# PSEUDOHOLOMORPHIC CURVES IN $\mathbb{S}^{6}$ AND $\mathbb{S}^{5}$ 

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#### Abstract

The octonionic cross product on $\mathbb{R}^{7}$ induces a nearly Kähler structure on $\mathbb{S}^{6}$, the analogue of the Kähler structure of $\mathbb{S}^{2}$ given by the usual (quaternionic) cross product on $\mathbb{R}^{3}$. Pseudoholomorphic curves with respect to this structure are the analogue of meromorphic functions. They are (super-)conformal minimal immersions. We reprove a theorem of Hashimoto [Tokyo J. Math. 23 (2000), 137-159] giving an intrinsic characterization of pseudoholomorphic curves in $\mathbb{S}^{6}$ and (beyond Hashimoto's work) $\mathbb{S}^{5}$. Instead of the Maurer-Cartan equations we use an embedding theorem into homogeneous spaces (here: $\mathbb{S}^{6}=G_{2} / S U_{3}$ ) involving the canonical connection.


## 1. Introduction

Minimal surfaces in the round 3 -sphere $\mathbb{S}^{3}$ have an intrinsic characterization: A metric $d s^{2}$ on a simply connected Riemann surface $M$ is the induced metric of a full conformal minimal immersion into $\mathbb{S}^{3}$ if and only if its Gaussian curvature $K$ satisfies $K \leq 1$ and

$$
\Delta \log (1-K)=4 K
$$

where $\Delta$ is the Laplacian of the metric $d s^{2} .{ }^{1}$ The formula goes back to Ricci [10, p. 340] who actually looked at surfaces of constant mean curvature 1 in euclidean 3 -space, but these are isometric to minimal surfaces in $\mathbb{S}^{3}$. There are similar ("Riccilike") formulas in other situations. In $\mathbb{S}^{4}$, superminimal surfaces (those with trivial associated family) are characterized by the equation (cf. [7, p. 191])

$$
\Delta \log (1-K)=6 K-2 .
$$

In the present paper, we give such characterizations for certain types of minimal surfaces in $\mathbb{S}^{5}$ and $\mathbb{S}^{6}$ :

$$
\begin{equation*}
\Delta \log (1-K)=6 K \tag{12.2}
\end{equation*}
$$

[^0]for so called pseudoholomorphic curves ${ }^{2}$ in $\mathbb{S}^{5}$ and
\[

$$
\begin{equation*}
\Delta \log (1-K)=6 K-1 \tag{11.6}
\end{equation*}
$$

\]

for superminimal pseudoholomorphic curves in $\mathbb{S}^{6}$ (see below). General pseudoholomorphic curves in $\mathbb{S}^{6}$ allow a similar characterization ([8]) which however depends on an additional structure, a holomorphic 6 -form $\Lambda$ on $M$ (which is zero in the superminimal case):

$$
\begin{equation*}
\Delta \log (1-K)-(6 K-1)=|\Lambda|^{2} /(1-K)^{2} \tag{11.2}
\end{equation*}
$$

A general theory of minimal surfaces in spheres allowing for Ricci-like characterizations was given in [13].

Pseudoholomorphic curves in $\mathbb{S}^{6}$ are the analogues of meromorphic functions on Riemann surfaces when $\mathbb{H}$ is replaced by $\mathbb{O}$. In fact, let $\mathbb{S} \in\left\{\mathbb{S}^{2}, \mathbb{S}^{6}\right\}$ be the unit sphere in the imaginary quaternions $\mathbb{H}^{\prime}$ or octonions $\mathbb{O}^{\prime}$, respectively. Left translation with the position vector $p \in \mathbb{S}$ induces an almost complex structure on $\mathbb{S}$ (which is integrable for $\mathbb{S}=\mathbb{S}^{2}$ ). For any Riemann surface $M$, a smooth mapping $f: M \rightarrow \mathbb{S}$ is pseudoholomorphic if its derivative $d f_{u}: T_{u} M \rightarrow T_{f(u)} \mathbb{S}^{6}$ is complex linear with respect to this almost complex structure. For $\mathbb{S}=\mathbb{S}^{2}$ these are the meromorphic functions on $M$. In the present paper we are dealing with the other case $\mathbb{S}=\mathbb{S}^{6}$. In particular, these maps are conformal and harmonic, hence (possibly branched) minimal immersions.

The subject was started by Bryant [3] who described pseudoholomorphic curves in terms of an adapted frame, called Frenet frame in analogy to curves in 3-space, and he gave examples for pseudoholomorphic curves on compact Riemann surfaces of any genus. Bolton, Vrancken and Woodword [2] characterized pseudoholomorphic curves among the minimal surfaces in $\mathbb{S}^{6}$. The intrinsic characterization (11.2) was given by Hashimoto [8].

In order to characterize immersions into a homogeneous space $f: M \rightarrow G / H$ one uses a lift, a map $F: M_{o} \rightarrow G$ (where $M_{o} \subset M$ is a contractible open subset) with $\pi \circ F=f$ for the canonical projection $\pi: G \rightarrow G / H$. The lift $F$ in turn can be described by the $\mathfrak{g}$-valued one-form $\alpha=F^{-1} d F .^{3}$ Vice versa, if an arbitrary $\mathfrak{g}$-valued one-form $\alpha$ on a simply connected manifold $M$ is given, we look for a map $F: M \rightarrow G$ with

$$
\begin{equation*}
d F=F \alpha \tag{1.1}
\end{equation*}
$$

This is an overdetermined system, and the local existence of solutions $F$ is equivalent to an integrability condition for the coefficient matrix $\alpha$, the Maurer-Cartan equation $d \alpha=[\alpha, \alpha]$. However, this system is very large. Following [3] and [8], we replace (1.1) by the equation

$$
\begin{equation*}
\nabla F=F \beta \tag{1.2}
\end{equation*}
$$

where $\nabla$ is a canonical $G$-connection on $G / H$ (holonomy in $G$ and parallel curvature and torsion). The advantage of (1.2) is that $\beta$ takes values in the smaller Lie algebra

[^1]$\mathfrak{h}$ rather than in $\mathfrak{g}$. The integrability condition for (1.2) is given by an embedding theorem into homogeneous spaces, cf. [6].

In the present paper, the transition from (1.1) to (1.2) is done more explicitly than in [8], where the proof of the existence part (cf. [8, p. 150]) is extremely short. Further, we try to replace computations on $G=\operatorname{Aut}(\mathbb{O})$ by computations in $\mathbb{O}$ and $\mathbb{O} \otimes \mathbb{C}$. Lastly, we derive some new consequences; in particular, we characterize pseudoholomorphic maps with values in $\mathbb{S}^{5} \subset \mathbb{S}^{6}$.

After recalling the necessary background on octonionic computations and pseudoholomorphic maps in the 6 -sphere (sections 2-6), we derive in section 9 the equations for the Frenet frame in terms of the canonical connection introduced in sections 7,8 . The main results are stated and proved in sections 11 for $\mathbb{S}^{6}$ and in 12 for $\mathbb{S}^{5}$ (a case which was not treated by Hashimoto [8]). We try to give complete computations with all details.

## 2. Octonions

A finite dimensional algebra $\mathbb{A}$ over $\mathbb{R}$ with unit 1 and euclidean inner product is called "normed" if $|a b|=|a||b|$ for any $a, b \in \mathbb{A}$. We have an orthogonal decomposition $\mathbb{A}=\mathbb{R} \cdot 1 \oplus \mathbb{A}^{\prime}$, where $\mathbb{A}^{\prime}$ is called the space of imaginary elements of $\mathbb{A}$. Every nonzero $a \in \mathbb{A}$ has an inverse $a^{-1}=\bar{a} /|a|^{2}$, where $\bar{a}=a_{o}-a^{\prime}$ for $a=a_{o}+a^{\prime}$ with $a_{o} \in \mathbb{R}$ and $a^{\prime} \in \mathbb{A}^{\prime}$. There are only four normed algebras: $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ (real and complex numbers, quaternions, and octonions), and the octonions $\mathbb{O} \cong \mathbb{R}^{8}$ contain all the others. Octonions are not associative, but still computations are easy if one observes the following three rules which follow almost immediately from the equation $|a b|=|a||b|:{ }^{4}$
(1) Any unit vector $a \in \mathbb{O}^{\prime}$ generates a subalgebra isomorphic to $\mathbb{C}$ where $a$ plays the rôle of $i$.
(2) Any two orthonormal $a, b \in \mathbb{O}^{\prime}$ generate a subalgebra isomorphic to $\mathbb{H}$ where $a, b, a b$ play the rôles of $i, j, k$; they are associative and anti-commutative, $a b=-b a$.
(3) Any three orthonormal $a, b, c \in \mathbb{O}^{\prime}$ with $c \perp a b$ ("normed Cayley triples") generate the algebra $\mathbb{O}$; they are anti-associative, $a(b c)=-(a b) c$.
Let $1, i, j, k, l, i l, j l, k l$ be the standard basis of $\mathbb{O}=\mathbb{H}+\mathbb{H} l$. Then $(i, j, l)$ is a normed Cayley triple, and so is its image ( $\alpha i, \alpha j, \alpha l$ ) under any automorphism $\alpha$ of $\mathbb{O} ;$ note that $\alpha$ is orthogonal. ${ }^{5}$ Vice versa, given any normed Cayley triple ( $a, b, c$ ), there is precisely one automorphism $\alpha$ of $\mathbb{O}$ with $a=\alpha i, b=\alpha j, c=\alpha l$. Thus the space of normed Cayley triples is a manifold of dimension $6+5+3=14$ on which

[^2]the exceptional group $G_{2}=\operatorname{Aut}(\mathbb{O}) \subset S O_{7}$ acts simply transitively. In particular, $G_{2}$ acts transitively on $\mathbb{S}^{6}$.

We will also need the complexified octonions $\mathbb{O}_{c}=\mathbb{O} \otimes \mathbb{C}=\mathbb{O} \oplus \mathfrak{i}($ (we distinguish $\mathrm{i}=\sqrt{-1}$ from $i \in \mathbb{O}$ ). This is no longer a division algebra: there are zero divisors, e.g., $1+\mathrm{i} a$ for any $a \in \mathbb{S}^{6} \subset \mathbb{O}^{\prime}$. However, analytic formulas which hold in $\mathbb{O}$ extend to $\mathbb{O}_{c}$; e.g., for $a \in \mathbb{O}^{\prime}$ and $b \in \mathbb{O}$ we have (using rule (2))

$$
a(a b)=a^{2} b=-\langle a, a\rangle b,
$$

and this remains true for $a \in \mathbb{O}_{c}^{\prime}, b \in \mathbb{O}_{c}$, where $\langle$,$\rangle is the complexified inner$ product. In particular $a(a b)=0$ when $\langle a, a\rangle=0$. Other useful formulas which extend for all $a, b, c \in \mathbb{O}_{c}$ are

$$
\langle a b, a c\rangle=\langle a, a\rangle\langle b, c\rangle
$$

and the antisymmetry of $\langle a b, c\rangle$ in all three variables.
As $\mathbb{O}$ is decomposed into planes that are invariant under left multiplication with $\mathbb{C} \subset \mathbb{O}$, we may decompose $\mathbb{O}_{c}$ into free $\mathbb{C}_{c}$-modules, where $\mathbb{C}_{c}=\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of $\mathbb{C}$. A complex Cayley triple is a triple $(a, b, c)$ in $\mathbb{O}_{c}^{\prime}$ where $a$ lies in $\mathbb{C}_{c}$ (or in an isomorphic subalgebra) and where $b, c$ belong to two perpendicular $\mathbb{C}_{c}$-modules. Like its real analogue, a complex Cayley triple is anti-associative, $(a b) c=-a(b c)$.

