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# Bounds for Invariance Pressure*

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**Abstract**

This paper provides an upper bound for the invariance pressure of control sets with nonempty interior and a lower bound for sets with finite volume. In the special case of the control set of a hyperbolic linear control system on $\mathbb{R}^d$ this yields an explicit formula. Further applications to linear control systems on Lie groups and to inner control sets are discussed.

*Keywords:* Invariance pressure, invariance entropy, control sets

## 1. Introduction

The notion of invariance pressure generalizes invariance entropy by adding potentials $f$ on the control range. It has been introduced and analyzed in Colonius, Cossich and Santana [5, 6]. Zhong and Huang [19] show that invariance pressure can be characterized as a dimension-like notion within the framework due to Pesin. A basic reference for invariance entropy is Kawan's monograph [17]; here also the relation to minimal data rates is explained which gives the main motivation from applications. Further references include the seminal paper Nair, Evans, Mareels and Moran [18] as well as Colonius and Kawan [8] and Da Silva and Kawan [12], [13]. In the latter

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paper, robustness properties in the hyperbolic case are proved. Huang and Zhong [15] show that several generalized notions of invariance entropy fit into the dimension-theoretic framework.

The main results of the present paper are upper and lower bounds for the invariance pressure of compact subsets $K$ in a control set $D$ with nonvoid interior and compact closure as well as a formula for the invariance pressure in the case of hyperbolic linear control systems on $\mathbb{R}^d$ where a unique control set with nonvoid interior exists. We also give applications for inner control sets and for certain linear systems on Lie groups. Invariance entropy of these systems has been analyzed by Da Silva [10].

Section 2 collects results on linearization of control systems and on the notion of invariance pressure. Upper and lower bounds for invariance pressure are given in Sections 3 and 4, respectively. Section 5 presents a formula for the invariance pressure of linear control systems on $\mathbb{R}^d$ and Section 6 discusses applications to linear systems on Lie groups and for inner control sets.

## 2. Preliminaries

In this section we first recall basic notions for control systems on manifolds and their linearization. Then the concepts of invariance pressure and outer invariance pressure are presented as well as some of their properties.

### *2.1. Control systems and linearization*

Throughout the paper $M$ will denote a smooth manifold, that is, a connected, second-countable, topological Hausdorff manifold endowed with a $C^\infty$ differentiable structure. A continuous-time ***control system*** on a smooth manifold $M$ is a family of ordinary differential equations

$$\dot{x}(t) = F(x(t), \omega(t)), \omega \in \mathcal{U}, \tag{1}$$

on $M$ which is parametrized by measurable functions $\omega : \mathbb{R} \to \mathbb{R}^m$, $\omega(t) \in U \subset \mathbb{R}^m$ almost everywhere, called ***controls*** forming the set $\mathcal{U}$ of ***admissible control functions***, where $U \subset \mathbb{R}^m$ is a compact set, the ***control range***. The function $F : M \times \mathbb{R}^m \to TM$ is a $C^1$-map such that for each $u \in U$, $F_u(\cdot) := F(\cdot, u)$ is a smooth vector field on $M$. For each $x \in M$ and $\omega \in \mathcal{U}$, we suppose that there exists an unique solution $\varphi(t, x, \omega)$ which is defined for all $t \in \mathbb{R}$. We usually refer to the solution $\varphi(\cdot, x, \omega)$ as a ***trajectory*** of $x$ with control function $\omega$ and write $\varphi_t(x, \omega) = \varphi(t, x, \omega)$ where convenient. We need several notions characterizing controllability properties of subsets of the state space $M$ of system (1). For $x \in M$ and $t > 0$, the ***set of points reachable from*** $x$ ***up to time*** $t$ and the ***set of points controllable to*** $x$ ***within time*** $t$ are given by

$$\mathcal{O}^+_{\leq t}(x) := \{y \in M;\ \text{ there are } s \in [0,t] \text{ and } \omega \in \mathcal{U} \text{ with } \varphi(s,x,\omega) = y\},$$

and

$$\mathcal{O}^-_{\leq t}(x) := \{y \in M;\ \text{ there are } s \in [0,t] \text{ and } \omega \in \mathcal{U} \text{ with } \varphi(s,y,\omega) = x\},$$

respectively. The ***positive*** and ***negative orbits from*** $x \in M$ are

$$\mathcal{O}^+(x) := \bigcup_{t>0} \mathcal{O}^+_{\leq t}(x) \text{ and } \mathcal{O}^-(x) := \bigcup_{t>0} \mathcal{O}^-_{\leq t}(x),$$

respectively.

A key concept of this paper is presented in the following definition.

**Definition 1.** *A subset $D$ of $M$ is a* ***control set*** *if*

*(i) for each $x \in D$, there exists $\omega \in \mathcal{U}$ with $\varphi(\mathbb{R}_+, x, \omega) \subset D$ (controlled invariance);*

*(ii) for each $x \in D$ one has $D \subset \overline{\mathcal{O}^+(x)}$ (approximate controllability);*

*(iii) $D$ is maximal with these properties.*

If for all $t > 0$ the sets $\mathcal{O}^-_{\leq t}(x)$ and $\mathcal{O}^+_{\leq t}(x)$ have nonempty interior, we say that system (1) is ***locally accessible from*** $x \in M$. We are mainly interested in control sets with nonvoid interior which are locally accessible from all $x \in \mathrm{int}D$, since here a general theory can be developed. In particular, they enjoy the property $\mathrm{int}D \subset \mathcal{O}^+(x)$ for all $x \in D$, cf. Colonius and Kliemann [9, Lemma 3.2.13].

Next we recall some basic concepts and results on linearization of a control system on a smooth Riemannian manifold $(M, g)$, cf. Kawan [17].

**Definition 2.** *For a control-trajectory pair $(\omega(\cdot), \varphi(\cdot, x, \omega))$ the linearized system is given by*

$$\frac{Dz}{dt}(t) = A(t)z(t) + B(t)\mu(t),\ \ \mu \in L^\infty(\mathbb{R}, \mathbb{R}^m), \tag{2}$$

*where $A(t) := \nabla F_{\omega(t)}(\varphi(t,x,\omega))$ and $B(t) := D_2F(\varphi(t,x,\omega), \omega(t))$.*

The derivative on the left-hand side of (2) is the covariant derivative of $z(\cdot)$ along $\varphi(\cdot, x, \omega)$ and $D_2$ is the derivative with respect to second component. A solution of (2) corresponding to $\mu \in L^\infty(\mathbb{R}, \mathbb{R}^m)$ with initial value $\lambda \in T_x M$ is a locally absolutely continuous vector field $z = \phi^{x,\omega}(\cdot, \lambda, \mu) : \mathbb{R} \to TM$ along $\varphi(\cdot, x, \omega)$ with $z(0) = \lambda$, satisfying the differential equation (2) for almost all $t \in \mathbb{R}$.

The next proposition presents some properties of linearized systems.

