltonian (1) via the Bethe ansatz [17]. In presence of the longitudinal trap potential $V(x)$, we determine the inhomogeneous profile of the particle density $n(x)$ in the spirit of the local density approximation (LDA) [9] by minimizing

$$E[n] = \int dx \{ \epsilon_{\text{hom}}[n(x)] + V(x)n(x) - \mu n(x) \}. \quad (2)$$

This is adequate for slow variations of V compared to the Fermi wavelength and for large particle numbers, $N \gg 1$, where we can ignore Friedel oscillations occurring on the scale of the Fermi wavelength $\pi/2k_F$ [13]. The chemical potential μ ensures that $N = \int dx n(x)$ and determines the length $2R$ of the gas cloud through $V(R) = \mu$, independent

of $V(|x| < R)$ and the form of $\epsilon_{\text{hom}}(n)$, provided $\partial_n \epsilon_{\text{hom}}(n=0) = 0$ [22]. Assuming now a harmonic trap potential $V(x) = \frac{m}{2} \omega_T^2 x^2$ of frequency ω_T , one finds from Eq. (2)

$$\partial_n \epsilon_{\text{hom}}[n(x)] + \frac{m}{2} \omega_T^2 x^2 - \mu = 0. \quad (3)$$

The resulting density profile $n(x)$ is depicted in Fig. 1. For spinless Bosons a similar calculation based on the Bethe ansatz has been carried out in Ref. [23].

To proceed we now focus on two regions of particular interest. Near the center of the trap, where the density n is large enough to validate perturbative expressions in the interaction strength, one finds to leading order in c/n [24]

$$\epsilon_{\text{pert}}(n) = \frac{\pi^2 n^3}{24m} \left(1 + \frac{c}{\pi^2 n} \right). \quad (4)$$

Inserting (4) into (3) gives a modified Thomas-Fermi profile

$$n_{\text{pert}}(\xi) = \frac{2m}{\pi} v_{F0} \left(\sqrt{1 + u^2 - \xi^2} - u \right). \quad (5)$$

Here we have introduced the dimensionless coordinate $\xi = x/R$ and interaction strength $u = c/m\pi v_{F0}$ where $v_{F0} = R\omega_T$ is the Fermi velocity of a homogeneous, non-interacting system at chemical potential μ . On the other hand, near the trap edges, $c \gg n$ so that we can evaluate the Bethe-ansatz solution to leading order in n/c , yielding [9]

$$\epsilon_{\text{edge}}(n) = \frac{\pi^2 n^3}{6m} \left(1 - \frac{4n}{c} \ln 2 \right). \quad (6)$$

Inserting (6) into (3) gives

$$n_{\text{edge}}(\xi) = \frac{2m}{\pi} \frac{v_{F0}}{2} \left[\sqrt{1 - \xi^2} + \frac{8 \ln 2}{3\pi^2 u} (1 - \xi^2) \right]. \quad (7)$$

As seen in Fig. 1, the analytic forms (5) and (7) describe the trap density accurately in their regions of validity. We

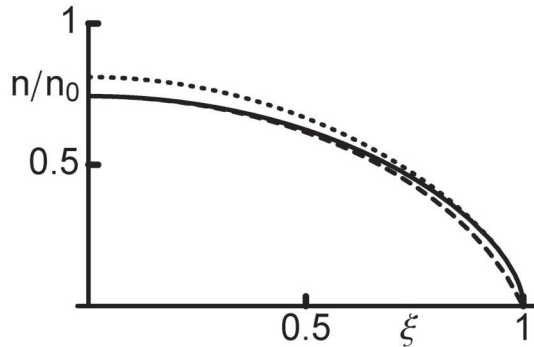


FIG. 1. LDA density profile of interacting Fermions in a trap of length $2R$ for dimensionless interaction strength $u = 0.3$, using the full Bethe-ansatz equations numerically (solid line), Eq. (5) (dashed line), and Eq. (7) (dotted line); $n_0 = \frac{2}{\pi} m \omega_T R$ is the density of a noninteracting Fermi gas at the same μ .

stress that $n_{\text{edge}}(\xi)$ vanishes with infinite slope at the edges $\xi \rightarrow \pm 1$, which is not seen from the perturbative expression $n_{\text{pert}}(\xi)$.