## 3. The nearly Kähler structure on $\mathbb{S}^{6}$

The 6 -sphere $\mathbb{S}^{6}$ plays a similar rôle for the octonions $\mathbb{O}$ as the 2 -sphere $\mathbb{S}^{2}$ for the quaternions $\mathbb{H}$ : they are unit spheres $\mathbb{S} \subset \mathbb{A}^{\prime}$, where $\mathbb{A}^{\prime}$ denotes the imaginary part of the division algebra $\mathbb{A}=\mathbb{O}, \mathbb{H}$, respectively. Each $p \in \mathbb{S}$ satisfies $\left(L_{p}\right)^{2}=-I$, where $L_{p}: x \mapsto p x$ denotes the left multiplication with $p$. Hence $L_{p}$ is a complex structure preserving the plane Span $\{1, p\}$ and its orthogonal complement, the tangent space $T_{p} \mathbb{S}$. Thus $J_{p}:=L_{p} \mid T_{p} \mathbb{S}$ is a complex structure on $T_{p} \mathbb{S}$ and defines an almost complex structure $J$ on $\mathbb{S}$. It is convenient to use the cross product $a \times b$ which is the imaginary $\left(\mathbb{A}^{\prime}-\right)$ part of the product $a b$ for any $a, b \in \mathbb{A}^{\prime}$ :

$$
a \times b=(a b)^{\prime}= \begin{cases}a b & \text { when } a \perp b \\ 0 & \text { when } a, b \text { are linearly dependent } .\end{cases}
$$

Then each $J_{p}$ extends to a linear map on $\mathbb{A}^{\prime}$,

$$
\begin{equation*}
J_{p}(v)=p \times v \tag{3.1}
\end{equation*}
$$

and the derivative of the matrix-valued linear map $J: \mathbb{A}^{\prime} \rightarrow \operatorname{End}\left(\mathbb{A}^{\prime}\right): p \mapsto J_{p}$ is $\left(\partial_{v} J\right) w=v \times w$. Denoting by $D=\partial^{T}$ the Levi-Civita derivative on $\mathbb{S}$, we have

$$
\begin{equation*}
\left(D_{v} J\right) w=(v \times w)_{p^{\perp}}=v \times w-\langle v \times w, p\rangle p, \tag{3.2}
\end{equation*}
$$

where $p \in \mathbb{S}$ is the position vector and $v, w \in T_{p} \mathbb{S}=p^{\perp}$. In particular $\left(\partial_{v} J\right) v=$ $v \times v=0$ and therefore

$$
\begin{equation*}
\left(D_{v} J\right) v=0 \tag{3.3}
\end{equation*}
$$

A Riemannian manifold with an almost complex structure $J$ with this property is called nearly Kähler. ${ }^{6}$

An orthogonal linear map $g$ on $\mathbb{O}^{\prime}$ which preserves the almost complex structure $J$ satisfies $g J_{p}(v)=J_{g p}(g v)$ for any $p, v \in \mathbb{O}^{\prime}$ with $v \perp p$. By (3.1) this is equivalent to $g(p v)=(g p)(g v)$, which holds if and only if $g \in G_{2}=\operatorname{Aut}(\mathbb{O}) \subset S O_{7}$. Thus $G_{2}$ is precisely the group of isometries $g$ on $\mathbb{S}^{6}$ which are pseudoholomorphic, that is their differentials $d g_{p}: T_{p} \mathbb{S}^{6} \rightarrow T_{g p} \mathbb{S}^{6}$ are complex linear with respect to the complex structures given by $J$ on the tangent spaces of $\mathbb{S}^{6}$. The stabilizer subgroup $H=\left(G_{2}\right)_{p}$ of any $p \in \mathbb{S}^{6}$ (say: $p=l$ ) preserves the tangent space $T_{p} \mathbb{S}^{6}$ and its complex structure $J_{p}$, making $T_{p} \mathbb{S}^{6}$ a 3 -dimensional complex vector space. Identifying $\left(T_{p} \mathbb{S}^{6}, J_{p}\right)$ with $\mathbb{C}^{3}$ we obtain $H \subset U_{3}$. But $H$ preserves also the antisymmetric 3 -form $\langle u v, w\rangle$ on $T_{p} \mathbb{S}^{6}$, which can be viewed as the real part of a complex determinant, thus $H \subset S U_{3}$, and by dimension reasons we have equality $H=S U_{3}$.

## 4. Pseudoholomorphic curves

Let $M$ be a Riemann surface. A smooth map $f: M \rightarrow \mathbb{S}^{6}$ is called pseudoholomorphic if it is holomorphic with respect to this almost complex structure $J_{p} v=p \times v$. In other words, if $z=x+i y$ is a conformal coordinate on $M$, the corresponding partial derivatives $f_{x}, f_{y}$ satisfy

$$
\begin{equation*}
f \times f_{x}=f_{y}, \quad f \times f_{y}=-f_{x} \tag{4.1}
\end{equation*}
$$

Clearly, such map is conformal since $\left|f_{x}\right|=\left|f_{y}\right|$ and $f_{x} \perp f_{y}$. Further $f$ is harmonic, that is $f_{x x}+f_{y y}$ is a normal vector, a multiple of $f$. In fact, differentiating (4.1) we obtain

$$
\begin{aligned}
& f_{y y}=\left(f \times f_{x}\right)_{y}=f_{y} \times f_{x}+f \times f_{x y}, \\
& f_{x x}=-\left(f \times f_{y}\right)_{x}=-f_{x} \times f_{y}-f \times f_{y x},
\end{aligned}
$$

and hence

$$
\begin{align*}
f_{y y}+f_{x x} & =2 f_{y} \times f_{x},  \tag{4.2}\\
f_{y y}-f_{x x} & =2 f \times f_{x y} . \tag{4.3}
\end{align*}
$$

Equation (4.2) shows that $f$ is harmonic: $f_{y} \times f_{x}$ is proportional to $f$ since by (4.1), $f, f_{x}, f_{y}$ span a quaternion subalgebra wherever $d f \neq 0$, but see Remark 4.2 below. Moreover,

$$
\begin{equation*}
f_{y x}=\left(f \times f_{x}\right)_{x}=f \times f_{x x}=J f_{x x} \tag{4.4}
\end{equation*}
$$

It is convenient to use the complex derivatives $f_{z}=\frac{1}{2}\left(f_{x}-\mathrm{i} f_{y}\right)$ and

$$
f_{z z}=\frac{1}{4}\left(\left(f_{x}-\mathrm{i} f_{y}\right)_{x}-\mathrm{i}\left(f_{x}-\mathrm{i} f_{y}\right)_{y}\right)=\frac{1}{4}\left(f_{x x}-f_{y y}-2 \mathrm{i} f_{x y}\right) \stackrel{(4.3)}{=}-\frac{1}{2}(J+\mathrm{i}) f_{x y}
$$

Hence

$$
\begin{align*}
f_{z} & =-(J+\mathrm{i}) f_{y} / 2  \tag{4.5}\\
f_{z z} & =-(J+\mathrm{i}) f_{x y} / 2
\end{align*}
$$

[^3]Since $(J-\mathrm{i})(J+\mathrm{i})=0$, these vectors belong to the i-eigenspace $E_{+}$of $J_{f}: v \mapsto f \times v$ on $T_{f} \mathbb{S}^{6}$. This is an isotropic subspace, i.e., $\langle v, v\rangle=0$ for all $v \in E_{+}$: If $v=(J+\mathrm{i}) a$, then $\langle v, v\rangle=\langle J a, J a\rangle-\langle a, a\rangle+2 \mathrm{i}\langle J a, a\rangle=0$.
Lemma 4.1. Putting $\lambda=\left\langle f_{z}, f_{\bar{z}}\right\rangle=\left|f_{z}\right|^{2}$ and $l=\log \lambda$, we have

$$
\begin{align*}
f_{z z} & =f_{z z}^{\perp}+l_{z} f_{z} \\
\left(f_{z}\right)_{\bar{z}} & =-\lambda f  \tag{4.6}\\
\left(f_{z z}^{\perp}\right)_{\bar{z}} & =-\left(\lambda+l_{z \bar{z}}\right) f_{z} .
\end{align*}
$$

Proof. To prove the first equation we note that $\left\langle f_{z z}, f_{z}\right\rangle=\frac{1}{2}\left\langle f_{z}, f_{z}\right\rangle_{z}=0$ and $\left\langle f_{z z}, f_{\bar{z}}\right\rangle=\lambda_{z}-\left\langle f_{z}, f_{z \bar{z}}\right\rangle=\lambda_{z}$, since $\left\langle f_{z}, f_{z \bar{z}}\right\rangle=\frac{1}{2}\left\langle f_{z}, f_{z}\right\rangle_{\bar{z}}=0$. Hence $f_{z z}-f_{z z}^{\perp}=$ $f_{z z}^{T}=\frac{1}{\lambda}\left\langle f_{z z}, f_{\bar{z}}\right\rangle f_{z}=\left(\lambda_{z} / \lambda\right) f_{z}=l_{z} f_{z}$.

The second equation follows since $4 f_{z \bar{z}}=\left(f_{x}-\mathrm{i} f_{y}\right)_{x}+\mathrm{i}\left(f_{x}-\mathrm{i} f_{y}\right)_{y}=f_{x x}+f_{y y}$, and this is a multiple of $f$. To determine the multiple we we compute the inner product $\left\langle f_{z \bar{z}}, f\right\rangle=\left\langle f_{z}, f\right\rangle_{\bar{z}}-\left\langle f_{z}, f_{\bar{z}}\right\rangle=-\lambda$, since $\left\langle f_{z}, f\right\rangle=\frac{1}{2}\langle f, f\rangle_{z}=0$. This shows the second equality.

The third equality follows from the two previous ones: From $f_{z z}^{\perp}=f_{z z}-l_{z} f_{z}$ we have $\left(f_{z z}^{\perp}\right)_{\bar{z}}=f_{z z \bar{z}}-\left(l_{z} f_{z}\right)_{\bar{z}}=-(\lambda f)_{z}-l_{z \bar{z}} f_{z}+l_{z} \lambda f=-\left(\lambda+l_{z \bar{z}}\right) f_{z}$, using $\lambda_{z}=l_{z} \lambda$.

Remark 4.2. As a consequence, $f_{z}$ and $f_{z z}^{\perp}$ are holomorphic sections of the complexified tangent and normal bundles $T^{c}$ and $N^{c}$ of $f: M \rightarrow \mathbb{S}^{6}$, since $\left(f_{z}\right)_{\bar{z}}$ and $\left(f_{z z}^{\perp}\right)_{\bar{z}}$ have zero projection to $T^{c}$ and $N^{c}$, respectively. Thus the zeros of $f_{z}$ are isolated and the isotropic subbundles $T^{\prime}=\mathbb{C} f_{z}$ and $N_{1}^{\prime}=\mathbb{C} f_{z z}^{\perp}$ are well defined even at possible zeros of these sections, and by isotropy the same holds for the real bundles $T$ and $N_{1}$, the tangent bundle and the first normal bundle of $f$. Hence along $f$, the tangent bundle of $\mathbb{S}^{6}$ splits into three $J$-invariant orthogonal plane bundles, $f^{*}\left(T \mathbb{S}^{6}\right)=T \oplus N_{1} \oplus N_{2}$.