**Proposition 3.** *Let $(\omega(\cdot), \varphi(\cdot, x, \omega))$ be a control-trajectory pair with corresponding linearization (2). Then the following statements hold:*

*(i) For all $\tau > 0$ the mapping $\varphi_\tau : M \times L^\infty([0, \tau], \mathbb{R}^m) \to M, (x, \omega) \mapsto \varphi(\tau, x, \omega)$ is continuously (Fréchet) differentiable.*

*(ii) For every initial value $\lambda \in T_x M$ and every $\mu \in L^\infty(\mathbb{R}, \mathbb{R}^m)$ there exists a unique solution $\phi^{x,\omega}(\cdot, \lambda, \mu) : \mathbb{R} \to TM$ of (2) satisfying*

$$\phi^{x,\omega}(0, \lambda, \mu) = \lambda, \phi^{x,\omega}(t, \lambda, \mu) = D\varphi_t(x, \omega)(\lambda, \mu), t \in \mathbb{R}, \tag{3}$$

*for $(\lambda, \mu) \in T_x M \times L^\infty(\mathbb{R}, \mathbb{R}^m)$, where $D$ stands for the total derivative of $\varphi_t : M \times L^\infty(\mathbb{R}, \mathbb{R}^m) \to M$ which consists of the derivative $d_x\varphi_t(\cdot, \omega) : T_x M \to T_{\varphi(t,x,\omega)}M$ in the first, and the Fréchet derivative of $\varphi_t(x, \cdot) : L^\infty(\mathbb{R}, \mathbb{R}^m) \to T_{\varphi(t,x,\omega)}M$ in the second component.*

*(iii) For every $\tau > 0$ the map $\phi^{x,\omega}(\tau, \cdot, \cdot) : T_x M \times L^\infty([0, \tau], \mathbb{R}^m) \to T_{\varphi(\tau,x,\omega)}M$ is linear and continuous.*

*(iv) For each $t \in \mathbb{R}$ abbreviate $\phi_t^{x,\omega} := \phi^{\varphi(t,x,\omega),\omega(t+\cdot)}$. Then for all $t, s \in \mathbb{R}$, $\lambda \in T_x M$ and $\mu \in L^\infty(\mathbb{R}, \mathbb{R}^m)$,*

$$\phi_s^{x,\omega}(t, \phi^{x,\omega}(s, \lambda, \mu), \Theta_s\mu) = \phi^{x,\omega}(t + s, \lambda, \mu),$$

*and, in particular,*

$$\phi_s^{x,\omega}(t, \phi^{x,\omega}(s, \lambda, \mathbf{0}), \mathbf{0}) = \phi^{x,\omega}(t + s, \lambda, \mathbf{0}).$$

Now we present the notion of regularity of a control-trajectory pair.

**Definition 4.** *Consider some $(x, \omega, \tau) \in M \times \mathcal{U} \times (0, \infty)$ and let $y := \varphi(\tau, x, \omega)$. The linearization along $(\omega(\cdot), \varphi(\cdot, x, \omega))$ is **controllable on** $[0, \tau]$ if for each $\lambda_1 \in T_x M$ and $\lambda_2 \in T_y M$ there exists $\mu \in L^\infty([0, \tau], \mathbb{R}^m)$ with*

$$\phi^{x,\omega}(\tau, \lambda_1, \mu) = \lambda_2.$$

*In this case, we say that the control-trajectory pair $(\omega(\cdot), \varphi(\cdot, x, \omega))$ is **regular** on $[0, \tau]$.*

A control-trajectory pair $(\omega(\cdot), \varphi(\cdot, x, \omega))$ is called $\tau$-periodic, $\tau \geq 0$, if $(\varphi(t+\tau, x, \omega), \omega(t+\tau)) = (\varphi(t, x, \omega), \omega(t))$ for all $t \in \mathbb{R}$, or equivalently if $\varphi(\tau, x, \omega) = x$ and $\Theta_\tau \omega = \omega$, where $(\Theta_\tau \omega)(t) = \omega(t+\tau), t \in \mathbb{R}$, is the $\tau$-shift on $\mathcal{U}$. A periodic regular control-trajectory pair enjoys the property described in the following proposition (cf. [17, Proposition 1.30]).

**Proposition 5.** *Let $(\omega(\cdot), \varphi(\cdot, x, \omega))$ be a $\tau$-periodic control-trajectory pair which is regular on $[0, \tau]$. Then there exists $C > 0$ such that for every $\lambda \in T_x M$ there is $\mu \in L^\infty([0, \tau], \mathbb{R}^m)$ with $\phi^{x,\omega}(\tau, \lambda, \mu) = 0_x$ and $\|\mu\|_{[0,\tau]} \leq C|\lambda|$, where $\|\cdot\|_{[0,\tau]}$ denotes the $L^\infty$-norm.*

For a $\tau$-periodic control-trajectory pair $(\omega(\cdot), \varphi(\cdot, x, \omega))$ the Floquet or Lyapunov exponents are given by

$$\lim_{t\to\infty} \frac{1}{t} \log \|\phi^{x,\omega}(t, \lambda, \mathbf{0})\| = \lim_{n\to\infty} \frac{1}{n\tau} \log \|\phi^{x,\omega}(n\tau, \lambda, \mathbf{0})\|, \lambda \in T_x M. \quad (4)$$

These limits exist and the Lyapunov exponents are denoted by $\rho_1(\omega, x), \dots, \rho_r(\omega, x)$ with $1 \leq r := r(\omega, x) \leq d = \dim M$. The Lyapunov spaces are given by

$$L_j(\omega, x) = \left\{ \lambda \in T_x M;\ \lim_{t\to\pm\infty} \frac{1}{t} \log \|\phi^{x,\omega}(t, \lambda, \mathbf{0})\| = \rho_j(\omega, x) \right\}, j = 1, \dots, r,$$

with dimensions $d_j(\omega, x)$. They yield the decomposition

$$T_x M = L_1(\omega, x) \oplus \cdots \oplus L_r(\omega, x).$$

*2.2. Invariance pressure*

In this subsection we recall the concepts of invariance and outer invariance pressure introduced in Colonius, Cossich and Santana [5, 6] and some of their properties.

A pair $(K, Q)$ of nonempty subsets of a smooth Riemannian manifold $M$ is called ***admissible*** if $K$ is compact and for each $x \in K$ there exists $\omega \in \mathcal{U}$ such that $\varphi(\mathbb{R}_+, x, \omega) \subset Q$. For an admissible pair $(K, Q)$ and $\tau > 0$, a $(\tau, K, Q)$-***spanning set*** $\mathcal{S}$ is a subset of $\mathcal{U}$ such that for all $x \in K$ there is $\omega \in \mathcal{S}$ with $\varphi(t, x, \omega) \in Q$ for all $t \in [0, \tau]$. Denote by $C(U, \mathbb{R})$ the set of continuous function $f : U \to \mathbb{R}$ which we call ***potentials***.

For a potential $f \in C(U,\mathbb{R})$ denote $(S_\tau f)(\omega) := \int_0^\tau f(\omega(t))dt$ and

$$a_\tau(f,K,Q) := \inf\left\{\sum_{\omega\in\mathcal{S}} e^{(S_\tau f)(\omega)};\ \mathcal{S}\ (\tau,K,Q)\text{-spanning}\right\}.$$

The ***invariance pressure*** $P_{inv}(f,K,Q)$ of control system (1) is defined by

$$P_{inv}(f,K,Q) := \limsup_{\tau\to\infty}\frac{1}{\tau}\log a_\tau(f,K,Q).$$

Given an admissible pair $(K,Q)$ such that $Q$ is closed in $M$ and a metric $\varrho$ on $M$ which is compatible with the Riemannian structure, we define the ***outer invariance pressure*** of $(K,Q)$ by

$$P_{out}(f,K,Q) := \lim_{\varepsilon\to 0} P_{inv}(f,K,N_\varepsilon(Q)),$$

where $N_\varepsilon(Q) = \{y \in M;\ \exists\ x \in Q \text{ with } \varrho(x,y) < \varepsilon\}$ denotes the $\varepsilon$ - neighborhood of $Q$.

