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Regularization by noise for the stochastic transport equation

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(joint work with Franco Flandoli, Massimiliano Gubinelli, Mario Maurelli)

We discuss several aspects of regularity and uniqueness for weak (L^∞-) solutions to the (deterministic and) stochastic transport equation

$$\text{(sTE)} \qquad du = b \cdot \nabla u dt + \sigma \nabla u \circ dW_t$$

on $[0,T] \times \mathbb{R}^d$ with initial values $u_0 \colon \mathbb{R}^d \to \mathbb{R}$ for $t = 0$. Here, $b \colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a deterministic vector field (the drift), σ a real number, $(W_t)_{t \geq 0}$ a Brownian motion in $\mathbb{R}^d$, $u \colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$ the (random) unknown, and the stochastic term is interpreted in the Stratonovich sense.

For the deterministic equation ($\sigma = 0$) the following dichotomy is well-known. If the drift b is sufficiently regular, then the associated equation of characteristics generates a flow $\Phi_t \colon \mathbb{R}^d \to \mathbb{R}^d$ of diffeomorphisms and the initial value problem to (sTE) admits (in suitable function spaces) a unique solution, which preserves C^1-regularity of the initial values and allows for the representation formula $u(t,x) = u_0(\Phi_t^{-1}(x))$, see [2]. In contrast, if b is not regular enough (such as only Hölder continuous in space), then multiple solutions may exist and solutions may blow up from smooth initial data in finite time (which, on the level of the equation of characteristics, means non-uniqueness or coalescence of the trajectories). For the stochastic equation ($\sigma \neq 0$) it turns out that noise can lead to a non-trivial regularization effect, namely that the formation of non-uniqueness and of singularities is prevented.

Similar phenomena of regularization due to noise were observed for different types of partial differential equations, for instance, for reaction diffusion equations in [6], for the transport equation (sTE) in [4, 3, 8] or for stochastic conservation laws in [5]. The main goal in these papers consists in understanding the effect of regularization due to the stochastic term. This requires in particular to find a suitable noise term, as simple as possible, for which the regularization effect takes place, while imposing as little regularity as possible on the deterministic terms.

In the case of the stochastic transport equation (sTE) we work in [1] with multiplicative Stratonovich noise and a mere integrability assumption on the drift (known from fluid dynamics as the Ladyzhenskaya–Prodi–Serrin condition). More precisely, we assume

$$b \in L^q([0,T]; L^p(\mathbb{R}^d, \mathbb{R}^d)) \qquad \text{for } p, q \in (2, \infty) \text{ such that } \frac{d}{p} + \frac{2}{q} \leq 1$$

and in particular, we do not assume any kind of differentiability or Hölder continuity. A scaling argument shows that this integrability condition is subcritical for strict inequality and critical for equality in $\frac{d}{p} + \frac{2}{q} \leq 1$, meaning that the Gaussian velocity field dominates the drift or that it is comparable to the drift, which suggests its optimality. In this setting, we prove in the purely stochastic case $\sigma \neq 0$ the conservation of Sobolev regularity of the initial values in the sense of

$$u_0 \in W^{1,2m}(\mathbb{R}^d) \quad \Rightarrow \quad \sup_{t\in[0,T]} E\big[\|u(t,\cdot)\|^m_{W^{1,m}(\mathbb{R}^d)}\big] < \infty$$

for $m \in 2\mathbb{N}$ (up to a restriction of the growth at infinity). The techniques needed to reach the critical case are of analytic nature and rely crucially on parabolic equations satisfied by moments of first derivatives of the solution. This is opposite to the previous works [4, 3, 8] based on stochastic flows and their regularity in terms of weak differentiability (which, by means of the result [7], is only known to be true in the subcritical case). Our approach covers in fact stochastic generalized transport equations, containing in particular the stochastic continuity equation

$$dv = \mathrm{div}(bv)dt + \sigma \mathrm{div}(v \circ dW_t)$$

which is in duality correspondence with the stochastic transport equation. By a duality approach in the stochastic setting, this allows to apply our regularity results in order to prove also the restoration of wellposedness for (sTE) provided that both b and $\mathrm{div}\, b$ satisfy the integrability assumption $L^q([0,T]; L^p(\mathbb{R}^d, \mathbb{R}^d))$ (which in fact guarantees weak differentiability of the solutions to the stochastic continuity equation).

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