The full $(+\mathrm{i})$-eigenspace $E_{+}=T_{f}^{\prime} \mathbb{S}^{6}$ is spanned by

$$
\begin{align*}
& F_{1}=f_{z} \\
& F_{2}=f_{z z}^{\perp}  \tag{4.7}\\
& F_{3}=\overline{F_{1} F_{2}}=f_{\bar{z}} \times f_{\bar{z} \bar{z}}
\end{align*}
$$

The third line follows since $\left(f, F_{1}, F_{2}\right)$ is a complex Cayley triple, hence $f\left(F_{1} F_{2}\right)=$ $-\left(f F_{1}\right) F_{2}=-\mathrm{i} F_{1} F_{2}$ and therefore $\overline{F_{1} F_{2}} \in E_{+}$. In analogy to the theory of curves in euclidean space $\mathbb{R}^{3}$, we will call $F=\left(F_{1}, F_{2}, F_{3}\right)$ the Frenet frame of $f$, as was suggested in [3].

The three vectors $F_{1}, F_{2}, F_{3}$ together with their complex conjugates $\bar{F}_{1}, \bar{F}_{2}, \bar{F}_{3}$ form bases of the complexified bundles $T^{c}, N_{1}^{c}, N_{2}^{c}$, respectively, and the only nonzero inner products are

$$
\begin{equation*}
\left\langle F_{1}, \bar{F}_{1}\right\rangle=: \lambda, \quad\left\langle F_{2}, \bar{F}_{2}\right\rangle=: \mu, \quad\left\langle F_{3}, \bar{F}_{3}\right\rangle=2 \lambda \mu \tag{4.8}
\end{equation*}
$$

The last equality is seen as follows: If $F_{1}=(f+\mathrm{i}) a$ and $F_{2}=(f+\mathrm{i}) b$, then $F_{1} F_{2}=(f a+\mathrm{i} a)(f b+\mathrm{i} b)=(f a)(f b)-a b+\mathrm{i}((f a) b+a(f b))$. If $(f, a, b)$ is an
(unnormed) Cayley triple, then so is $(f, f a, b)$, and $(f a)(f b)=-((f a) f) b=-a b$ (using $|f|=1$ ) while $a(f b)=-(a f) b=(f a) b$. Thus $F_{1} F_{2}=-2 a b+2 \mathrm{i}(f a) b$, and $\left|F_{1} F_{2}\right|^{2}=8|a|^{2}|b|^{2}$ while $\left|F_{1}\right|^{2}\left|F_{2}\right|^{2}=4|a|^{2}|b|^{2}$.

Remark 4.3. Later we will also use the normalized Frenet frame

$$
\begin{equation*}
F_{1}^{o}=F_{1} / \sqrt{\lambda}, \quad F_{2}^{o}=F_{2} / \sqrt{\mu}, \quad F_{3}^{o}=F_{3} / \sqrt{2 \lambda \mu} \tag{4.9}
\end{equation*}
$$

Corollary 4.4. Let $f: M \rightarrow \mathbb{S}^{6}$ be a pseudoholomorphic map and $z$ a conformal coordinate on $M$. Then $\mu=\left|f_{z z}^{\perp}\right|^{2}$ depends on $\lambda=\left|f_{z}\right|^{2}$ :

$$
\begin{equation*}
\mu=\lambda^{2}(1-K)=\lambda\left(\lambda+l_{z \bar{z}}\right), \quad \text { where } l=\log \lambda . \tag{4.10}
\end{equation*}
$$

Proof. From $\left\langle f_{z z}^{\perp}, f_{\bar{z}}\right\rangle=0$ we obtain, using the third equation of (4.6):

$$
0=\left\langle f_{z z}^{\perp}, f_{\bar{z}}\right\rangle_{\bar{z}}=-\left(\lambda+l_{z \bar{z}}\right)\left\langle f_{z}, f_{\bar{z}}\right\rangle+\left\langle f_{z z}^{\perp}, f_{\bar{z} \bar{z}}\right\rangle=-\left(\lambda+l_{z \bar{z}}\right) \lambda+\mu .
$$

The first equality in (4.10) follows since the Gaussian curvature $K$ of the induced metric $d s^{2}=2 \lambda \cdot d z d \bar{z}$ on $M$ satisfies

$$
\lambda K=-(\log \lambda)_{z \bar{z}}=-l_{z \bar{z}}
$$

thus $\lambda(1-K)=\lambda+l_{z \bar{z}}$.
Remark 4.5. Equation (4.10) is just the Gauss equation $(G)$ for the conformal minimal immersion $f: M \rightarrow \mathbb{S}^{6}$ :

$$
4 \lambda^{2}(K-1)=\left|f_{x}\right|^{2}\left|f_{y}\right|^{2}(K-1) \stackrel{(G)}{=}\left\langle f_{x x}^{\perp}, f_{y y}^{\perp}\right\rangle-\left|f_{x y}^{\perp}\right|^{2} \stackrel{*}{=}-2\left|f_{x x}^{\perp}\right|^{2}=-4 \mu
$$

For " $\stackrel{*}{=}$ " recall that $f_{y y}^{\perp}=-f_{x x}^{\perp}$ (harmonicity) and $f_{x y}=J f_{x x}$, see (4.4). Further we have used (4.5) to see

$$
\begin{aligned}
& 2 \lambda=2\left|f_{z}\right|^{2}=\left|f_{x}\right|^{2}=\left|f_{y}\right|^{2} \\
& 2 \mu=2\left|f_{z z}^{\perp}\right|^{2}=\left|f_{x x}^{\perp}\right|^{2}=\left|f_{x y}^{\perp}\right|^{2}
\end{aligned}
$$

## 5. The generalized Hopf differentials

For any conformal harmonic map $f: M \rightarrow \mathbb{S}^{n}$ on a Riemann surface $M$ one considers the higher fundamental forms

$$
A_{k}\left(v_{1}, \ldots, v_{k}\right)=\left(\partial_{v_{1}} \ldots \partial_{v_{k}} f\right)^{N_{k-1}}
$$

for arbitrary tangent vectors $v_{1}, \ldots, v_{k}$, where $N_{0}=T$ is the tangent space and $N_{k-1}$ for $k \geq 2$ the ( $k-1$ )-th normal space ${ }^{7}$ of the surface $f$, and ( $)^{N_{k-1}}$ denotes the orthogonal projection into this space. Using a conformal coordinate $z$ on $M$, the harmonicity of $f$ yields the vanishing of all mixed components of $A_{k}$ (those involving both $d z$ and $d \bar{z}$ ). Thus

$$
A_{k}=B_{k}+\bar{B}_{k}, \quad \text { with } B_{k}=\left(\left(\frac{\partial}{\partial z}\right)^{k} f\right)^{N_{k-1}} d z^{k}
$$

[^4]see [11] for details. The generalized Hopf differential is the symmetric $2 k$-form on $M$ defined by
$$
\Lambda_{k}=\left\langle B_{k}, B_{k}\right\rangle
$$

The first Hopf differential $\Lambda_{1}=\left\langle f_{z}, f_{z}\right\rangle d z^{2}$ vanishes by conformality of $f$, and the second one $\Lambda_{2}=\left\langle f_{z z}^{\perp}, f_{z z}^{\perp}\right\rangle d z^{4}$ is the classical Hopf differential which is holomorphic for every conformal harmonic map. More generally, $\Lambda_{k}$ is holomorphic if $\Lambda_{1}, \ldots, \Lambda_{k-1}$ vanish everywhere, cf. [11]. If $M$ is compact of genus 0 , all holomorphic differentials vanish, hence all $\Lambda_{k}$ are zero. This is the superminimal case investigated first by Calabi [4].

In our case of pseudoholomorphic maps $f: M \rightarrow \mathbb{S}^{6}$, we have always $\Lambda_{2}=0$ since $f_{z z}^{\perp}$ lies in the isotropic space $E_{+}$. Therefore $\Lambda_{3}=\left\langle f_{z z z}^{N_{2}}, f_{z z z}^{N_{2}}\right\rangle d z^{6}$ is holomorphic. ${ }^{8}$ For completeness and to introduce notation we give a direct proof.

Lemma 5.1. Let $f: M \rightarrow \mathbb{S}^{6}$ be a pseudoholomorphic curve and $z$ a conformal coordinate on $M$. Then the function $h:=\left\langle f_{z z z}, f_{z z z}\right\rangle$ is holomorphic with

$$
\begin{equation*}
h=\left\langle f_{z z z}^{N_{2}}, f_{z z z}^{N_{2}}\right\rangle=\left\langle\left(F_{2}\right)_{z},\left(F_{2}\right)_{z}\right\rangle=\left\langle\left(F_{2}\right)_{z}^{N_{2}},\left(F_{2}\right)_{z}^{N_{2}}\right\rangle, \tag{5.1}
\end{equation*}
$$

and $\Lambda_{3}=h(z) d z^{6}$.
Proof. $\left\langle f_{z z z}, f_{z z z}\right\rangle_{\bar{z}}=2\left\langle f_{z z z \bar{z}}, f_{z z z}\right\rangle=-2\left\langle(\lambda f)_{z z}, f_{z z z}\right\rangle=0$ since $f_{z z z}$ is perpendicular to $f, f_{z}, f_{z z}$. In fact, $\left\langle f, f_{z z z}\right\rangle=\left\langle f, f_{z z}\right\rangle_{z}=0$ since $\left\langle f, f_{z z}\right\rangle=-\left\langle f_{z}, f_{z}\right\rangle=0$; further $\left\langle f_{z}, f_{z z z}\right\rangle=-\left\langle f_{z z}, f_{z z}\right\rangle=0$ and $\left\langle f_{z z}, f_{z z z}\right\rangle=\frac{1}{2}\left\langle f_{z z}, f_{z z}\right\rangle_{z}=0$. Thus $h$ is holomorphic and $h(z) d z^{6}$ defines a holomorphic 6 -form on $M$.

From (4.6) we have $f_{z z}=F_{2}+l_{z} f_{z}$, and thus $\left(f_{z z}-F_{2}\right)_{z}=\left(l_{z} f_{z}\right)_{z}$ belongs to the span of $f_{z}$ and $f_{z z}$, which is part of the isotropic subspace $E_{+}$. Further, since $f_{z z z} \perp f_{z}, f_{z z}$, we have $f_{z z z}-f_{z z z}^{N_{2}} \in \operatorname{Span}\left(f_{z}, f_{z z}\right)$. (The components of $f_{z z z}$ proportional to $f_{\bar{z}}, f_{\bar{z} \bar{z}}$ involve the inner products with $f_{z}, f_{z z}$, which are zero.) Thus $h=\left\langle f_{z z z}, f_{z z z}\right\rangle=\left\langle\left(F_{2}\right)_{z},\left(F_{2}\right)_{z}\right\rangle=\left\langle f_{z z z}^{N_{2}}, f_{z z z}^{N_{2}}\right\rangle$, and $h(z) d z^{6}=\Lambda_{3}$. Moreover, $\left(F_{2}\right)_{z} \perp f, F_{1}, F_{2}$, hence $\left(F_{2}\right)_{z}-\left(F_{2}\right)_{z}^{N_{2}} \in \operatorname{Span}\left\{F_{1}, F_{2}\right\}$, and this component does not contribute to the inner product $\left\langle\left(F_{2}\right)_{z},\left(F_{2}\right)_{z}\right\rangle$. This proves the last equality in (5.1).