Note that $-\infty < P_{out}(f,K,Q) \le P_{inv}(f,K,Q) \le \infty$ for every admissible pair $(K,Q)$ and all potentials $f$. For the potential $f = \mathbf{0}$, this reduces to the notion of invariance entropy, $P_{inv}(\mathbf{0},K,Q) = h_{inv}(K,Q)$ and $P_{out}(\mathbf{0},K,Q) = h_{out}(K,Q)$, cf. Kawan [17].

The next proposition presents some properties of the function $P_{inv}(\cdot,K,Q) : C(U,\mathbb{R}) \to \mathbb{R}$, cf. [6, Proposition 3.4].

**Proposition 6.** *The following assertions hold for an admissible pair $(K,Q)$, functions $f,g \in C(U,\mathbb{R})$ and $c \in \mathbb{R}$:*

*(i) $P_{inv}(f,K,Q) \le P_{inv}(g,K,Q)$ and $P_{out}(f,K,Q) \le P_{out}(g,K,Q)$ for $f \le g$.*

*(ii) $P_{inv}(f+c,K,Q) = P_{inv}(f,K,Q) + c$.*

*(iii) $h_{inv}(K,Q) + \min_{u\in U} f(u) \le P_{inv}(f,K,Q) \le h_{inv}(K,Q) + \max_{u\in U} f(u)$.*

Proposition 6 (iii) shows, in particular, that $P_{inv}(f,K,Q) < \infty$ if and only if $h_{inv}(K,Q) < \infty$. For general admissible pairs $(K,Q)$, one cannot guarantee the existence of finite $(\tau,K,Q)$-spanning sets $\mathcal{S}$. The following two remarks discuss the cardinality of spanning sets and relations to properties of invariance pressure.

**Remark 7.** *If there is no countable* $(\tau, K, Q)$*-spanning set, then* $a_\tau(f, K, Q) = \infty$ *(see Kawan [17, Example 2.3] for an example). If* $P_{inv}(f, K, Q) < \infty$*, then* $a_\tau(f, K, Q) < \infty$ *for every* $\tau > 0$*. Hence there is a* $(\tau, K, Q)$*-spanning set with* $\sum_{\omega\in\mathcal{S}} e^{(S_\tau f)(\omega)} < \infty$ *implying that there can be only countably many summands, i.e., there is a countable* $(\tau, K, Q)$*-spanning set. On the other hand, if for all* $\tau > 0$ *there is a countable* $(\tau, K, Q)$*-spanning set,* $a_\tau(f, K, Q) = \infty$ *is also possible. If every* $(\tau, K, Q)$*-spanning set* $\mathcal{S}$ *contains a finite* $(\tau, K, Q)$*-spanning subset* $\mathcal{S}'$*, then*

$$a_\tau(f, K, Q) = \inf \left\{ \sum_{\omega\in\mathcal{S}} e^{(S_\tau f)(\omega)};\ \mathcal{S} \text{ finite and } (\tau, K, Q)\text{-spanning} \right\}.$$

*This follows, since all summands satisfy* $e^{(S_\tau f)(\omega)} > 0$*, and hence the summands in* $\mathcal{S} \setminus \mathcal{S}'$ *can be omitted. This situation occurs e.g. if* $Q$ *is open where compactness of* $K$ *may be used. For the outer invariance entropy one considers* $(\tau, K, N_\varepsilon(Q))$*-spanning sets,* $\varepsilon > 0$*, and hence here it is also sufficient to consider finite* $(\tau, K, N_\varepsilon(Q))$*-spanning sets. In the definition of inner invariance pressure of discrete time systems, one considers sets which are* $(\tau, K, \mathrm{int}Q)$*-spanning. Here again finite spanning sets are sufficient.*

**Remark 8.** *The Lipschitz continuity property*

$$|P_{inv}(f, K, Q) - P_{inv}(g, K, Q)| \le \|f - g\|_\infty \text{ for } f, g \in C(U, \mathbb{R}),$$

*holds if* $h_{inv}(K, Q) < \infty$*. In fact, as seen in Remark 7, in this case there are for every* $\tau > 0$ *countable* $(\tau, K, Q)$*-spanning sets* $\mathcal{S}$ *with* $\sum_{\omega\in\mathcal{S}} e^{(S_\tau f)(\omega)} < \infty$*. The arguments used in [5, Proposition 13(iii)] to show Lipschitz continuity under the assumption that finite* $(\tau, K, Q)$*-spanning sets exist, can be applied in this situation observing that the elementary lemma [5, Lemma 12], on which the proof is based, is valid not only for finite but also for infinite sequences: Let* $a_i \ge 0, b_i > 0, i \in \mathbb{N}$*. Then for all* $n \in \mathbb{N}$

$$\frac{\sum_{i=1}^n a_i}{\sum_{i=1}^n b_i} \ge \min_{i=1,\dots,n} \frac{a_i}{b_i} \ge \inf_{i\in\mathbb{N}} \frac{a_i}{b_i},$$

*and one may take the limit for* $n \to \infty$*.*

The following proposition shows that in the definition of invariance pressure we can take the limit superior over times which are integer multiples of some fixed time step $\tau > 0$.

**Proposition 9.** *The invariance pressure satisfies for every* $\tau > 0$

$$P_{inv}(f,K,Q) = \limsup_{n\to\infty} \frac{1}{n\tau} \log a_{n\tau}(f,K,Q) \text{ for all } f \in C(U,\mathbb{R}). \tag{5}$$

**Proof.** Let $\tau_k \in (0,\infty), k \in \mathbb{N}$, with $\tau_k \to \infty$. Then for every $k \geq 1$ there exists $n_k \geq 1$ such that $n_k\tau \leq \tau_k \leq (n_k+1)\tau$ and $n_k \to \infty$ for $k \to \infty$. Since $\tilde{f}(u) := f(u) - \min f, u \in U$, is nonnegative, it follows that $a_{\tau_k}(\tilde{f},K,Q) \leq a_{(n_k+1)\tau}(\tilde{f},K,Q)$ and consequently $\frac{1}{\tau_k}\log a_{\tau_k}(\tilde{f},K,Q)$ is less than or equal to $\frac{1}{n_k\tau}\log a_{(n_k+1)\tau}(\tilde{f},K,Q)$. Hence

$$\begin{aligned}
\limsup_{k\to\infty} \frac{1}{\tau_k} \log a_{\tau_k}(\tilde{f},K,Q) &\leq \limsup_{k\to\infty} \frac{1}{n_k\tau} \log a_{(n_k+1)\tau}(\tilde{f},K,Q) \\
&= \limsup_{k\to\infty} \frac{n_k+1}{n_k} \frac{1}{(n_k+1)\tau} \log a_{(n_k+1)\tau}(\tilde{f},K,Q) \\
&\leq \limsup_{n\to\infty} \frac{1}{n\tau} \log a_{n\tau}(\tilde{f},K,Q).
\end{aligned}$$

This shows that

$$P_{inv}(f - \min f, K, Q) \leq \limsup_{n\to\infty} \frac{1}{n\tau} \log a_{n\tau}(f - \min f, K, Q).$$

Using $a_{n\tau}(\tilde{f},K,Q) = a_{n\tau}(f,K,Q) - \min f$ and Proposition 6 (ii) we obtain

$$P_{inv}(f,K,Q) \leq \limsup_{n\to\infty} \frac{1}{n\tau} \log a_{n\tau}(f,K,Q).$$

The converse inequality is obvious. ■

For the proof of the following proposition see [6, Corollary 15].