The inhomogeneous TLL Hamiltonian is of the form

$$H_{\text{TLL}} = \frac{1}{2} \sum_{\nu=\rho,\sigma} \int dx [v_{J\nu}(x) (\Theta'_\nu(x))^2 + v_{N\nu}(x) (\Phi'_\nu(x))^2], \quad (8)$$

where primes denote spatial derivatives, and Θ_ν and Φ_ν are the usual Bosonic phase fields in the charge ($\nu = \rho$) and spin ($\nu = \sigma$) sectors, obeying $[\Phi_\nu(x), \Theta'_{\nu'}(x')] = i\delta_{\nu\nu'} \delta(x - x')$. They define density excitations $\nu = \langle \Phi'_\nu \rangle / \sqrt{\pi}$. The velocity parameters $v_{N\nu}(x)$ follow from exact compressibility relations [12,21,25]

$$v_{N\nu} = \frac{2}{\pi} \frac{\partial^2 \epsilon_{\text{hom}}}{\partial \nu^2}, \quad (9)$$

while symmetries fix the values of velocity parameters $v_{J\nu}$: $v_{J\rho} = \pi n / 2m$ owing to Galilei invariance and $v_{J\sigma} = v_{N\sigma}$ as a consequence of SU(2) invariance in the spin sector. The model (8) can be justified if $\nu \ll n$ which, evidently, fails too close to the trap edges where $n \rightarrow 0$, see below.

After separating off the time dependence $\sim e^{i\omega t}$ in the Heisenberg equations of motion, the eigenvalue equations for the density excitations become

$$-v_{N\nu} v_{J\nu} \nu'' - (2v'_{N\nu} v_{J\nu} + v_{N\nu} v'_{J\nu}) \nu' - (v'_{N\nu} v'_{J\nu} + v''_{N\nu} v_{J\nu}) \nu = \omega^2 \nu. \quad (10)$$

As boundary conditions, we require that charge and spin currents vanish at the trap edges, $j_\nu(x = \pm R) = 0$. In view of (8) and the identity $j_\nu = \Theta'_\nu / \sqrt{\pi}$, this boundary condition becomes $v_{J\nu} v'_{N\nu} \nu + v_{J\nu} v_{N\nu} \nu' = 0$ at $\xi = \pm 1$ which, together with (10), governs the dynamics of density wave packets. In the homogeneous case, $v'_{(N/J)\nu}$ and $v''_{(N/J)\nu}$ vanish and (10) simplifies to a wave equation. In the inhomogeneous case, without interactions, all four TLL velocity parameters are equal and given by $v_F(\xi) = v_{F0} \sqrt{1 - \xi^2}$, which follows from Eqs. (3) and (9) for $\epsilon_{\text{hom}}(n) = \pi^2 n^3 / 24m$. Then the solutions of (10) [26]

$$\nu_{\text{nonint}}(x, \omega) = \frac{1}{\sqrt{R^2 - x^2}} \cos(\bar{\omega} \arccos \xi), \quad (11)$$

with integer $\bar{\omega} = \omega / \omega_T$, constitute a complete orthonormal set with regard to the measure $\frac{2R}{\pi \sqrt{R^2 - x^2}}$ [27].

Near the trap center, we obtain from Eqs. (5) and (9) to leading order in u

$$v_{J\rho} = v_F = v_{F0} \left(\sqrt{1 + u^2 - \xi^2} - u \right) \quad (12)$$

$$v_{N\rho} = v_F + u v_{F0} = v_{F0} \sqrt{1 + u^2 - \xi^2} \quad (13)$$

$$v_\sigma = \sqrt{v_F(v_F - uv_{F0})} \approx v_{\sigma 0} \sqrt{1 - x^2/R_\sigma^2}, \quad (14)$$

where $v_{\sigma 0} = v_{F0} \sqrt{1 - 3u}$ and $R_\sigma = R \sqrt{1 - 3u/2}$. The solutions of (10) now become

$$\rho_n(x) = \frac{n/2}{\sqrt{(1+u^2)R^2 - x^2}} C_n^{(h/2)}\left(\frac{x}{R_2}\right) \quad (15)$$

$$\sigma_n(x) = \frac{1}{\sqrt{R_\sigma^2 - x^2}} \cos\left(n \arccos \frac{x}{R_\sigma}\right), \quad (16)$$

where the $C_n^{(\alpha)}$ denote ultraspherical polynomials [28], $R_2 = R \sqrt{1 - u/2}$, $h = u/(2 - u)$, and the eigenfrequencies are $\omega_{\rho,n} = \omega_T \sqrt{n(n+h)/(1+h)}$, $\omega_{\sigma,n} = \omega_T n \sqrt{(1-3u)(1-3u/2)}$. Thus, in the charge sector the spectrum is no more equidistant (see also Refs. [9,22]), although the lowest mode ($n=1$) remains unaffected by interactions in accordance with Kohn's theorem [29]. Equations (15) and (16) describe how weak interactions smoothly deform the noninteracting solution (11); note that $C_n^{(0)}(\xi) = \frac{2}{n} \cos(n \arccos \xi)$. The inequalities $R_2 \omega_{\rho,n} > n v_{F0}$ and $R_\sigma \omega_{\sigma,n} < n v_{F0}$ reflect the enhanced and suppressed dynamics of charges and spins, respectively.