## 6. The derivatives of the Frenet frame

Proposition 6.1. Let $f: M \rightarrow \mathbb{S}^{6}$ be a pseudoholomorphic curve with Frenet frame $F_{1}, F_{2}, F_{3}$ as in (4.7), corresponding to a conformal coordinate $z$ on M. Let $\lambda=\left|F_{1}\right|^{2}, \mu=\left|F_{2}\right|^{2}$ and $l=\log \lambda, m=\log \mu$. Then:

[^5]\[

$$
\begin{array}{rlrl}
\left(F_{1}\right)_{z} & =l_{z} F_{1} & +F_{2}, & \\
\left(F_{2}\right)_{z} & = & m_{z} F_{2} & +(\mathrm{i} h / 2 \lambda \mu) F_{3} \\
\left(F_{3}\right)_{z} & = & & -(\mathrm{i} / 2) \bar{F}_{3}, \\
\left(F_{1}\right)_{\bar{z}} & = & \mathrm{i} \lambda \bar{F}_{2}, \\
\left(F_{2}\right)_{\bar{z}} & =-\frac{\mu}{\lambda} F_{1}, & -\lambda f, \\
\left(F_{3}\right)_{\bar{z}} & & & \\
& & & \\
(\mathrm{i} \bar{h} / \mu) F_{2} & +\left(l_{\bar{z}}+m_{\bar{z}}\right) F_{3} . &
\end{array}
$$
\]

Proof. The equations for $\left(F_{1}\right)_{z},\left(F_{1}\right)_{\bar{z}}$, and $\left(F_{2}\right)_{\bar{z}}$ follow directly from (4.6) using $\lambda+l_{z \bar{z}}=\mu / \lambda$, see (4.10). The equation for $\left(F_{3}\right)_{z}=\overline{\left(f_{z} \times f_{z z}^{\perp}\right)_{\bar{z}}}$ is proved as follows:

$$
\begin{aligned}
\left(f_{z} \times f_{z z}^{\perp}\right)_{\bar{z}} & =f_{z \bar{z}} \times f_{z z}^{\perp}+f_{z} \times\left(f_{z z}^{\perp}\right)_{\bar{z}} \\
& \stackrel{(4.6)}{=}-\lambda f \times f_{z z}^{\perp}-\frac{\mu}{\lambda} f_{z} \times f_{z} \\
& =-i \lambda f_{z z}^{\perp} .
\end{aligned}
$$

The equations for $\left(F_{2}\right)_{z}$ and $\left(F_{3}\right)_{\bar{z}}$ are proved in the subsequent two lemmas.

## Lemma 6.2.

$$
\begin{equation*}
\left(f_{z z}^{\perp}\right)_{z}=m_{z} f_{z z}^{\perp}+\mathrm{i} h /(2 \lambda \mu) f_{\bar{z}} \times f_{\bar{z} \bar{z}}-(\mathrm{i} / 2) f_{z} \times f_{z z} \tag{6.1}
\end{equation*}
$$

where $l=\log \lambda$ and $m=\log \mu$.
Proof.

$$
\begin{array}{cccccc}
\left\langle\left(f_{z z}^{\perp}\right)_{z}, f_{z}\right\rangle & = & -\left\langle f_{z z}^{\perp}, f_{z z}\right\rangle & = & 0, \\
\left\langle\left(f_{z z}^{\perp}\right)_{z}, f_{\bar{z}}\right\rangle & = & -\left\langle f_{z \bar{z}}^{\perp}, f_{\bar{z} z}\right\rangle=\left\langle f_{z z}^{\perp}, \lambda f\right\rangle & = & 0, \\
\left\langle\left(f_{z z}^{\perp}\right)_{z}, f_{z z}^{\perp}\right\rangle & = & (1 / 2)\left\langle f_{z z}^{\perp}, f_{z z}^{\perp}\right\rangle_{z} & = & 0, \\
\left\langle\left(f_{z z}^{\perp}\right)_{z}, f_{\bar{z} \bar{z}}^{\perp}\right\rangle & = & \left\langle f_{z \bar{z}}^{\perp}, f_{\bar{z} \bar{z}}^{\perp}\right\rangle_{z}+\left\langle f_{z \bar{z}}^{\perp},(\lambda f)_{\bar{z}}\right\rangle & = & \mu_{z}, \\
\left\langle\left(f_{z z}^{\perp}\right)_{z}, f_{\bar{z}} \times f_{\bar{z} \bar{z}}^{\perp}\right\rangle & = & \left\langle f_{z z}^{\perp}, \lambda f \times f_{\bar{z} \bar{z}}^{\perp}+f_{\bar{z}} \times(\lambda f)_{\bar{z}}\right\rangle & = & -\mathrm{i} \lambda \mu . \tag{e}
\end{array}
$$

Equation (e) tells us

$$
\left\langle\left(F_{2}\right)_{z}, F_{3}\right\rangle=-\mathrm{i} \lambda \mu .
$$

It remains to compute $\left\langle\left(F_{2}\right)_{z}, \bar{F}_{3}\right\rangle$, using

$$
h=\left\langle\left(F_{2}\right)_{z}^{N_{2}},\left(F_{2}\right)_{z}^{N_{2}}\right\rangle .
$$

We have

$$
2 \lambda \mu\left(F_{2}\right)_{z}^{N_{2}}=\left\langle\left(F_{2}\right)_{z}, \bar{F}_{3}\right\rangle F_{3}+\left\langle\left(F_{2}\right)_{z}, F_{3}\right\rangle \bar{F}_{3}
$$

and hence

$$
\begin{aligned}
(2 \lambda \mu)^{2} h & =2\left\langle\left(F_{2}\right)_{z}, \bar{F}_{3}\right\rangle \cdot\left\langle\left(F_{2}\right)_{z}, F_{3}\right\rangle \cdot\left\langle F_{3}, \bar{F}_{3}\right\rangle \\
& =2\left\langle\left(F_{2}\right)_{z}, \bar{F}_{3}\right\rangle \cdot(-\mathrm{i} \lambda \mu) \cdot 2 \lambda \mu,
\end{aligned}
$$

from which we find the missing equation:

$$
\begin{equation*}
\left\langle\left(F_{2}\right)_{z}, \bar{F}_{3}\right\rangle=\mathrm{i} h . \tag{f}
\end{equation*}
$$

From (a), (b), (c) we see ${ }^{9}$ that $\left(F_{2}\right)_{z}=a F_{2}+b F_{3}+c \bar{F}_{3}$, and further

$$
\begin{aligned}
& a\left\langle F_{2}, \bar{F}_{2}\right\rangle=\left\langle\left(F_{2}\right)_{z}, \bar{F}_{2}\right\rangle \stackrel{(\mathrm{d})}{=} \mu_{z}, \\
& b\left\langle F_{3}, \bar{F}_{3}\right\rangle=\left\langle\left(F_{2}\right)_{z}, \bar{F}_{3}\right\rangle \stackrel{(\mathrm{f})}{=} \mathrm{i} h, \\
& c\left\langle\bar{F}_{3}, F_{3}\right\rangle=\left\langle\left(F_{2}\right)_{z}, F_{3}\right\rangle \stackrel{\left(e^{\prime}\right)}{=}-\mathrm{i} \lambda \mu .
\end{aligned}
$$

Thus

$$
\begin{array}{ll}
a=\mu_{z} / \mu \quad= & m_{z} \\
b= & \mathrm{i} h /(2 \lambda \mu), \\
c=-\mathrm{i} \lambda \mu /(2 \lambda \mu)= & -\mathrm{i} / 2
\end{array}
$$

## Lemma 6.3.

$$
\begin{align*}
\left(f_{z} \times f_{z z}\right)_{z} & =-(\mathrm{i} h / \mu) f_{\bar{z} \bar{z}}^{\perp}+\left(l_{z}+m_{z}\right) f_{z} \times f_{z z}  \tag{6.2}\\
\left(F_{3}\right)_{\bar{z}} & =(\mathrm{i} \bar{h} / \mu) F_{2}+\left(l_{\bar{z}}+m_{\bar{z}}\right) F_{3} . \tag{6.3}
\end{align*}
$$

Proof. We compute the components of $\left(f_{z} \times f_{z z}\right)_{z}$. Using $f_{z} \times f_{z z} \in N_{2}^{c} \perp T^{c} \oplus N_{1}^{c}$, we obtain:

$$
\begin{array}{rlrl}
\left\langle\left(f_{z} \times f_{z z}\right)_{z}, f_{z}\right\rangle & =-\left\langle f_{z} \times f_{z z}, f_{z z}\right\rangle & & =0, \\
\left\langle\left(f_{z} \times f_{z z}\right)_{z}, f_{\bar{z}}\right\rangle & =\left\langle f_{z} \times f_{z z}, \lambda f\right\rangle & & =0, \\
\left\langle\left(f_{z} \times f_{z z}\right)_{z}, f_{z z}^{\perp}\right\rangle & =-\left\langle f_{z} \times f_{z z},\left(f_{z z}^{\perp}\right)_{z}\right\rangle & & \stackrel{(6.1)}{=}-\mathrm{i} h, \\
\left\langle\left(f_{z} \times f_{z z}\right)_{z}, f_{\bar{z} \bar{z}}\right\rangle & =\left\langle f_{z} \times f_{z z},(\lambda f)_{\bar{z}}\right\rangle & & =0, \\
\left\langle\left(f_{z} \times f_{z z}\right)_{z}, f_{z} \times f_{z z}\right\rangle & =\left\langle f_{z} \times f_{z z}, f_{z} \times f_{z z}\right\rangle_{z} / 2 & =0, \\
\left\langle\left(f_{z} \times f_{z z}\right)_{z}, f_{\bar{z}} \times f_{\bar{z} \bar{z}}\right\rangle \stackrel{*}{=}\left\langle f_{z} \times f_{z z}, f_{\bar{z}} \times f_{\bar{z} \bar{z}}\right\rangle_{z} & & =2(\lambda \mu)_{z},
\end{array}
$$

where " $\stackrel{*}{=}$ " follows since $\left(f_{\bar{z}} \times f_{\bar{z} \bar{z}}^{\perp}\right)_{z}=\mathrm{i} \lambda f_{\bar{z} \bar{z}}^{\perp} \perp N_{2}$. Thus we obtain $\left(f_{z} \times f_{z z}\right)_{z}=$ $a f_{\bar{z} \bar{z}}^{\perp}+b f_{z} \times f_{z z}$ with

$$
\begin{aligned}
& a \cdot \mu=\left\langle\left(f_{z} \times f_{z z}\right)_{z}, f_{z z}^{\perp}\right\rangle \quad=-\mathrm{i} h, \\
& b \cdot 2 \lambda \mu=\left\langle\left(f_{z} \times f_{z z}\right)_{z}, f_{\bar{z}} \times f_{\bar{z} \bar{z}}\right\rangle=2(\lambda \mu)_{z},
\end{aligned}
$$

which shows that $a=-\mathrm{i} h / \mu$ and $b=\log (\lambda \mu)_{z}=l_{z}+m_{z}$. Equation (6.3) follows applying complex conjugation and using (4.7).