**Proposition 10.** *Let* $K_1, K_2$ *be two compact sets with nonempty interior contained in a control set* $D \subset M$. *Then* $(K_1,Q)$ *and* $(K_2,Q)$ *are admissible pairs and for all* $f \in C(U,\mathbb{R})$ *we have*

$$P_{inv}(f,K_1,Q) = P_{inv}(f,K_2,Q).$$

## 3. An upper bound on control sets

Our goal in this section is to obtain an upper bound for the invariance pressure of a control set. We consider a smooth control system (1) on a Riemannian manifold $(M, g)$ under our standard assumptions.

In the following theorem, given a periodic control-trajectory pair $(\omega(\cdot), \varphi(\cdot, x, \omega))$,the different Lyapunov exponents at $(x, \omega)$ are denoted by $\rho_1(x, \omega), \ldots, \rho_r(x, \omega)$, $r = r(x, \omega)$, with Lyapunov spaces of dimensions $d_1(x, \omega), \ldots, d_r(x, \omega)$, respectively.

**Theorem 11.** *Let $D \subset M$ be a control set with nonempty interior and compact closure for control system (1). Then for every compact set $K \subset D$ and every set $Q \supset D$, the pair $(K, Q)$ is admissible and for all potentials $f \in C(U, \mathbb{R})$ the invariance pressure satisfies*

$$P_{inv}(f, K, Q) \leq \inf_{(T,x,\omega)} \left\{ \sum_{j=1}^{r(x,\omega)} \max\{0, d_j(x, \omega)\rho_j(x, \omega)\} + \frac{1}{T} \int_0^T f(\omega(s))ds \right\},$$

*where the infimum is taken over all $(T, x, \omega) \in (0, \infty) \times \mathrm{int}D \times \mathcal{U}$ such that the control-trajectory pair $(\omega(\cdot), \varphi(\cdot, x, \omega))$ is $T$-periodic and regular and the values $\omega(t), t \in [0, T]$, are in a compact subset of* $\mathrm{int}U$.

**Remark 12.** *For $f \equiv 0$, the statement of the theorem reduces to Kawan [16, Theorem 4.3],*

$$h_{inv}(K, Q) = P_{inv}(\mathbf{0}, K, Q) \leq \inf_{(T,x,\omega)} \left\{ \sum_{j=1}^{r(x,\omega)} \max\{0, d_j(x, \omega)\rho_j(x, \omega)\} \right\}.$$

**Proof.** The theorem will follow by an extension of the proof given in [16, Theorem 4.3] for invariance entropy. First we briefly sketch the construction in [16, pp. 740-745], then we indicate the new arguments needed for invariance pressure.

By Proposition 10 one can choose $K$ as an arbitrary compact subset of $D$ with nonvoid interior. Let $(\omega_0(\cdot), \varphi(\cdot, x_0, \omega_0))$ be a $T$-periodic and regular control-trajectory pair as in the statement of the theorem. Then fix real numbers $\varepsilon > 0$ and

$$S_0 > \sum_{j=1}^{r} \max(0, d_j\rho_j),$$

where $d_j = d_j(x_0, \omega_0)$ and $\rho_j = \rho_j(x_0, \omega_0), j = 1, \dots, r$. An ingenious and lengthy construction in [16] provides a closed ball $K := \mathrm{cl}(B_{b_0}(x_0)) \subset D$ with radius $b_0 > 0$ centered at $x_0$ with the following properties: For some $\tau = kT, k \in \mathbb{N}$, and arbitrary $n \in \mathbb{N}$ one finds a set $\mathcal{S}_n$ of $(n\tau, K, Q)$-spanning controls $\omega \in \mathcal{S}_n$ satisfying

$$\|\omega - \omega_0\|_{[0,n\tau]} \leq C b_0 \sqrt{d}, \tag{6}$$

where $C > 0$ is a constant and $b_0 > 0$ can be taken arbitrarily small (see [16, formula (4.17)]: the elements of $\mathcal{S}_n$ are $n$-fold concatenations of the controls denoted there by $u_x$). The cardinality $\#\mathcal{S}_n$ of $\mathcal{S}_n$ is bounded by

$$\frac{1}{n\tau} \log \#\mathcal{S}_n \leq S_0 + \varepsilon, \tag{7}$$

cf. [16, estimates on middle of p. 745].

In order to get a bound for the invariance pressure we need the following additional arguments: Let $f \in C(U, \mathbb{R})$ be a potential. Since $f$ is defined on the compact set $U$, its uniform continuity implies that there exists $\delta > 0$ such that $|u - v| < \delta$ implies $|f(u) - f(v)| < \varepsilon$. Take $b_0 > 0$ small enough such that $C b_0 \sqrt{d} < \delta$. By (6) every $\omega \in \mathcal{S}_n$ satisfies $|\omega(t) - \omega_0(t)| \leq \|\omega - \omega_0\|_{[0,n\tau]} < \delta$ for almost all $t \in [0, n\tau]$. Hence it follows that $|f(\omega(t)) - f(\omega_0(t))| < \varepsilon$ for almost all $t \in [0, n\tau]$.

Now we can estimate

$$\begin{aligned}
\frac{1}{n\tau} \log a_{n\tau}(f, K, Q) &\leq \frac{1}{n\tau} \log \sum_{\omega \in \mathcal{S}_n} e^{(S_{n\tau} f)(\omega)} = \frac{1}{n\tau} \log \sum_{\omega \in \mathcal{S}_n} e^{\int_0^{n\tau} f(\omega(t))dt} \\
&= \frac{1}{n\tau} \log \sum_{\omega \in \mathcal{S}_n} e^{\int_0^{n\tau} f(\omega_0(t))dt + \int_0^{n\tau} [f(\omega(t)) - f(\omega_0(t))]dt} \\
&\leq \frac{1}{n\tau} \log \left[ \sum_{\omega \in \mathcal{S}_n} e^{\int_0^{n\tau} f(\omega_0(t))dt} \cdot e^{\int_0^{n\tau} \varepsilon dt} \right] \\
&= \frac{1}{n\tau} \log \left( \#\mathcal{S}_n e^{\int_0^{n\tau} f(\omega_0(t))dt} \right) + \frac{1}{n\tau} \log e^{\int_0^{n\tau} \varepsilon dt} \\
&= \frac{1}{n\tau} \log \#\mathcal{S}_n + \frac{1}{n\tau} \int_0^{n\tau} f(\omega_0(t))dt + \varepsilon \\
&< S_0 + \frac{1}{T} \int_0^T f(\omega_0(t))dt + 2\varepsilon.
\end{aligned}$$

For the last inequality we have used (7) and $T$-periodicity of $\omega_0$. By Proposition 9 this implies

$$P_{inv}(f,K,Q) = \limsup_{n\to\infty} \frac{1}{n\tau} \log a_{n\tau}(f,K,Q) \leq S_0 + \frac{1}{T}\int_0^T f(\omega_0(t))dt + 2\varepsilon.$$

Since $\varepsilon > 0$ can be chosen arbitrarily small and $S_0$ arbitrarily close to $\sum_{j=1}^r \max(0, d_j\rho_j)$, the assertion of the theorem follows. ∎

**Remark 13.** *In Kawan [17, Section 5.2] and Da Silva and Kawan [12, Section 3.2] one finds more information on regular periodic control-trajectory pairs.*

## 4. A lower bound

Again we consider a smooth control system (1) on a Riemannian manifold $(M, g)$ under our standard assumptions. The next theorem presents a lower bound for the invariance pressure of admissible pairs $(K, Q)$.