Near the trap edges, in the nonperturbative regime where $\sqrt{R^2 - x^2} \ll uR$ (but still $R - |x| \gg k_F^{-1}$), interactions alter the above picture, even qualitatively. Using Eqs. (6) and (7), and employing the Bethe-ansatz result by Coll [20], we get from (9) to leading order in $u^{-1} \sqrt{1 - \xi^2}$

$$v_{J\rho} = v_{F0} \left[\sqrt{1 - \xi^2}/2 + \beta(1 - \xi^2)/2 \right] \quad (17)$$

$$v_{N\rho} = v_{F0} \left[2\sqrt{1 - \xi^2} - 4\beta(1 - \xi^2) \right] \quad (18)$$

$$v_\sigma = \frac{4\pi m v_F^2}{3c} = \frac{v_{F0}}{3u} (1 - \xi^2), \quad (19)$$

where $\beta = 8 \ln 2 / (3\pi^2 u)$. Disregarding for the moment subleading terms $\propto \beta$ in Eqs. (17) and (18), we see that the charge sector exponent $K_\rho = \sqrt{v_{J\rho}/v_{N\rho}}$ approaches the value 1/2 close to the edges, which is consistent with the limiting value of K_ρ in the Hubbard model when the filling goes to zero [30]. The plasmon velocity $\sqrt{v_{N\rho} v_{J\rho}}$, on the other hand, takes exactly the value of the noninteracting system [cf. Eqs. (12) and (13) for $u \rightarrow 0$]. This amazing property originates from the fact that the particle density is reduced by a factor 1/2 at infinite interactions, compared to the noninteracting system. Furthermore, since Eq. (10) only contains products $v_{N\rho} v_{J\rho}$ and their spatial derivatives, we conclude that even the whole charge dynamics coincides exactly with the noninteracting dynamics (11) sufficiently close to the trap edges.

To leading nontrivial order in β , Eq. (10) can be transformed into a damped Mathieu equation that is solved by

$$\rho_n(x) = \frac{\exp(\frac{3}{2}\beta\sqrt{1-\xi^2})}{\sqrt{R^2 - x^2}} \begin{cases} se_{2n}\left(q, \frac{1}{2} \arcsin \xi\right), & n \text{ odd} \\ ce_{2n}\left(q, \frac{1}{2} \arcsin \xi\right), & n \text{ even} \end{cases}, \quad (20)$$

with the Mathieu functions se_n and ce_n . The eigenenergies ω_n follow from the characteristic values [28] $a_{2n}(q_n) = 4\omega_n^2 - 9\beta^2$ of the Mathieu functions while $q_n = -\beta(2\omega_n^2 - 1)$. When $\beta\sqrt{1 - \xi^2} \rightarrow 0$, the $\rho_n(x)$ turn into the noninteracting solutions (11), in agreement with the above observations for the vicinity of the trap edges.

In the spin sector the corresponding solutions,

$$\sigma(x, \omega) = \frac{R}{R^2 - x^2} \left\{ \frac{\sin}{\cos} \right\} \left(\frac{3u\omega}{\omega_T} \operatorname{atanh} \xi \right), \quad (21)$$

differ qualitatively from the noninteracting solutions (11), as well as from the weakly interacting ones (16). They are rapidly oscillating near the trap edges which manifests extremely slow spatial propagation of spin density waves, as we discuss now.