[^6]
## 7. The canonical $G_{2}$ Connection

The three vectors $F_{1}=f_{z}, F_{2}=f_{z \bar{z}}^{\perp}, F_{3}=f_{\bar{z}} \times f_{\bar{z} \bar{z}}$ defined in (4.7) (spanning the isotropic subspace $E_{+}=\left\{v \in \mathbb{O}_{c}^{\prime}: f \times v=\mathrm{i} v\right\}$ ) are positive real multiples of $i-\mathrm{i} l i, j-\mathrm{i} l j, k-\mathrm{i} l k$, up to transformation with some element of $G_{2}=\operatorname{Aut}(\mathbb{O})$. Thus, up to positive factors, $F=\left(F_{1}, F_{2}, F_{3}\right)$ can be considered as a moving $G_{2^{-}}$ frame, a section of the $S U_{3}$-principal bundle $G_{2} \rightarrow G_{2} / S U_{3}=\mathbb{S}^{6}$, pulled back to $M$ via $f$. But as we see from Proposition 6.1, the derivative $D F$ cannot be expressed in terms of $F$ alone; one also needs $\bar{F}$. The reason is that the covariant derivative on $\mathbb{S}^{6}$ relies on the Levi-Civita parallel displacements which unfortunately do not preserve $J$, they do not belong to $G_{2}$. Therefore we will use another connection $\nabla$ on $\mathbb{S}^{6}$, whose parallel displacements belong to $G_{2}$ : a $G_{2}$-connection or hermitian connection. Thus we will derive formulas of the type $\nabla^{\prime} F=F B^{\prime}$ and $\nabla^{\prime \prime} F=F B^{\prime \prime}$ for some complex $(3 \times 3)$-matrices $B^{\prime}, B^{\prime \prime}$. It turns out that $B^{\prime}, B^{\prime \prime}$ depend only on the metric coefficients of the surface $f$ and some given holomorphic 6 -form $\Lambda_{3}$; this will prove existence and uniqueness of pseudoholomorphic maps.

A $G_{2}$-connection $\nabla=D+A$ needs to make $J$ parallel,

$$
0=\nabla_{v} J=D_{v} J+\left[A_{v}, J\right]
$$

where $\left(D_{v} J\right) w=(v \times w)_{p^{\perp}}$ for $v, w \in T_{p} \mathbb{S}^{6}=p^{\perp}$. Thus $\left[A_{v}, J\right]=-D_{v} J$. We may split $A_{v}=A_{v}^{+}+A_{v}^{-}$, where $A_{v}^{+}$commutes with $J$ and $A_{v}^{-}$anticommutes with $J$. Then $-D_{v} J=\left[A_{v}, J\right]=\left[A_{v}^{-}, J\right]=2 A_{v}^{-} J$, hence $A_{v}^{-}=\frac{1}{2}\left(D_{v} J\right) J$ while $A_{v}^{+}$is unrestricted.

Among the $G_{2}$-connections there is the canonical connection (see also [1]), which has the additional property that $G_{2}$ acts on $\mathbb{S}^{6}$ by affine transformations: $\nabla_{g V}(g W)=g\left(\nabla_{V} W\right)$ for any $g \in G_{2}$ and any two tangent vector fields $V, W$ on $\mathbb{S}^{6}$. Clearly $G_{2} \subset S O_{7}$ is affine also for the Levi-Civita connection $D$, hence it keeps $A=\nabla-D$ invariant. In particular, fixing a base point $p \in \mathbb{S}^{6}$, say $p=l$, the tensor $A$ at $p$ is invariant under the isotropy group $S U_{3}$ at $l$. Thus the map

$$
v \mapsto A_{v}^{+}: T_{p} \mathbb{S}^{6}=\mathbb{C}^{3} \rightarrow \mathbb{C}^{3 \times 3}
$$

is $S U_{3}$-equivariant. The group $S U_{3}$ acts on the matrix space $\mathbb{C}^{3 \times 3}$ by conjugation, splitting it into two equivalent subrepresentations (hermitian and antihermitian matrices), both of which are irreducible up to a one-dimensional fixed space. Thus there is no nonzero equivariant linear map $\mathbb{C}^{3} \rightarrow \mathbb{C}^{3 \times 3}$. Therefore the canonical connection satisfies $A_{v}^{+}=0$, hence $A_{v}=A_{v}^{-}=\frac{1}{2}\left(D_{v} J\right) J$ and therefore

$$
\nabla_{v}=D_{v}+A_{v}, \quad 2 A_{v}=\left(D_{v} J\right) J
$$

Now $\nabla_{v} J=\left[\nabla_{v}, J\right]=\left[D_{v}, J\right]+\left[A_{v}, J\right]=0$.

## 8. Canonical torsion and curvature on $\mathbb{S}^{6}$

It is well known that a canonical connection has parallel torsion and curvature tensors, which we are going to compute now. Let us put

$$
S_{v}=D_{v} J
$$

Since $J_{p} v=p \times v$ for any $p \in \mathbb{S}^{6}$ and $v \in T_{p} \mathbb{S}^{6}=p^{\perp}$, we have $S_{v} w=\left(D_{v} J\right) w=$ $(v \times w)_{p^{\perp}}=v \times w-\langle p, v \times w\rangle p$, and since $\langle p, v \times w\rangle=\langle p \times v, w\rangle=\langle J v, w\rangle$, we obtain

$$
\begin{equation*}
S_{v} w=v \times w-\langle J v, w\rangle p=(v w)^{T_{p} \mathbb{S}^{6}} \tag{8.1}
\end{equation*}
$$

where $p$ is the position vector, $v, w \in T_{p} \mathbb{S}^{6}$ and ()$^{T_{p} \mathbb{S}^{6}}$ denotes the projection onto $T_{p} \mathbb{S}^{6}$. Using the fact that the parallel displacements of $\nabla$ belong to the group $G_{2}$, which preserves the cross product and the inner product, it is clear that $S$ is a $\nabla$-parallel tensor (see [1, Lemma 2.4] for a direct proof). Note that $2 A=S J=-J S$, since $0=D\left(J^{2}\right)=S J+J S$. Further, $S_{v} w=-S_{w} v$ by (3.3).

The torsion tensor of $\nabla$ is

$$
T(v, w)=\nabla_{v} w-\nabla_{w} v-[v, w]=A_{v} w-A_{w} v .
$$

We have $2 A_{v} w=S_{v} J w=-J S_{v} w$, and thus $A_{v} w=-A_{w} v$. Hence

$$
\begin{equation*}
T(v, w)=S_{v} J w=-J S_{v} w \tag{8.2}
\end{equation*}
$$

which shows again that $T$ is $\nabla$-parallel since so are $S$ and $J$.
We want to compute $S$ in terms our frame $(F, \bar{F})$. By (4.9), (4.5), and (4.7), $F_{j}$ is a real multiple of

$$
F_{j}^{o}=\left(e_{j}-\mathrm{i} f e_{j}\right) / \sqrt{2}
$$

where $e_{1}, e_{2}, e_{3} \in \mathbb{O}^{\prime}$ is an orthonormal 3 -frame perpendicular to $f$ with $e_{3}=e_{1} e_{2}$. Since

$$
\begin{aligned}
\left(e_{i}-\mathrm{i} f e_{i}\right)\left(e_{j}-\mathrm{i} f e_{j}\right) & =2\left(e_{k}+\mathrm{i} f e_{k}\right), \\
\left(e_{i}-\mathrm{i} f e_{i}\right)\left(e_{j}+\mathrm{i} f e_{j}\right) & =0, \\
\left(e_{i}-\mathrm{i} f e_{i}\right)\left(e_{i}+\mathrm{i} f e_{i}\right) & =-2+2 \mathrm{i} f
\end{aligned}
$$

for $(i, j, k)=(1,2,3)$ up to cyclic permutations, we have from (8.1)

$$
S_{F_{i}^{o}} F_{j}^{o}=\sqrt{2} \bar{F}_{k}^{o}, \quad S_{F_{i}^{o}} \bar{F}_{j}^{o}=0, \quad S_{F_{i}^{o}} \bar{F}_{i}^{o}=0
$$

The real factors are given by (4.9). Thus

## Lemma 8.1.

$$
\begin{align*}
S_{F_{1}} F_{2} & =\bar{F}_{3}, \\
S_{F_{2}} F_{3} & =2 \mu \bar{F}_{1}, \\
S_{F_{3}} F_{1} & =2 \lambda \bar{F}_{2},  \tag{8.3}\\
S_{F_{j}} \bar{F}_{k} & =0, \quad \forall j, k
\end{align*}
$$

Recalling $2 A=S J$ and $J F_{j}=\mathrm{i} F_{j}$, we obtain:
Corollary 8.2. For $A^{\prime}=A_{F_{1}}$ and $A^{\prime \prime}=A_{\bar{F}_{1}}$ we have

$$
\begin{aligned}
2 A^{\prime} F_{1} & =0 \\
2 A^{\prime} F_{2} & =\mathrm{i} \bar{F}_{3}, \\
2 A^{\prime} F_{3} & =-2 \mathrm{i} \lambda \bar{F}_{2}, \\
2 A^{\prime \prime} F_{1} & =0 \\
2 A^{\prime \prime} F_{2} & =0, \\
2 A^{\prime \prime} F_{3} & =0
\end{aligned}
$$

Next we compute the curvature tensor $R$ of $\nabla$; see also [9, Cor. 3.4]. From $\nabla_{v}=D_{v}+A_{v}$ we obtain when $[v, w]=0$ :

$$
R_{v w}=\left[\nabla_{v}, \nabla_{w}\right]=\left[D_{v}, D_{w}\right]+D_{v} A_{w}-D_{w} A_{v}+\left[A_{v}, A_{w}\right] .
$$

Here $\left[D_{v}, D_{w}\right]=R^{o}$ is the curvature tensor of the sphere $\mathbb{S}^{6}$,

$$
\begin{equation*}
R_{v w}^{o} x=\langle x, w\rangle v-\langle x, v\rangle w . \tag{8.4}
\end{equation*}
$$

Now $2 A_{w}=\left(D_{w} J\right) J=S_{w} J$, hence $2 D_{v} A_{w}=D_{v}\left(S_{w} J\right)=\left(D_{v} D_{w} J\right) J+S_{w} S_{v}$. Thus

$$
2\left(D_{v} A_{w}-D_{w} A_{v}\right)=\left[D_{v}, D_{w}\right] J+\left[S_{w}, S_{v}\right]
$$

and moreover

$$
4\left[A_{v}, A_{w}\right]=\left[S_{v} J, S_{w} J\right]=\left[S_{v}, S_{w}\right],
$$

since $S_{v} J S_{w} J=-S_{v} J J S_{w}=S_{v} S_{w}$. Thus

$$
\begin{equation*}
R_{v w}=R_{v w}^{o}+(1 / 2)\left[R_{v w}^{o}, J\right]-(1 / 4)\left[S_{v}, S_{w}\right] \tag{8.5}
\end{equation*}
$$

Since $R_{o}$ is determined by the metric, which is parallel, and since $J$ and $S$ are parallel, we see directly that $R$ is parallel.
Lemma 8.3. For $R_{1 \overline{1}}:=R_{F_{1} \bar{F}_{1}}=\left[\nabla_{F_{1}}, \nabla_{\bar{F}_{1}}\right]=\left[\nabla^{\prime}, \nabla^{\prime \prime}\right]$ we have

$$
\begin{array}{lll}
R_{1 \overline{1}} F_{1}=\lambda F_{1}, & R_{1 \overline{1}} F_{2}=-\frac{\lambda}{2} F_{2}, & R_{1 \overline{1}} F_{3}=-\frac{\lambda}{2} F_{3} \\
R_{1 \overline{1}} \bar{F}_{1}=-\lambda \bar{F}_{1}, & R_{1 \overline{1}} \bar{F}_{2}=\frac{\lambda}{2} \bar{F}_{2}, & R_{1 \overline{1}} \bar{F}_{3}=\frac{\lambda}{2} \bar{F}_{3} . \tag{8.6}
\end{array}
$$

Proof. The first line follows from (8.5) with (8.4) and (8.3), where we put $v=F_{1}$ and $w=\bar{F}_{1}$. Applying $R_{1 \overline{1}}^{o}=R_{F_{1} \bar{F}_{1}}^{o}$ to $F_{1}, F_{2}, F_{3}$ we observe $\left\langle F_{1}, F_{j}\right\rangle=0$ and $\left\langle F_{1}, \bar{F}_{j}\right\rangle=\lambda \delta_{1 j}$, hence

$$
R_{1 \overline{1}}^{o} F_{1}=\lambda F_{1}, \quad \text { while } \quad R_{1 \overline{1}} F_{2}=0, R_{1 \overline{1}} F_{3}=0
$$

In particular, $R_{1 \overline{1}}$ commutes with $J$, and consequently the second term on the right hand side of (8.5) vanishes, $\left[R_{1 \overline{1}}, J\right]=0$. It remains to compute $\left[S_{F_{1}}, S_{\bar{F}_{1}}\right]$ :

$$
\begin{array}{lll}
S_{F_{1}}: & F_{2} \mapsto \bar{F}_{3}, & F_{3} \mapsto-2 \lambda \bar{F}_{2}, \\
S_{\bar{F}_{1}}: & \bar{F}_{2} \mapsto F_{3}, & \bar{F}_{3} \mapsto-2 \lambda F_{2},
\end{array}
$$

while $F_{1}, \bar{F}_{1}$ are mapped to 0 . Thus $\left[S_{F_{1}}, S_{\bar{F}_{1}}\right]$ has eigenvalues $-2 \lambda$ for $F_{2}, F_{3}$ and $2 \lambda$ for $\bar{F}_{2}, \bar{F}_{3}$, while $F_{1}, \bar{F}_{1}$ are mapped to 0 . Now the first line of (8.6) follows from (8.5).

