Symmetric translation nets

By *Dieter Jungnickel* at Gießen

1. Introduction

Let Σ be an incidence structure with a parallelism $\|$ (i.e. an equivalence relation on the block set of Σ such that each $\|$-class partitions the point set of Σ). Σ is called an $(s, r; \mu)$-net provided that any two non-parallel blocks intersect in precisely μ points and that there are r parallel classes each of which has s blocks in it. It is easily seen that an $(s, r; \mu)$-net is the same as an affine 1-$(s^2\mu, s\mu, r)$-design. Such structures have found considerable interest; in case $\mu=1$ they are just the well-known Bruck nets (see [4], [5]) which are equivalent to mutually orthogonal sets of Latin squares providing a very important tool in the construction of block designs. We mention two special classes of $(s, r; \mu)$-nets: A *complete* (s, μ)-net is an $(s, r; \mu)$-net with $r=\frac{s^2\mu-1}{s-1}$. This is the maximum value of r one may have in any $(s, r; \mu)$-net and it is reached in fact precisely when simultaneously Σ is an (affine) 2-design. For a proof, see e.g. the survey paper of Mavron [14] who also discusses the second special class of nets we want to consider, i.e. *symmetric* (s, μ)-nets: These are $(s, s\mu; \mu)$-nets whose dual is likewise an $(s, s\mu; \mu)$-net.

The author has studied nets with various types of collineation groups in a series of papers (for references, cf. [9]). Here we will be concerned once more with *translation* nets, i.e. with $(s, r; \mu)$-nets admitting a collineation group G acting regularly on the point set and fixing each parallel class of Σ. Translation nets have been studied in case $\mu=1$ by Sprague [16] and in general by the author in [9]. There we have given lower and upper bounds on r for various types of translation groups (abelian, nilpotent,...). The main result of this paper was that the maximum value for r in a non-elementary abelian p-group is roughly only the p-th part of that in the corresponding elementary abelian group (see [9], Section 4). Using this and the result of Schulz [15] who proved that any translation 2-design has a p-group (in fact of exponent p) as its translation group we obtained:

Theorem 1. *An affine translation 2-design with parameters (s, μ) and translation group G exists iff*

(i) *s is a prime power and μ a power of s;*

(ii) *G is elementary abelian.*

0075-4102/82/0335-0010$02.00

Previously, this was only known if one had required the existence of a non-trivial central dilatation. Regarding symmetric nets, we had shown

Lemma 1. *A symmetric translation (s, μ)-net with nilpotent translation group G exists iff*

(i) *s and μ are powers of the same prime p;*

(ii) *G is elementary abelian.*

We conjectured that the hypothesis that G be nilpotent is unnecessary in this case, too. We shall now use arguments in analogy to those of Schulz [15] to show the following

Lemma 2. *The translation group of a symmetric translation net is necessarily a p-group.*

Combining Lemmas 1 and 2, we immediately have

Theorem 2. *A symmetric translation (s, μ)-net with translation group G exists iff*

(i) *s and μ are powers of the same prime p;*

(ii) *G is elementary abelian.*

The constructive part of this assertion was already given in [10]. We will now proceed to proving Lemma 2 using the classification of all finite groups with a partition (i.e. a set of subgroups which pairwise intersect in 1 only and which cover the group).

2. The proof

Let Σ be a symmetric translation (s, μ)-net with translation group G. Then Σ may be represented in the following way (see [9]): points are the elements of G and blocks are the cosets of a family $\mathcal{U} = \{U_1, \ldots, U_{s\mu}\}$ of subgroups of G satisfying

(1) $|U_i| = s\mu$ for $i = 1, \ldots, s\mu$;

(2) $|U_i \cap U_j| = \mu$ whenever $i \neq j$;

and then also

(3) $U_i U_j = G$ whenever $i \neq j$.

(In fact one chooses the U_i to be the stabilizers in G of the blocks through 1). We now denote by N the set of all points not joined to 1, i.e.