**Theorem 14.** *Let $(K, Q)$ be an admissible pair where both $K$ and $Q$ have positive and finite volume. Then for every $f \in C(U, \mathbb{R})$*

$$P_{inv}(f,K,Q) \geq \limsup_{\tau\to\infty} \frac{1}{\tau}\left( \inf_{(x,\omega)} \int_0^\tau f(\omega(s))ds + \max\{0, \inf_{(x,\omega)} \int_0^\tau \mathrm{div} F_{\omega(s)}(\varphi(s,x,\omega))ds\}\right),$$

*where both infima are taken over all $(x,\omega) \in K \times \mathcal{U}$ with $\varphi([0,\tau],x,\omega) \subset Q$.*

**Proof.** First observe that by Remark 7 we may assume that for all $\tau > 0$ there exists a countable $(\tau, K, Q)$-spanning set, since otherwise $P_{inv}(f, K, Q) = \infty$, and the infimum in $a_\tau(f, K, Q)$ may be taken over all countable $(\tau, K, Q)$-spanning sets $\mathcal{S}$. For each $\omega$ in a countable $(\tau, K, Q)$-spanning set $\mathcal{S}$ define

$$K_\omega := \{x \in K; \varphi([0,\tau],x,\omega) \subset Q\}.$$

Thus $K = \bigcup_{\omega\in\mathcal{S}} K_\omega$. Since $Q$ is Borel measurable, each set $K_\omega$ is measurable as the countable intersection of measurable sets,

$$K_\omega = K \cap \bigcap_{t\in[0,\tau]\cap\mathbb{Q}} \varphi_{t,\omega}^{-1}(Q).$$

Then

$$\begin{aligned}\mathrm{vol}(Q) &\geq \mathrm{vol}(\varphi_{\tau,\omega}(K_\omega)) = \int_{\varphi_{\tau,\omega}(K_\omega)} \mathrm{dvol} = \int_{K_\omega} \left|\det \mathrm{d}_x\varphi_{\tau,\omega}\right| \mathrm{dvol} \\ &\geq \mathrm{vol}(K_\omega) \inf_{(x,\omega)} \left|\det \mathrm{d}_x\varphi_{\tau,\omega}\right|,\end{aligned}$$

where the infimum is taken over all $(x,\omega) \in K \times \mathcal{U}$ with $\varphi([0,\tau],x,\omega) \subset Q$. Abbreviating with the same infima

$$\alpha(\tau) := \inf_{(x,\omega)} \left|\det \mathrm{d}_x\varphi_{\tau,\omega}\right|, \ \beta(\tau) := \inf_{(x,\omega)} S_\tau(f)(\omega),$$

we find

$$\begin{aligned}e^{\beta(\tau)}\mathrm{vol}(K) &\leq \sum_{\omega\in\mathcal{S}} e^{(S_\tau f)(\omega)}\mathrm{vol}(K_\omega) \leq \sup_{\omega\in\mathcal{S}} \mathrm{vol}(K_\omega) \sum_{\omega\in\mathcal{S}} e^{(S_\tau f)(\omega)} \\ &\leq \frac{\mathrm{vol}(Q)}{\max\{1,\alpha(\tau)\}} \sum_{\omega\in\mathcal{S}} e^{(S_\tau f)(\omega)}.\end{aligned}$$

Since this holds for every countable $(\tau,K,Q)$-spanning set $\mathcal{S}$, we find

$$\begin{aligned}a_\tau(f,K,Q) &= \inf\left\{\sum_{\omega\in\mathcal{S}} e^{(S_\tau f)(\omega)};\ \mathcal{S} \text{ countable } (\tau,K,Q)\text{-spanning}\right\} \\ &\geq \frac{\mathrm{vol}(K)}{\mathrm{vol}(Q)} e^{\beta(\tau)} \max\{1,\alpha(\tau)\},\end{aligned}$$

Since for each $t \geq 0$ and each control $\omega \in \mathcal{U}$ the map $\varphi_{t,\omega} : M \to M$ is a diffeomorphism, Liouville's formula shows

$$\log\det \mathrm{d}_x\varphi_{\tau,\omega} = \int_0^\tau \mathrm{div}F_{\omega(s)}(\varphi(s,x,\omega))ds. \tag{8}$$

Now the assertion of the theorem follows from

$$P_{inv}(f,K,Q) = \limsup_{\tau\to\infty}\frac{1}{\tau}\log a_\tau(f,K,Q)$$

$$\geq \limsup_{\tau\to\infty}\frac{1}{\tau}\left(\beta(\tau) + \log\max\{1,\alpha(\tau)\}\right)$$

$$= \limsup_{\tau\to\infty}\frac{1}{\tau}\left(\inf_{(x,\omega)}\int_0^\tau f(\omega(s))ds + \max\{0,\inf_{(x,\omega)}\int_0^\tau \mathrm{div}F_{\omega(s)}(\varphi(s,x,\omega))ds\}\right).$$

■

## 5. Linear control systems

In this section we consider linear control systems on $\mathbb{R}^d$ with restricted controls. Here a unique control set $D$ with nonvoid interior exists and the previous bounds on the invariance pressure are sharpened to provide a formula for the invariance pressure of $D$.

Linear control systems on $\mathbb{R}^d$ have the form

$$\dot{x}(t) = Ax(t) + B\omega(t),\ \omega \in \mathcal{U}, \tag{9}$$

with $A \in \mathbb{R}^{d\times d}$ and $B \in \mathbb{R}^{d\times m}$ and we suppose that the set $\mathcal{U}$ of control functions is as for (1).

For system (9) there exists a unique control set $D$ with nonvoid interior, if, without control constraint, the system is controllable (which holds if and only if $\mathrm{rank}[B, AB, \ldots, A^{d-1}B] = d$) and the control range $U$ is a compact neighborhood of the origin in $\mathbb{R}^m$. It is convex with $0 \in \mathrm{int}D$, and it is bounded if and only if $A$ is hyperbolic, i.e., there is no eigenvalue of $A$ with vanishing imaginary part (cf. Hinrichsen and Pritchard [14, Theorems 6.2.22 and 6.2.23], Colonius and Kliemann [9, Example 3.2.16]). Then the state space $\mathbb{R}^d$ can be decomposed into the direct sum of the stable subspace $E^s$ and the unstable subspace $E^u$ which are the direct sums of all generalized real eigenspaces for the eigenvalues $\lambda$ with $\mathrm{Re}\,\lambda < 0$ and $\mathrm{Re}\,\lambda > 0$, resp. Let $\pi : \mathbb{R}^d \to E^u$ be the projection along $E^s$. We obtain the following estimates, where $\lambda_j$ denote the $r$ eigenvalues of $A$ with algebraic multiplicities $d_j$.