For the second line we just observe that $R_{\overline{1} 1}=-R_{1 \overline{1}}$ and therefore $R_{1 \overline{1}} \bar{F}_{j}=$ $\overline{R_{\overline{1} 1} F_{j}}=-\overline{R_{1 \overline{1}} F_{j}}$.

## 9. Structure equations

From Proposition 6.1 and Corollary 8.2 we obtain the derivatives of the Frenet frame:

Proposition 9.1. Let $M$ be a Riemann surface and $f: M \rightarrow \mathbb{S}^{6}$ a pseudoholomorphic curve. Let $\nabla$ denote the canonical $G_{2}$-connection on $\mathbb{S}^{6}$ and let $\nabla^{\prime}=\nabla_{\partial / \partial z}$ and $\nabla^{\prime \prime}=\nabla_{\partial / \partial \bar{z}}$. Let $F_{1}=f_{z}, F_{2}=f_{z \bar{z}}^{\perp}, F_{3}=f_{\bar{z}} \times f_{\bar{z} \bar{z}}$ be the Frenet frame of $f$. Then

$$
\begin{array}{rlrl}
\nabla^{\prime} F_{1} & =l_{z} F_{1} & \quad+F_{2} \\
\nabla^{\prime} F_{2} & = & m_{z} F_{2}+\frac{\mathrm{i} h}{2 \lambda \mu} F_{3} \\
\nabla^{\prime} F_{3} & =0 \\
\nabla^{\prime \prime} F_{1} & =0 \\
\nabla^{\prime \prime} F_{2} & =-\frac{\mu}{\lambda} F_{1} \\
\nabla^{\prime \prime} F_{3} & = & (i \bar{h} / \mu) F_{2}+(l+m)_{\bar{z}} F_{3} .
\end{array}
$$

Corollary 9.2. The frame $F=\left(f_{z}, f_{z \bar{z}}^{\perp}, f_{\bar{z}} \times f_{\bar{z} \bar{z}}\right)$ of $E_{+}=\left\{v \in \mathbb{O}_{c}^{\prime}: f \times v=\mathrm{i} v\right\}$ solves the differential equations

$$
\begin{equation*}
\nabla^{\prime} F=F B^{\prime}, \quad \nabla^{\prime \prime} F=F B^{\prime \prime} \tag{9.1}
\end{equation*}
$$

with

$$
B^{\prime}=\left(\begin{array}{ccc}
l_{z} & 0 & 0  \tag{9.2}\\
1 & m_{z} & 0 \\
0 & \frac{i}{2 \lambda \mu} & 0
\end{array}\right), \quad B^{\prime \prime}=\left(\begin{array}{ccc}
0 & -\frac{\mu}{\lambda} & 0 \\
0 & 0 & \mathrm{i} \bar{h} / \mu \\
0 & 0 & (l+m)_{\bar{z}}
\end{array}\right) .
$$

Remark 9.3. In the superminimal case $h=0$ we see that $\nabla F_{3}$ is a multiple of $F_{3}$. In our analogy with the Frenet frame of a space curve $c$, the third vector $F_{3}$ corresponds to the binormal $f_{3}=f_{1} \times f_{2}$, where $f_{1}=c^{\prime}$ and $f_{2}=\left(c^{\prime \prime}\right)^{\perp}$, and $f_{3}^{\prime}$ is proportional to $f_{3}$ if and only if the torsion of $c$ vanishes (which means that $c$ is a planar curve). Thus Bryant [3] calls superminimal pseudoholomorphic curves torsion free. However, they are not "planar" in any sense: a weak analogue of planes would be a pseudoholomorphic embedding of a complex 2-dimensional manifold into $\mathbb{S}^{6}$, but there are none. This makes these mappings particularly interesting.
Remark 9.4. One might wonder why the matrices $B^{\prime}, B^{\prime \prime}$ obviously do not belong to $\mathfrak{s u}_{3}$. The reason is that the frame $F$ is not normalized. This can easily be corrected by passing to the normalized frame $F^{o}$ with $F=F^{o} D$, where $D=$
$\operatorname{diag}(\sqrt{\lambda}, \sqrt{\mu}, \sqrt{2 \lambda \mu})$. We have $\nabla F=\nabla\left(F^{o} D\right)=\left(\nabla F^{o}\right) D+F^{o} \partial D$ and $F B=$ $F^{o} D B$. Thus from $\nabla F=F B$ we obtain $\nabla F^{o}=F^{o} B_{o}$, with

$$
\begin{equation*}
B_{o}=D B D^{-1}-(\partial D) D^{-1} \tag{9.3}
\end{equation*}
$$

We have

$$
\begin{align*}
D B^{\prime} D^{-1} & =\left(\begin{array}{lll}
\sqrt{\lambda} & & \\
& \sqrt{\mu} & \\
& & \sqrt{2 \lambda \mu}
\end{array}\right)\left(\begin{array}{ccc}
l_{z} & 0 & 0 \\
1 & m_{z} & 0 \\
0 & \frac{i}{2 \lambda \mu} & 0
\end{array}\right)\left(\begin{array}{lll}
\frac{1}{\sqrt{\lambda}} & & \\
& \frac{1}{\sqrt{\mu}} & \\
& & \frac{1}{\sqrt{2 \lambda \mu}}
\end{array}\right) \\
& =\left(\begin{array}{ccc}
l_{z} & & \\
\frac{\sqrt{\mu}}{\sqrt{\lambda}} & m_{z} & \\
& \frac{\mathrm{i} h}{\mu \sqrt{2 \lambda}} & 0
\end{array}\right), \\
\left(\partial_{z} D\right) D^{-1} & =\frac{1}{2} \operatorname{diag}\left(l_{z}, m_{z}, l_{z}+m_{z}\right), \text { hence by }(9.3), \\
B_{o}^{\prime} & =\left(\begin{array}{ccc}
\frac{1}{2} l_{z} & 0 & 0 \\
\frac{\sqrt{\mu}}{\sqrt{\lambda}} & \frac{1}{2} m_{z} & 0 \\
0 & \frac{i h}{\mu \sqrt{2 \lambda}} & -\frac{1}{2}\left(l_{z}+m_{z}\right)
\end{array}\right) . \tag{9.4}
\end{align*}
$$

Similarly,

$$
B_{o}^{\prime \prime}=\left(\begin{array}{ccc}
-\frac{1}{2} l_{\bar{z}} & -\frac{\sqrt{\mu}}{\sqrt{\lambda}} & 0  \tag{9.5}\\
0 & -\frac{1}{2} m_{\bar{z}} & \frac{i \bar{h}}{\mu \sqrt{2 \lambda}} \\
0 & 0 & \frac{1}{2}\left(l_{\bar{z}}+m_{\bar{z}}\right)
\end{array}\right)=-\left(B_{o}^{\prime}\right)^{*} .
$$

Recall that $\nabla_{x}=\nabla^{\prime}+\nabla^{\prime \prime}$ and $\nabla_{y}=\mathrm{i}\left(\nabla^{\prime}-\nabla^{\prime \prime}\right)$, where $z=x+\mathrm{i} y$ is the conformal coordinate. Thus

$$
\nabla_{x} F^{o}=F^{o}\left(B_{o}^{\prime}+B_{o}^{\prime \prime}\right), \quad \nabla_{y} F^{o}=\mathrm{i} F^{o}\left(B_{o}^{\prime}-B_{o}^{\prime \prime}\right),
$$

and the matrices $B_{o}^{\prime}+B_{o}^{\prime \prime}$ and $\mathrm{i}\left(B_{o}^{\prime}-B_{o}^{\prime \prime}\right)$ belong to $\mathfrak{s u}_{3}$.

## 10. Integrability conditions

The coefficients of $B^{\prime}$ and $B^{\prime \prime}$ still must satisfy some relations, the integrability conditions for the overdetermined system (9.1). In fact,

$$
\begin{aligned}
& \nabla^{\prime} \nabla^{\prime \prime} F=\nabla^{\prime}\left(F B^{\prime \prime}\right)=F B^{\prime} B^{\prime \prime}+F B_{z}^{\prime \prime}, \\
& \nabla^{\prime \prime} \nabla^{\prime} F=\nabla^{\prime \prime}\left(F B^{\prime}\right)=F B^{\prime \prime} B^{\prime}+F B_{\bar{z}}^{\prime},
\end{aligned}
$$

which implies

$$
\left[\nabla^{\prime}, \nabla^{\prime \prime}\right] F=F\left(\left[B^{\prime}, B^{\prime \prime}\right]+B_{z}^{\prime \prime}-B_{\bar{z}}^{\prime}\right) .
$$

On the other hand, we have seen in Lemma 8.3:

$$
\left[\nabla^{\prime}, \nabla^{\prime \prime}\right] F=R_{1 \overline{1}} F=F \operatorname{diag}\left(\lambda,-\frac{\lambda}{2},-\frac{\lambda}{2}\right)
$$

Thus an integrability condition for (9.1) is

$$
\begin{equation*}
\operatorname{diag}\left(\lambda,-\frac{\lambda}{2},-\frac{\lambda}{2}\right)=R_{1 \overline{1}}=\left[B^{\prime}, B^{\prime \prime}\right]+\left(B^{\prime \prime}\right)_{z}-\left(B^{\prime}\right)_{\bar{z}} \tag{10.1}
\end{equation*}
$$