With the eigenfunctions for charge and spin densities at hand, we are in the position to solve for the time evolution of an initial wave packet $\nu_0(x)$ in the trap center, or near the

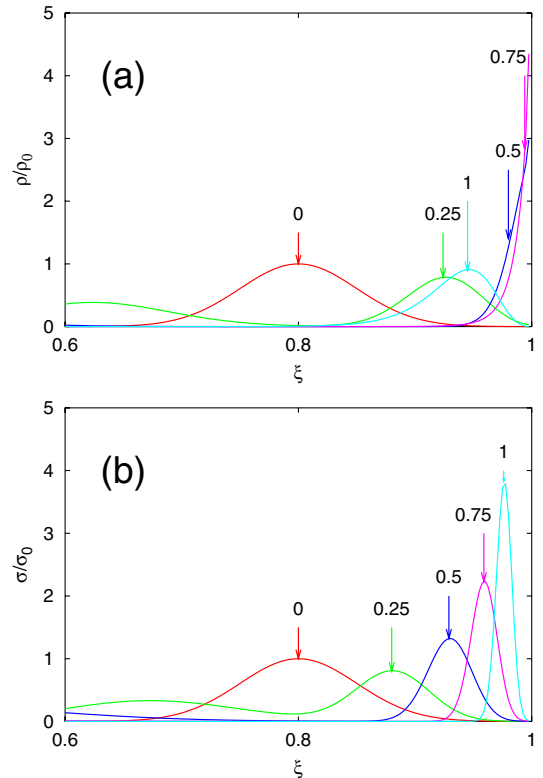


FIG. 2 (color online). Propagation of a charge density (a) and a spin density (b) excitation for dimensionless interaction strength $u = 0.3$. Time steps are $\Delta t \omega_T = 0.25$ as indicated. The charge wave is reflected at the edge of the trap, similar to the noninteracting case, while the spin wave exponentially slows down without reflection.

edge. After preparation, the packet splits into left- and right-moving parts, ν_+ and ν_- , and the charge and spin constituents separate. Near the edge the time evolution follows

$$\rho_{\pm}(x, t) \approx \frac{1}{2} \sqrt{\frac{R^2 - x'^2}{R^2 - x^2}} \rho_0(x')$$

$$x' = R \cos(\arccos \xi \pm \omega_T t) \quad (22)$$

$$\sigma_{\pm}(x, t) = \frac{1}{2} \frac{R^2 - x'^2}{R^2 - x^2} \sigma_0(x')$$

$$x' = R \tanh\left(\operatorname{atanh} \xi \pm \frac{\omega_T}{3u} t\right). \quad (23)$$

Subleading corrections of order β require a summation over the eigenfunctions (20) and can no longer be expressed in the simple form (22).

It follows from Eq. (22) that the charge density exhibits a temporal evolution that roughly resembles that in the absence of interactions: the right-moving part of the initial wave packet slows down somewhat due to the decreasing Fermi velocity, and then is reflected at the edge [cf. Fig. 2(a)]. We thus expect charge density excitations to keep oscillating in the trap before damping mechanisms set in. On the contrary, the right-moving part of the spin density wave is slowed down even exponentially fast, see Eq. (23), and is not reflected [cf. Fig. 2(b)]. We mention that for long times Eq. (23) gives $\sigma_{\pm}(x, t \rightarrow \infty) \rightarrow A_0 \delta(x \pm R)$ where $A_0 = \int dx \sigma_0(x)$ is the conserved magnitude of the initial spin wave packet. This follows from the finite slope of ν_{σ} at $\xi = \pm 1$, as opposed to the infinite slope $\nu'_{\rho(N/I)}$ at $\xi = \pm 1$ in the charge sector. However, our analysis is only valid up to times of order $t_c \approx \frac{u}{\omega_T} \times [\ln(\frac{2mR\omega_T}{\pi\sigma_0(\xi_0)} \sqrt{1 - \xi_0^2}) + 3 \ln \frac{2}{1 + \xi_0}]$ since then $\sigma_{\pm}(x, t > t_c)$ exceeds $n(x)$. This time is large for small initial displacements σ_0 so that a dramatic separation of the spin and charge peaks will have occurred well before effects on the length scale of the Fermi wavelength, that have been disregarded here, become relevant.

In conclusion, we have studied interacting fermionic atoms of two spin species in an effectively one-dimensional harmonic trap by exploiting solvability by the Bethe ansatz for contact forces. We have evaluated the particle density profile within the local density approximation. Near the trap center we confirm the occurrence of spin-charge separation. Near the edges of the trap, interactions affect the dynamics in an unexpectedly drastic way. While in the immediate vicinity of an edge charge density, waves move as if they were noninteracting with reflection at the edge, spin density waves are not reflected and accumulate at the edge. This establishes an even more pronounced spin-charge separation than in homogeneous systems. Experimentally one could confirm our predictions, e.g., by selectively evaporating parts of the “up” spin component ρ_{\uparrow} in a small region of space and observe

the right- and left-moving spin and charge density waves separating from one another (since $2\rho_{\uparrow} = \rho + \sigma$), both in the bulk and near the edge of the trap.

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