**Lemma 15.** *Consider a linear control system in $\mathbb{R}^d$ of the form (9) and assume that the pair $(A, B)$ is controllable, that $A$ is hyperbolic and the control range $U$ is a compact neighborhood of the origin. Let $D$ be the unique control set with nonvoid interior. Then for every compact set $K \subset D$ with positive Lebesgue measure every potential $f \in C(U, \mathbb{R})$ satisfies*

$$\inf_{(T',x',\omega')} \frac{1}{T'} \int_0^{T'} f(\omega'(s))ds \leq P_{inv}(f, K, D) - \sum_{j=1}^{r} d_j \max\{0, \mathrm{Re}\,\lambda_j\}$$
$$\leq \inf_{(T,x,\omega)} \frac{1}{T} \int_0^{T} f(\omega(s))ds,$$

*where the first infimum is taken over all $(T', x', \omega') \in (0, \infty) \times \pi K \times \mathcal{U}$ with $\pi\varphi([0, T'], x', \omega') \subset \pi D$ and the second infimum is taken over all $(T, x, \omega) \in (0, \infty) \times \mathrm{int}D \times \mathcal{U}$ such that the control-trajectory pair $(\omega(\cdot), \varphi(\cdot, x, \omega))$ is $T$-periodic and the values $\omega(t), t \in [0, T]$, are in a compact subset of* $\mathrm{int}U$.

**Proof.** The hypotheses imply that $0 \in \mathrm{int} D \subset \mathbb{R}^d$ and the Lebesgue measures of $K$ and $D$ (which coincide with the volumes) are finite and positive. Theorem 11 yields

$$P_{inv}(f, K, D) \leq \inf_{(T,x,\omega)} \left\{ \sum_{j=1}^{r(x,\omega)} \max\{0, d_j(x,\omega)\rho_j(x,\omega)\} + \frac{1}{T}\int_0^T f(\omega(s))ds \right\}, \tag{10}$$

where the infimum is taken over all $T > 0$ and all $(x, \omega) \in \mathrm{int}D \times \mathcal{U}$ such that the control-trajectory pair $(\omega(\cdot), \varphi(\cdot, x, \omega))$ is $T$-periodic and the values $\omega(t), t \in [0, T]$, are in a compact subset of $\mathrm{int}U$. Note that this control-trajectory pair is regular, since we assume that $(A, B)$ is controllable. By Floquet theory it follows (cf. [6, Proposition 20]) that for all $T$-periodic $(\omega(\cdot), \varphi(\cdot, x, \omega))$

$$\sum_{j=1}^{r(x,\omega)} \max\{0, d_j(x,\omega)\rho_j(x,\omega)\} = \sum_{j=1}^{r} \max\{0, d_j \operatorname{Re} \lambda_j\},$$

where the sum is over the $r$ eigenvalues $\lambda_j$ of $A$ with multiplicities $d_j$. Hence

$$P_{inv}(f, K, D) \leq \inf_{(T,x,\omega)} \frac{1}{T}\int_0^T f(\omega(s))ds + \sum_{j=1}^{r} d_j \max\{0, \operatorname{Re}\lambda_j\},$$

where the infimum is taken over all $(T, x, \omega)$ as in (10). This proves the second inequality.

Hence it remains to prove the first inequality. By Theorem 14

$$\begin{aligned} &P_{inv}(f, K, D) \\ &\geq \limsup_{\tau\to\infty} \frac{1}{\tau}\left( \inf_{(x,\omega)} \int_0^\tau f(\omega(s))ds + \max\left\{0, \inf_{(x,\omega)} \int_0^\tau \mathrm{div} F_{\omega(s)}(\varphi(s,x,\omega))ds \right\}\right) \\ &\geq \inf_{(T,x,\omega)} \frac{1}{T}\int_0^T f(\omega(s))ds + \max\left\{0, \inf_{(T,x,\omega)} \frac{1}{T}\int_0^T \mathrm{div} F_{\omega(s)}(\varphi(s,x,\omega))ds\right\}, \end{aligned}$$

where both infima in the second line are taken over all pairs $(x, \omega) \in K \times \mathcal{U}$ with $\varphi([0, \tau], x, \omega) \subset D$ and both infima in the third line are taken over all $(T, x, \omega) \in (0, \infty) \times K \times \mathcal{U}$ with $\varphi([0, T], x, \omega) \subset D$. In the considered linear case one has $\mathrm{d}_x\varphi_{T,\omega} = A$ and

$$\int_0^T \mathrm{div} F_{\omega(s)}(\varphi(s,x,\omega))ds = \log\det \mathrm{d}_x\varphi_{T,\omega} = T\sum_{j=1}^{r} d_j \operatorname{Re}\lambda_j,$$

where the sum is over the $r$ eigenvalues $\lambda_j$ of $A$ with multiplicities $d_j$.

**Step 1:** Suppose that $\operatorname{Re}\lambda_j > 0$ for all $j$. Then it follows that

$$\begin{aligned} &P_{inv}(f,K,D) \\ &\geq \inf_{(T,x,\omega)} \frac{1}{T}\int_0^T f(\omega(s))ds + \sum_{j=1}^r d_j \operatorname{Re}\lambda_j, \end{aligned}$$

where the infimum is taken over all $(T,x,\omega) \in (0,\infty) \times K \times \mathcal{U}$ with $\varphi([0,T],x,\omega) \subset D$.

**Step 2:** Next we treat the general case, where also eigenvalues with negative real part are allowed. Recall that $\pi : \mathbb{R}^d \to E^u$ denotes the projection onto the unstable subspace $E^u$ along the stable subspace $E^s$.

Since these subspaces are $A$-invariant, this defines a (time-invariant) semi-conjugacy between system (9) and the system on $E^u$ given by

$$\dot{y}(t) = A_{|E^u}\, y(t) + \pi B u(t), u \in \mathcal{U}, \tag{11}$$

with trajectories $\pi\varphi(\cdot,x',\omega')$, and the sets $K$ and $D$ are mapped to $\pi K$ and $\pi D$, resp. Then $\pi K$ and $\pi D$ have positive volume and form an admissible pair (cf. Kawan [17, proof of Theorem 3.1]). One easily proves that (cf. [6, Proposition 10])

$$P_{inv}(f,K,Q) \geq P_{inv}(f,\pi K,\pi Q),$$

since every $(\tau,K,D)$-spanning set yields a $(\tau,\pi K,\pi D)$-spanning set. Similarly as in Step 1, Theorem 14 applied to system (11) implies that

$$P_{inv}(f,\pi K,\pi D) \geq \inf_{(T',x',\omega')} \frac{1}{T'}\int_0^{T'} f(\omega'(s))ds + \sum_{j=1}^r d_j \max\{0,\operatorname{Re}\lambda_j\},$$

where the infimum is taken over all $(T',x',\omega') \in \mathbb{R}_+ \times \pi K \times \mathcal{U}$ with

$$\pi\varphi([0,T'],x',\omega') \subset \pi D.$$

■

Next we show that the two infima in the lemma above actually coincide again using hyperbolicity of $A$ in a crucial way. This provides the announced formula for the invariance pressure involving the $r$ eigenvalues $\lambda_j$ of $A$ with algebraic multiplicities $d_j$.