The commutator [ $B^{\prime}, B^{\prime \prime}$ ] equals

$$
\left[\left(\begin{array}{ccc}
l_{z} & 0 & 0 \\
1 & m_{z} & 0 \\
0 & \frac{i}{2 \lambda \mu} & 0
\end{array}\right),\left(\begin{array}{ccc}
0 & -\frac{\mu}{\lambda} & 0 \\
0 & 0 & \mathrm{i} \bar{h} / \mu \\
0 & 0 & (l+m)_{\bar{z}}
\end{array}\right)\right]=\left(\begin{array}{ccc}
\frac{\mu}{\lambda} & \frac{\mu}{\lambda}\left(m_{z}-l_{z}\right) & 0 \\
0 & -\frac{\mu}{\lambda}+\frac{|h|^{2}}{2 \lambda \mu^{2}} & \mathrm{i} \bar{h} m_{z} / \mu \\
0 & -\frac{i h}{2 \lambda \mu}(l+m)_{\bar{z}} & -\frac{|h|^{2}}{2 \lambda \mu^{2}}
\end{array}\right)
$$

and the derivatives are

$$
\left(B^{\prime \prime}\right)_{z}=\left(\begin{array}{ccc}
0 & -\left(\frac{\mu}{\lambda}\right)_{z} & 0 \\
0 & 0 & \mathrm{i} \bar{h}\left(\frac{1}{\mu}\right)_{z} \\
0 & 0 & (l+m)_{\bar{z} z}
\end{array}\right), \quad\left(B^{\prime}\right)_{\bar{z}}=\left(\begin{array}{ccc}
l_{z \bar{z}} & 0 & 0 \\
0 & m_{z \bar{z}} & 0 \\
0 & \frac{\mathrm{i} h}{2}\left(\frac{1}{\lambda \mu}\right)_{\bar{z}} & 0
\end{array}\right) .
$$

Since

$$
\left(\frac{1}{\mu}\right)_{z}=-\frac{m_{z}}{\mu}, \quad\left(\frac{1}{\lambda \mu}\right)_{\bar{z}}=-\frac{(l+m)_{\bar{z}}}{\lambda \mu}, \quad\left(\frac{\mu}{\lambda}\right)_{z}=(m-l)_{z} \frac{\mu}{\lambda},
$$

we obtain from (10.1):

$$
\begin{align*}
\operatorname{diag}\left(\lambda,-\frac{\lambda}{2},\right. & \left.-\frac{\lambda}{2}\right) \\
& =\operatorname{diag}\left(\frac{\mu}{\lambda}-l_{z \bar{z}}, \frac{|h|^{2}}{2 \lambda \mu^{2}}-\frac{\mu}{\lambda}-m_{z \bar{z}},(l+m)_{z \bar{z}}-\frac{|h|^{2}}{2 \lambda \mu^{2}}\right) . \tag{10.2}
\end{align*}
$$

Lemma 10.1. Let $\lambda, \mu$ be absolute value type functions on $M$ such that

$$
\begin{equation*}
\mu=\lambda\left(\lambda+l_{z \bar{z}}\right) \tag{10.3}
\end{equation*}
$$

and let $h: M \rightarrow \mathbb{C}$ be a holomorphic function. Then (10.2) is satisfied if and only if

$$
\begin{equation*}
|h|^{2}=\lambda^{2} \mu^{2}+2 \lambda \mu^{2}(l+m)_{z \bar{z}} . \tag{10.4}
\end{equation*}
$$

Proof. The condition (10.3) is equivalent to the equality in the first entry, and moreover, the equalities in the second and third entries become the same. The equality in the third entry is (10.4).
Lemma 10.2. If $F$ is the Frenet frame of a pseudoholomorphic curve $f: M \rightarrow \mathbb{S}^{6}$ with Gaussian curvature $K$ and $h=\left\langle f_{z z z}, f_{z z z}\right\rangle$, then (10.4) is equivalent to

$$
\begin{equation*}
|h|^{2}=\lambda^{6}(1-K)^{2}(\Delta \log (1-K)+1-6 K), \tag{10.5}
\end{equation*}
$$

where $\Delta$ is the Laplacian of the induced metric on $M$.
Proof. We have

$$
l+m=\log (\lambda \mu) \stackrel{(4.10)}{=} \log \left(\lambda^{3}(1-K)\right)=3 \log \lambda+\log (1-K) .
$$

Further, from $(\log \lambda)_{z \bar{z}}=-\lambda K$ and $\mu=\lambda^{2}(1-K)(c f .(4.10))$ and $\partial_{z} \partial_{\bar{z}}=\frac{1}{2} \lambda \Delta$ we obtain

$$
\begin{aligned}
2(l+m)_{z \bar{z}} & =-6 \lambda K+\lambda \Delta \log (1-K) \\
2 \lambda \mu^{2}(l+m)_{z \bar{z}} & =\lambda^{2} \mu^{2}(-6 K+\Delta \log (1-K)) \\
\lambda^{2} \mu^{2}+2 \lambda \mu^{2}(l+m)_{z \bar{z}} & =\lambda^{2} \mu^{2}(1-6 K+\Delta \log (1-K)) \\
& =\lambda^{6}(1-K)^{2}(1-6 K+\Delta \log (1-K)) .
\end{aligned}
$$

Thus the conditions (10.4) and (10.5) are the same.

## 11. Existence of pseudoholomorphic curves

Let $M \subset \mathbb{C}$ be an open domain. Suppose that on $M$ a holomorphic function $h$ and absolute value type functions $\lambda, \mu$ are given satisfying (4.10) and (10.4),

$$
\begin{aligned}
\mu & =\lambda\left(\lambda+l_{z \bar{z}}\right), \\
|h|^{2} & =\lambda^{2} \mu^{2}+2 \lambda \mu^{2}(l+m)_{z \bar{z}},
\end{aligned}
$$

where $l=\log \lambda$ and $m=\log \mu$. Over $M$ we consider the trivial vector bundle $E=M \times \mathbb{O}_{c}^{\prime}$ with a connection $\nabla$ defined by

$$
\nabla^{\prime} F=F B^{\prime} \quad \text { and } \quad \nabla^{\prime \prime} F=F B^{\prime \prime}
$$

where $B^{\prime}, B^{\prime \prime}$ are given in (9.2),

$$
B^{\prime}=\left(\begin{array}{ccc}
l_{z} & 0 & 0 \\
1 & m_{z} & 0 \\
0 & \frac{\mathrm{i} h}{2 \lambda \mu} & 0
\end{array}\right), \quad B^{\prime \prime}=\left(\begin{array}{ccc}
0 & -\left(\lambda+l_{z \bar{z}}\right) & 0 \\
0 & 0 & \mathrm{i} \bar{h} / \mu \\
0 & 0 & (l+m)_{\bar{z}}
\end{array}\right),
$$

and where $F=\left(F_{1}, F_{2}, F_{3}\right): M \rightarrow\left(\mathbb{O}_{c}^{\prime}\right)^{3}$ with

$$
\begin{array}{lll}
F_{1}=\sqrt{\lambda} F_{1}^{o}, & F_{2}=\sqrt{\mu} F_{2}^{o}, & F_{3}=\sqrt{2 \lambda \mu} F_{3}^{o}, \\
F_{1}^{o}=(\underline{i}-\underline{\mathrm{i}} \underline{i}) / \sqrt{2}, & F_{2}^{o}=(\underline{j}-\underline{\mathrm{i} l} \underline{j}) / \sqrt{2}, & F_{3}^{o}=(\underline{k}-\mathrm{i} \underline{l k}) / \sqrt{2},
\end{array}
$$

see (4.9). Here, $\underline{i}, \underline{j}, \underline{k}, \underline{l}, \underline{l}, \underline{l} \underline{j}, \underline{l} \underline{k}$ denote the basis of $\mathbb{O}_{c}^{\prime}$, considered as constant sections on $E$. In particular, the only nonzero derivatives are

$$
\begin{align*}
\nabla^{\prime} F_{1} & =l_{z} F_{1}+F_{2}, \\
\nabla^{\prime} F_{2} & =m_{z} F_{2}+\frac{\mathrm{i} h}{2 \lambda \mu} F_{3}, \\
\nabla^{\prime \prime} F_{2} & =-\left(\lambda+l_{z \bar{z}}\right) F_{1},  \tag{11.1}\\
\nabla^{\prime \prime} F_{3} & =\frac{\mathrm{i} \bar{h}}{\mu} F_{2}+(l+m)_{\bar{z}} F_{3} .
\end{align*}
$$

On $E$ we have the tensor fields $J, S, T, R$, where

$$
J v=\underline{l} \times v, \quad S_{v} w=(v \times w)^{T}
$$

and $T, R$ are given by (8.2), (8.5), (8.4). In order to apply the existence and uniqueness theorem in [6] we need $\nabla$ to be a metric connection and $J, S$ (and hence $T, R)$ to be parallel with respect to $\nabla$. This follows by passing to the normalized frame $F^{o}$ and using that $B_{o}^{\prime}+B_{o}^{\prime \prime}$ and $\mathrm{i}\left(B_{o}^{\prime}-B_{o}^{\prime \prime}\right)$ belong to the Lie algebra $\mathfrak{s u}_{3}$ acting on $\operatorname{Span}_{\mathbb{R}}(\underline{i}, \underline{j}, \underline{k}, \underline{l}, \underline{j}, \underline{l} \underline{)})=\mathbb{C}^{3}$ with $\underline{l}$ as complex structure, see (9.4), (9.5) (Remark 9.4). The holonomy group belongs to $S U_{3}$, which preserves the metric and the tensors $J$ and $S$, hence $R$.

We are ready now to prove Hashimoto's result [8].

Theorem 11.1. Let $M$ be a simply connected Riemann surface carrying a compatible Riemannian metric ds ${ }^{2}$, possibly with branch points, ${ }^{10}$ and a holomorphic 6 -form $\Lambda$. Let $K$ be the Gaussian curvature and $\Delta$ the Laplacian of ds ${ }^{2}$. Suppose that $1-K$ is an absolute value type function. Then there is a unique pseudoholomorphic curve $f: M \rightarrow \mathbb{S}^{6}$ (up to translation with elements of $G_{2}$ ) such that ds ${ }^{2}$ is the induced metric and $\Lambda=\Lambda_{3}$ is the third Hopf differential (see section 5) if and only if

$$
\begin{equation*}
(1-K)^{2}(\Delta \log (1-K)+1-6 K)=|\Lambda|^{2} . \tag{11.2}
\end{equation*}
$$

Proof. " $\Rightarrow$ " If such a pseudoholomorphic curve $f: M \rightarrow \mathbb{S}^{6}$ is given, then (11.2) is satisfied by Lemma 10.1 and (10.5); note that

$$
\begin{equation*}
|\Lambda|^{2}=|h|^{2} / \lambda^{6} \tag{11.3}
\end{equation*}
$$

" $\Leftarrow$ ": Let $\left(M, d s^{2}\right)$ and $\Lambda$ be given with (11.2). Choosing a conformal coordinate $z$ on some simply connected open subset $M_{o} \subset M$, we have $d s^{2}=2 \lambda d z d \bar{z}$ for some absolute value type function $\lambda$, and the curvature of $d s^{2}$ is $K=-l_{z \bar{z}} / \lambda$, where $l:=\log \lambda$. Moreover, $\Lambda=h(z) d z^{6}$ for some holomorphic function $h$ with (11.3). Further we define the absolute value type function

$$
\mu=\lambda\left(\lambda+l_{z \bar{z}}\right)=\lambda^{2}(1-K) .
$$

Using these functions, we consider the bundle $E=M_{o} \times T$ for

$$
T=\operatorname{Span}_{\mathbb{C}}(i, j, k, i l, j l, k l)
$$

with sections $F_{1}, F_{2}, F_{3}$ and a connection $\nabla$ as defined in (11.1) at the beginning of this section. By the main theorem of [6], there exist a smooth map $f: M \rightarrow \mathbb{S}^{6}$ and a bundle isomorphism $\Phi: E \rightarrow f^{*} T S$ preserving the metric and the tensors $J, S, R$ such that

$$
\begin{equation*}
\Phi \circ f_{z}=F_{1} \tag{11.4}
\end{equation*}
$$

if and only if

$$
\begin{array}{rlll}
\nabla^{\prime} \bar{F}_{1}-\nabla^{\prime \prime} F_{1} & = & T\left(F_{1}, \bar{F}_{1}\right) & = \\
{\left[\nabla^{\prime}, \nabla^{\prime \prime}\right] F} & = & R_{F_{1} \bar{F}_{1}} F & = \tag{11.5}
\end{array} F \operatorname{diag}\left(\lambda,-\frac{\lambda}{2},-\frac{\lambda}{2}\right) .
$$

The first equation holds by (11.1) since $\nabla^{\prime} \bar{F}_{1}=0=\nabla^{\prime \prime} F_{1}$.
The second equation comes down to (10.1) and (10.2) which in turn is equivalent to (10.5) or (11.2), by Lemma 10.1. This proves existence and uniqueness of a pair of maps $(f, \Phi)$ satisfying (11.4), and $f$ is pseudoholomorphic since $F_{1}$ and $f_{z}$ lie in the i-eigenspace of $J$. Moreover, $F=\left(F_{1}, F_{2}, F_{3}\right)$ becomes the Frenet frame along $f$ (via $\Phi$ ), using (11.1). In particular, from the " $\Rightarrow$ "-part we see $h=\left\langle\left(F_{2}\right)_{z},\left(F_{2}\right)_{z}\right\rangle$, cf. (5.1). This finishes the proof.