(4) $$N = \{g \in G:\ 1 \text{ and } g \text{ are not joined}\}.$$

As Σ is a symmetric net, not being joined it induces an equivalence relation on the point set; using this, one immediately sees that N is a subgroup (of order s) of G, as collineations preserve the property of being (not) joined.

Next recall that a *line* of Σ is the intersection of all blocks joining two given points (which are on a common block at all). Lines either are equal or intersect in at most 1 point. Now it is easily seen that N together with all stabilizers G_L (where L is any line through 1) forms a partition $\mathcal{P}$ of G. Assume first that $\mu=1$; then in fact $\mathcal{P}=\mathcal{U}\cup\{N\}$ is a congruence partition in the sense of André [1], i.e. $\mathcal{P}$ describes an affine translation plane. It is well-known that G then has to be an (elementary abelian) p-group.

Thus assume $\mu\neq 1$ henceforth. As $\mathcal{P}$ is a (non-trivial) partition of G, the results of Baer [2], [3], Kegel [11] and Suzuki [17] imply that G is one of the following: a p-group of order $>p$, a Frobenius group, an $HT(p)$-group or isomorphic to S_4, to $PGL(2,p^n)$ or $PSL(2,p^n)$ with $p^n\geqq 4$, or to a Suzuki group $Sz(q)$. We will eliminate all but the first case, thus proving Lemma 2.

First assume that G is a Frobenius group; let K be its kernel and H a Frobenius complement of G. Using Folgerung 4. 9 of Baer [2] H has no non-trivial partition; thus H is contained in a component X of $\mathcal{P}$. Let a denote the order of H and b the order of K; then $a|b-1$ and in particular $(a,b)=1$. Now assume first that $X\subset N$; then $a|s$ and $b=s\mu\dfrac{s}{a}$ contradicting $(a,b)=1$. Thus H is contained in a line stabilizer, hence in a block stabilizer U; therefore $a|s\mu$. Again $b=s\dfrac{s\mu}{a}$ and as $(a,b)=1$ we conclude that $(a,s)=1$, i.e. $a|\mu$, say $\mu=ac$. Then $b=s^2c$ and $(a,c)=1$. Now consider the set $\mathcal{V}=\{U_i\cap K\colon i=1,\dots,s\mu\}$. Note that each $U_i\cap K$ has order sc, as $U_iK=G$ for reasons of cardinality; also always $|(U_i\cap K)\cap(U_j\cap K)|=|(U_i\cap U_j)\cap K|=c \quad (i\neq j)$ using (2) and $(U_i\cap U_j)K=G$ (again for reasons of cardinality: $a|\mu$ and $s^2c=|K|$). But this shows that $\mathcal{V}$ satisfies conditions (1) and (2) with μ replaced by c; thus $\mathcal{V}$ defines an $(s, s\mu; c)$-net. But $N<K$, as no Frobenius complement may intersect N non-trivially (because of $(s,a)=1$); hence this new net still contains points which are not joined. But this implies (see [7]) that $r\leqq sc<s\mu$, a contradiction.

Next assume that G is an $HT(p)$-group, i.e. G is neither a Frobenius nor a p-group, and one has $[G:H_p(G)]=p$ (where $H_p(G)$ is the subgroup of G generated by all elements of order $\neq p$). According to a result of Hughes and Thompson [8], there exists an element g in $H=H_p(G)$ of order p. But H is nilpotent (see Kegel [12]) and thus the centre $Z(H)$ of H has order divisible by p and also by another prime (otherwise G would be a p-group). Using Folgerung 2. 3 of Baer [2] one has that H is contained in a component of $\mathcal{P}$; as H is a maximal subgroup of G it coincides with this component and then H also must be a component of $\mathcal{U}$; but this implies $\mu=1$, a contradiction.

If $G=S_4$, then necessarily $s=2$ and $\mu=6$; thus all components have order 12 which is absurd, as the only subgroup of S_4 of order 12 is A_4.