**Theorem 16.** *Consider a linear control system in $\mathbb{R}^d$ of the form (9) and assume that the pair $(A, B)$ is controllable, the matrix $A$ is hyperbolic and the control range $U$ is a compact neighborhood of the origin. Let $D$ be the unique control set with nonvoid interior. Then for every compact set $K \subset D$ with nonempty interior every potential $f \in C(U, \mathbb{R})$ satisfies*

$$P_{inv}(f, K, D) = \min_{u \in U} f(u) + \sum_{j=1}^{r} d_j \max\{0, \operatorname{Re} \lambda_j\}. \tag{12}$$

**Proof.** Let $\varepsilon > 0$ and consider $T_0 > 0$ and a control $\omega_0 \in \mathcal{U}$ satisfying

$$\frac{1}{T_0} \int_0^{T_0} f(\omega_0(s)) ds \le \inf_{(T'', \omega'') \in (0,\infty) \times \mathcal{U}} \frac{1}{T''} \int_0^{T''} f(\omega''(s)) ds + \varepsilon. \tag{13}$$

Since $f$ is continuous, there is a control value $u_0 \in U$ with

$$f(u_0) = \min_{u \in U} f(u) = \inf_{(T'', \omega'') \in (0,\infty) \times \mathcal{U}} \frac{1}{T''} \int_0^{T''} f(\omega''(s)) ds, \tag{14}$$

where the second equality holds trivially. There is a control $\omega_1$ in the set

$$\operatorname{int} \mathcal{U}_{|[0,T_0]} = \{\omega \in L^\infty([0, T_0]; \exists K \subset \operatorname{int} U \text{ compact with } \omega(t) \in K \text{ a.e.}\}$$

such that

$$\frac{1}{T_0} \int_0^{T_0} f(\omega_1(s)) ds \le \frac{1}{T_0} \int_0^{T_0} f(\omega_0(s)) ds + \varepsilon. \tag{15}$$

**Claim:** For every $T > 0$ and every control $\omega \in \mathcal{U}$ there exists $x_1 \in \mathbb{R}^d$ with $\varphi(T, x_1, \omega) = x_1$.

In fact, hyperbolicity of $A$ implies that the matrix $I - e^{AT}$ is invertible, and hence there is a unique solution $x(T, \omega)$ of

$$\left(I - e^{AT}\right) x(T, \omega) = \varphi(T, 0, \omega).$$

Now the variation-of-constants formula shows the claim:

$$x(T, \omega) = e^{AT} x(T, \omega) + \varphi(T, 0, \omega) = \varphi(T, x(T, \omega), \omega).$$

Applying this to $T_0$ and $\omega_1$ we find a point $x_1 := x(T_0, \omega_1) = \varphi(T_0, x_1, \omega_1)$. Since $\omega_1 \in \operatorname{int} \mathcal{U}_{|[0,T_0]}$ every point in a neighborhood of $x_1$ can be reached in time $T_0$ from $x_1$. This follows, since by controllability the map

$$L_\infty([0, T_0], \mathbb{R}^m) \to \mathbb{R}^d, \omega \mapsto \varphi(T_0, 0, \omega)$$

is a linear surjective map, hence maps open sets to open sets, and the same is true for the map

$$\omega \mapsto \varphi(T_0, x_1, \omega) = e^{AT_1} + \varphi(T_0, 0, \omega).$$

Analogously, $x_1$ can be reached from every point in a neighborhood of $x_1$ in time $T_0$. Hence in the intersection of these two neighborhoods every point can be steered in time $2T_0$ into every other point. This shows that $x_1$ is in the interior of the (unique) control set $D$, and the corresponding trajectory $\varphi(t, x_1, \omega_1), t \in [0, T_0]$, remains in the interior of $D$. Extending $\omega_1(t), t \in [0, T_0]$, to a $T_0$-periodic control, again denoted by $\omega_1$ we find that the control-trajectory pair $(\omega_1(\cdot), \varphi(\cdot, x_1, \omega_1))$ is $T_0$-periodic, the trajectory is contained in int$D$ and the values $\omega_1(t), t \in [0, T_0]$, are in a compact subset of int$U$. By (13) and (15) it follows that

$$\begin{aligned} &\inf_{(T'',\omega'')} \frac{1}{T''} \int_0^{T''} f(\omega''(s))ds \geq \frac{1}{T_0} \int_0^{T_0} f(\omega_0(s))ds - \varepsilon \\ &\geq \frac{1}{T_0} \int_0^{T_0} f(\omega_1(s))ds - 2\varepsilon \geq \inf_{(T,x,\omega)} \frac{1}{T} \int_0^{T} f(\omega(s))ds - 2\varepsilon, \end{aligned}$$

where the infimum in the last line is taken over all $(T, x, \omega) \in (0, \infty) \times D \times \mathcal{U}$ such that the control-trajectory pair $(\omega(\cdot), \varphi(\cdot, x, \omega))$ is $T$-periodic, the trajectory is contained in int$D$ and the values $\omega(t), t \in [0, T]$, are in a compact subset of int$U$.

Together with (14) and the inequalities in Lemma 15 this implies

$$\begin{aligned} \min_{u \in U} f(u) &= \inf_{(T'',\omega'')} \frac{1}{T''} \int_0^{T''} f(\omega''(s))ds \leq \inf_{(T',x',\omega')} \frac{1}{T'} \int_0^{T'} f(\omega'(s))ds \\ &\leq P_{inv}(f, K, D) - \sum_{j=1}^{r} d_j \max\{0, \operatorname{Re} \lambda_j\} \\ &\leq \inf_{(T,x,\omega)} \frac{1}{T} \int_0^{T} f(\omega(s))ds \\ &\leq \inf_{(T'',\omega'')} \frac{1}{T''} \int_0^{T''} f(\omega''(s))ds + 2\varepsilon \\ &= \min_{u \in U} f(u) + 2\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, assertion (12) follows. ■

**Remark 17.** *The proof of the* ***Claim*** *above follows arguments in the proof of Da Silva and Kawan [13, Theorem 20].*

**Remark 18.** *Theorem 16 improves [6, Theorem 6.2], where it had to be assumed additionally that the minimum of $f(u), u \in U$, is attained in an equilibrium.*

## 6. Further applications

In this section, we apply Theorem 11 to linear control systems on Lie groups and to inner control sets.

*6.1. Control sets and equilibrium pairs*

Given a control system (1), a pair $(u_0, x_0) \in U \times M$ is called an ***equilibrium pair*** if $F(x_0, u_0) = 0$, or equivalently, $\varphi(t, x_0, \bar{u}_0) = x_0$ for all $t \in \mathbb{R}$, where $\bar{u}_0(t) \equiv u_0$.

If $(u_0, x_0)$ is an equilibrium pair, the linearized system is an autonomous linear control system in $T_{x_0}M$ and the Lyapunov exponents at $(u_0, x_0)$ in the direction $\lambda \in T_{x_0}M \backslash \{0_{x_0}\}$ coincide with the real parts of the eigenvalues of $\nabla F_{u_0}(x_0) : T_{x_0}M \to T_{x_0}M$. Then regularity, i.e., controllability of the linearized system, can be checked by Kalman's rank condition.