[^7]Remark 11.2. Replacing $\Lambda$ by $e^{i \theta} \Lambda$ for some constant angle $\theta$ does not change the condition (11.2). This gives the associated family of the minimal surface $f$ which also consists of pseudoholomorphic curves.
Corollary 11.3. Let $\left(M, d s^{2}\right)$ be as in the assumptions of Theorem 11.1. Then there is a superminimal ("torsion free") pseudoholomorphic curve $f: M \rightarrow \mathbb{S}^{6}$, unique up to translations in $G_{2}$, with induced metric $d s^{2}$ if and only if

$$
\begin{equation*}
\Delta \log (1-K)=6 K-1 \tag{11.6}
\end{equation*}
$$

## 12. Pseudoholomorphic curves in $\mathbb{S}^{5}$

Another interesting special case is when a pseudoholomorphic curve $f: M \rightarrow \mathbb{S}^{6}$ actually takes values in some equator sphere $\mathbb{S}^{5} \subset \mathbb{S}^{6}$. We will call it a pseudoholomorphic curve in $\mathbb{S}^{5}$.
Lemma 12.1. Let $f: M \rightarrow \mathbb{S}^{6}$ be a pseudoholomorphic curve and $z$ a conformal coordinate on $M$. Then $f$ takes values in some great sphere $\mathbb{S}^{5} \subset \mathbb{S}^{6}$ if and only if

$$
|h|=\lambda \mu .
$$

Proof. Assume that $f$ lies in $\mathbb{S}^{5}$. Then there exists a constant unit vector $\xi$ (inside $N_{2}$ ) such that $\langle f, \xi\rangle=0$. Using $f_{z}, f_{z z}, f_{z z z} \perp \xi$ and (6.1) we obtain

$$
\lambda \mu\left\langle f_{z} \times f_{z z}, \xi\right\rangle=h\left\langle f_{\bar{z}} \times f_{\bar{z} \bar{z}}, \xi\right\rangle
$$

and by conjugation

$$
\lambda \mu\left\langle f_{\bar{z}} \times f_{\bar{z} \bar{z}}, \xi\right\rangle=\bar{h}\left\langle f_{z} \times f_{z z}, \xi\right\rangle
$$

Multiplying these two equations we find $|h|=\lambda \mu$.
Conversely, we assume that $|h|=\lambda \mu$. Then comparing (6.1) and its conjugate we obtain a linear relation between $\left(\left(f_{\overline{\bar{z}}}^{\perp}\right)_{\bar{z}}\right)^{N_{2}}$ and its conjugate:

$$
\begin{equation*}
\frac{h}{\lambda \mu}\left(\left(f_{\bar{z} \bar{z}}^{\perp}\right)_{\bar{z}}\right)^{N_{2}}=\left(\left(f_{z \bar{z}}^{\perp}\right)_{z}\right)^{N_{2}} . \tag{12.1}
\end{equation*}
$$

Thus the real and the imaginary part of $\left(\left(f_{z z}^{\perp}\right)_{z}\right)^{N_{2}}$ are linearly dependent, and hence there is a real unit vector $\xi \in N_{2}$ which is perpendicular to $\left(\left(f_{z z}^{\perp}\right)_{z}\right)^{N_{2}}$. Consequently, $\xi$ is perpendicular to all derivatives of $f$ up to third order, and hence $\xi_{z} \perp f, f_{z}, f_{\bar{z}}, f_{z z}, f_{\bar{z} \bar{z}}, \xi$. So $\xi_{z}$ must be a multiple of $\left(\left(f_{z z}^{\perp}\right)_{z}\right)^{N_{2}}$, and by (12.1) the same holds for $\xi_{\bar{z}}$. On the other hand, $\left\langle\xi_{\bar{z}},\left(f_{z \bar{z}}^{\perp}\right)_{z}^{N_{2}}\right\rangle=\left\langle\xi_{\bar{z}},\left(f_{z \bar{z}}^{\perp}\right)_{z}\right\rangle=$ $-\left\langle\xi,\left(f_{z z}^{\perp}\right)_{z \bar{z}}\right\rangle=0$ since from $f_{z z}^{\perp}=f_{z z}+l_{z} f_{z}$ we obtain $\left(f_{z z}^{\perp}\right)_{z \bar{z}}^{\perp}=f_{z z z \bar{z}}+\left(l_{z} f_{z}\right)_{z \bar{z}} \in$ $\operatorname{Span}\left(f, f_{z}, f_{z z}\right) \perp \xi_{z}$. Thus $\xi$ is a constant vector and we conclude that $f$ lies in $\mathbb{S}^{5}=\mathbb{S}^{6} \cap \xi^{\perp}$.

Theorem 12.2. Let $M$ be a simply connected Riemann surface with compatible metric ds ${ }^{2}$ (possibly with branch points), and let $K$ be its Gaussian curvature and $\Delta$ its Laplacian. Suppose that $1-K$ is an absolute value type function. Then there is an isometric pseudoholomorphic map $f: M \rightarrow \mathbb{S}^{5}$ if and only if

$$
\begin{equation*}
\Delta \log (1-K)=6 K \tag{12.2}
\end{equation*}
$$

In fact, up to translations with elements of $G_{2}$ there is precisely one associated family of such maps.

Proof. If $f: M \rightarrow \mathbb{S}^{5}$ is pseudoholomorphic with induced metric $d s^{2}=2 \lambda d z d \bar{z}$, we have $|h|=\lambda \mu$ and $|h|^{2}=\lambda^{2} \mu^{2}=\lambda^{6}(1-K)^{2}$ using $\mu=\lambda^{2}(1-K)$. Thus the integrability condition

$$
\begin{equation*}
\lambda^{6}(1-K)^{2}(\Delta \log (1-K)+1-6 K)=|h|^{2} \tag{10.5}
\end{equation*}
$$

becomes (12.2). Conversely, (12.2) becomes (10.5) when we put $|h|:=\mu \lambda=$ $\lambda^{3}(1-K)$. Then

$$
\Delta \log |h|=3 \Delta \log \lambda+\Delta \log (1-K)=0,
$$

using (12.2) and the relation between conformal factor and curvature, $\Delta \log \lambda=$ $-2 K$. Thus $\log |h|$ is harmonic, hence the real part of a holomorphic function, and $|h|$ is the absolute value of a holomorphic function $h$, uniquely determined up to some constant phase factor $e^{i \theta}$. Thus $\Lambda=h d z^{6}$ defines a holomorphic 6 -form, and we conclude from Theorem 11.1 that there is a pseudoholomorphic map $f: M \rightarrow \mathbb{S}^{6}$ with induced metric $d s^{2}$. Since $|h|=\lambda \mu$, we see from Lemma 12.1 that $f$ takes values in some great sphere $\mathbb{S}^{5} \subset \mathbb{S}^{6}$.

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    ${ }^{1}$ This condition makes sense even at the zeros of $1-K$. In fact, for a minimal surface in $\mathbb{S}^{3}$, the expression $1-K$ is a so called absolute value type function [5], the absolute value of a holomorphic function (which may have zeros) multiplied by a positive function. Then $\Delta \log (1-K)$ is still defined at the zeros of $1-K$.

[^1]:    ${ }^{2}$ The term "curve" means complex curve, parametrized on a Riemann surface.
    ${ }^{3}$ To simplify notation, we think of $G$ as a matrix group, $G \subset \mathbb{R}^{n \times n}$.

[^2]:    ${ }^{4}$ If $a \in \mathbb{O}^{\prime}$ and $|a|=1$, then $|1 \pm a|=\sqrt{2}$, hence $|1+a||1-a|=2$. On the other hand, $(1+a)((1-a) x)=(1-a) x+a(x-a x)=x-a(a x)$ for all $x \in \mathbb{O}$, and $|(1+a)((1-a) x)|=2|x|$. Thus $|x-a(a x)|=2|x|$. This is impossible unless the two vectors $x$ and $-a(a x)$ (which have equal length) are equal, $a(a x)=-x$. This shows rule (1); rules $(2),(3)$ can be proved similarly.
    ${ }^{5}$ Any automorphism of $\mathbb{O}$ is orthogonal: it preserves real and imaginary octonions since real octonions are real multiples of 1 and imaginary octonions are those which square to negative real multiples of 1. Thus an automorphism preserves the conjugation $a^{*}=\Re a-\Im a$ and also the norm $|a|^{2}=a^{*} a$ for any $a \in \mathbb{O}$.

[^3]:    ${ }^{6}$ In the case of $\mathbb{S}^{2}$ we even obtain $D J=0$ (Kähler property) since $v \times w$ is normal when $v, w$ are tangent vectors, hence $\left(D_{v} J\right) w=(v \times w)^{T}=0$.

[^4]:    ${ }^{7}$ Putting $E_{k}$ the span of all derivatives of $f$ with degree up to $k$ where $k \geq 2$, we define $N_{k-1}$ recursively as the orthogonal complement of $N_{k-2}$ in $E_{k}$, where $N_{0}$ is the tangent space, the span of the first derivatives.

[^5]:    ${ }^{8}$ A conformal harmonic map $f: M \rightarrow \mathbb{S}^{2 m}$ with all $\Lambda_{k}=0$ but the highest one $\Lambda_{m-1}$ (which then must be holomorphic) is called superconformal.

[^6]:    ${ }^{9}$ Recall that by (4.8) any $v \in T_{f} \mathbb{S}^{6}=f^{\perp}$ has the representation $v=w+\bar{w}$ with

    $$
    w=\left\langle v, \bar{F}_{1}\right\rangle F_{1} / \lambda+\left\langle v, \bar{F}_{2}\right\rangle F_{2} / \mu+\left\langle v, \bar{F}_{3}\right\rangle F_{3} /(2 \lambda \mu) .
    $$

[^7]:    ${ }^{10} \mathrm{~A}$ compatible Riemannian metric of a Riemann surface is locally of the type $d s^{2}=2 \lambda d z d \bar{z}$ for some conformal coordinate $z$ on $M$, where $\lambda$ is a positive function. If we allow for isolated zeros of $\lambda$ such that $\lambda$ is an absolute value type function, such zeros are called branch points of the metric $d s^{2}$.