Next let $G = PSL(2, q)$ where q is even. Then G has order $s^2\mu = (q+1)\,q(q-1)$ and contains elements of order $q+1$ and of order $q-1$ (see Dickson [6], § 260). Clearly no such element can be in N (otherwise $q+1$ (resp. $q-1$) would divide s, hence $(q+1)^2$ (resp. $(q-1)^2$) would divide $|G|$) and thus these elements are in components of $\mathcal{U}$; as all such components have the same order $s\mu$ we conclude that $(q+1)(q-1)|s\mu$, i.e. that $[G:U] \leqq q$ for $U \in \mathcal{U}$. But according Dickson [6], § 262 G has no subgroup of index $< q+1$ unless $q = 2$, which were absurd.

Next let $G = PSL(2, q)$ where $q = p^n$ is odd. Then G has order $s^2\mu = \frac{(q+1)\,q(q-1)}{2}$ and contains elements of orders $\frac{q+1}{2}$ and $\frac{q-1}{2}$ (see [6], § 260). As before one sees that $\frac{(q+1)(q-1)}{4}$ divides $s\mu$; but G also contains an element of order p and clearly $p|s\mu$. Thus $\frac{p(q+1)(q-1)}{4}$ divides $s\mu$ and one has $[G:U] \leqq \frac{2q}{p} < q$ for $U \in \mathcal{U}$. Again using [6], § 262 q is one of 3, 5, 7, 9 or 11; in these cases 3, 5, 7, 6, 11 are the smallest possible indices which immediately rule out all cases except $q = 9$. In this case $s = 12$, $\mu = 5$ and all components U of $\mathcal{U}$ are isomorphic to A_5 (see [6], § 260). But U contains an element of order $\frac{q-1}{2} = 4$ and A_5 does not contain such an element, a contradiction.

Now let $G = PGL(2, q)$. If q is even, then $PGL(2, q) = PSL(2, q)$ has already been seen to be impossible. Thus let $q = p^n$ be odd. Then G has order $s^2\mu = (q+1)\,q(q-1)$ and contains elements of orders p, $q-1$ and $q+1$. As before, one sees that $\frac{p(q^2-1)}{2}$ divides $s\mu$ (note that $(q+1, q-1) = 2$ in this case!). Choose a component U of $\mathcal{U}$ and put $U_0 := U \cap PSL(2, q)$. Then either $U \leqq PSL(2, q)$ and $[G:U] \leqq \frac{q}{p}$ which is immediately seen to be impossible using [6], § 262 again; or U_0 has order $\frac{|U|}{2}$ and we have $[PSL:U_0] \leqq \frac{2q}{p}$. As above, application of [6], § 262 leaves only the possibility $q = 9$; but then G contains an element of order $q-1 = 8$, and so PSL contains its square, i.e. an element of order 4, and one gets a contradiction as before. (U_0 then would be isomorphic to A_5).

Finally it remains to consider the Suzuki group $Sz(q)$ (for the Suzuki groups, see Suzuki [18] or Lüneburg [13]). Here $|G| = s^2\mu = (q^2+1)\,q^2(q-1)$ where $q = 2^{2a+1}$; furthermore G contains elements of orders 4, $q-1$, $q+r+1$, $q-r+1$ where $r^2 = 2q$, i.e. $r = 2^{a+1}$. As before one sees that $(q-1)(q+r+1)(q-r+1) = (q-1)(q^2+1)$ divides $s\mu$; and clearly also $4|s\mu$ (if g has order 4, then either $g \in N$ and $4|s$ or $g \in U$ (for some $U \in \mathcal{U}$) and $4|s\mu$). Thus one has $[G:U] \leqq \frac{q^2}{4}$ which is impossible in $Sz(q)$ (see [18] or [13]).

Hence indeed G is a p-group and the proof of Lemma 2 (and thus of Theorem 2) is complete.

Note added in proof. In the mean time, T. C. Hine and V. C. Mavron have given an elementary proof for Theorems 1 and 2 in their paper "Translations of symmetric and complete nets", to appear in Math. Z.

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Mathematisches Institut, Justus-Liebig-Universität Gießen, Arndtstr. 2, D-6300 Gießen

Eingegangen 26. Januar 1982