**Corollary 19.** *Let $D \subset M$ be a control set with nonempty interior and let $f \in C(U, \mathbb{R})$. Suppose that there is a regular equilibrium pair $(u_0, x_0) \in \text{int}U \times \text{int}D$. Then for every compact set $K \subset D$ and every set $Q \supset D$ we have*

$$P_{inv}(f, K, Q) \leq \sum_{\lambda \in \sigma(\nabla F_{u_0}(x_0))} \max\{0, d_\lambda \operatorname{Re}(\lambda)\} + f(u_0),$$

*where $d_\lambda$ is the algebraic multiplicity of the eigenvalue $\lambda$ in the spectrum $\sigma(\nabla F_{u_0}(x_0))$.*

**Proof.** Since $(u_0, x_0)$ is a regular equilibrium pair, the control-trajectory pair $(\bar{u}_0(\cdot), \varphi(\cdot, x_0, \bar{u}_0))$ is $T$-periodic and regular for every $T > 0$. By Theorem 11 we obtain

$$\begin{aligned} P_{inv}(f, K, Q) &\leq \inf_{(T,x,\omega)} \left\{ \sum_{j=1}^{r(x,\omega)} \max\{0, d_j(x,\omega)\rho_j(x,\omega)\} + \frac{1}{T}\int_0^T f(\omega(s))ds \right\} \\ &\leq \sum_{\lambda \in \sigma(\nabla F_{\omega_0}(x_0))} \max\{0, d_\lambda \operatorname{Re}(\lambda)\} + f(u_0). \end{aligned}$$

■

*6.2. Control sets of linear control systems on Lie groups*

In this subsection we consider ***linear control systems on a connected Lie group*** $G$ introduced in Ayala and San Martin [2] and Ayala and Tirao [4].

They are given by a family of ordinary differential equations on $G$ of the form

$$\dot{x}(t) = \mathcal{X}(x(t)) + \sum_{j=1}^{m} \omega_j(t) X_j(x(t)),\ \omega = (\omega_1, \dots, \omega_m) \in \mathcal{U}, \tag{16}$$

where the drift vector field $\mathcal{X}$, called the ***linear vector field***, is an infinitesimal automorphism, i.e., its solutions are a family of automorphisms of the group, and the $X_j$ are right invariant vector fields. Note that the linear control systems of the form (9) are a special case with $G = \mathbb{R}^d$.

Their controllability properties have been analyzed in Da Silva [11], Ayala, Da Silva and Zsigmond [3] and Ayala and Da Silva [1]. In particular, the existence and uniqueness of control sets for general systems of the form (16) has been analyzed in [3]. If 0 is in the interior of the control range $U$ and the reachable set $\mathcal{O}^+(e_G)$ from the neutral element $e_G$ is open (this holds e.g. if $e_G \in \operatorname{int}\mathcal{O}^+(e_G)$), then there exists a control set $D$ containing $e_G$ in the interior. Sufficient conditions for boundedness and uniqueness of $D$ are given in [3, Theorem 3.9] and [3, Corollary 3.12], respectively.

Along with system (16) comes an associated derivation $\mathcal{D}$ of the Lie algebra $\mathfrak{g}$ of $G$ which is given by

$$\mathcal{D}(Y) = -\operatorname{ad}(\mathcal{X})(Y) := [\mathcal{X}, Y](e_G).$$

**Corollary 20.** *Consider the linear control system (16) on a Lie group $G$. Suppose that $D$ is a control set with $e_G \in \operatorname{int}D$ and compact closure $\overline{D}$ and let $K \subset D \subset Q$. Let $f \in C(U, \mathbb{R})$ be a potential. If the equilibrium pair $(0, e_G) \in \operatorname{int}U \times \operatorname{int}D$ is regular, then*

$$P_{inv}(f, K, Q) \leq \sum_{\lambda \in \sigma(\mathcal{D})} \max\{0, d_\lambda \operatorname{Re}(\lambda)\} + f(0).$$

*If furthermore $K$ has positive Haar measure and $f(0) = \min_{u \in U} f(u)$, then*

$$P_{inv}(f, K, Q) = P_{out}(f, K, Q) = \sum_{\lambda \in \sigma(\mathcal{D})} \max\{0, d_\lambda \operatorname{Re}(\lambda)\} + f(0).$$

**Proof.** Note that the right hand side of the system is given by $F(x,u) = \mathcal{X}(x) + \sum_{i=1}^m u_i X_i(x)$ and hence $F_0(x) := F(x,0) = \mathcal{X}(x)$. Let $(\phi, U)$ be a local coordinate neighborhood of $e_G$ and pick a left invariant vector field $Y$ in the Lie algebra $\mathfrak{g}$ of $G$. Then we can express $\mathcal{X}$ in terms of $(\phi, U)$ by

$$\mathcal{X}(h) = \sum_{i=1}^{d} y_i(h) \frac{\partial}{\partial x_i}.$$

Note that $\mathcal{X}(e_G) = 0$ implies $y_i(e_G) = 0$ for every $i \in \{1, \ldots, d\}$, hence the Levi-Civita connection $\nabla$ satisfies

$$(\nabla_{\mathcal{X}} Y)(e_G) = \sum_{i=1}^{d} y_i(e_G) \left( \nabla_{\frac{\partial}{\partial x_i}} Y \right) (e_G) = 0.$$

Since $\nabla$ is symmetric, we have

$$(\nabla_Y F_0)(e_G) = (\nabla_Y \mathcal{X})(e_G) = (\nabla_{\mathcal{X}} Y - [\mathcal{X}, Y])(e_G) = -[\mathcal{X}, Y] = \mathcal{D}(Y).$$

Since this holds for every $Y \in \mathfrak{g}$, we have $\nabla F_0(e_G) = \mathcal{D}$. By Corollary 19 we obtain

$$P_{inv}(f, K, Q) \leq \sum_{\lambda \in \sigma(\mathcal{D})} \max\{0, d_\lambda \operatorname{Re}(\lambda)\} + f(0).$$

Now, suppose that $K$ has positive Haar measure. By Da Silva [10, Theorem 4.3], we know that

$$h_{out}(K, Q) \geq \sum_{\lambda \in \sigma(\mathcal{D})} \max\{0, d_\lambda \operatorname{Re}(\lambda)\}.$$

Define $\tilde{f}(u) = f(u) - f(0), u \in U$. Since $\tilde{f} \geq 0$ Proposition 6(i) implies that

$$P_{inv}(\tilde{f}, K, Q) \geq P_{out}(\tilde{f}, K, Q) \geq h_{out}(K, Q) \geq \sum_{\lambda \in \sigma(\mathcal{D})} \max\{0, d_\lambda \operatorname{Re}(\lambda)\}.$$

Proposition 6(ii) implies $P_{inv}(\tilde{f}, K, Q) = P_{inv}(f, K, Q) - \min f$, hence this yields

$$P_{inv}(f, K, Q) = P_{out}(f, K, Q) = \sum_{\lambda \in \sigma(\mathcal{D})} \max\{0, d_\lambda \operatorname{Re}(\lambda)\} + \min f.$$

■

*6.3. Inner control sets*

This section presents an application of Theorem 11 to the class of ***inner control sets*** as defined (with small changes) in Kawan [17, Definition 2.6]. This nomenclature refers to a control set $D \subset M$ for which there exists a decreasing family of compact and convex sets $\{U_\rho\}_{\rho\in[0,1]}$ in $\mathbb{R}^m$ (i.e., $U_{\rho_2} \subset U_{\rho_1}$ for $\rho_1 < \rho_2$), such that for every $\rho \in [0,1]$ system $(1)_\rho$ with control range $U_\rho$ (instead of $U$ in (1)) has a control set $D_\rho$ with nonvoid interior and compact closure, and the following conditions are satisfied:

(i) $U = U_0$ and $D = D_1$;

(ii) $\overline{D_{\rho_2}} \subset \operatorname{int} D_{\rho_1}$ whenever $\rho_1 < \rho_2$;

(iii) for every neighborhood $W$ of $\overline{D}$ there is $\rho \in [0,1)$ with $\overline{D_\rho} \subset W$.

We will estimate the outer invariance pressure of the set $Q = \overline{D}$ for the system with control range $U = U_0$. Note that, in general, $D$ is not a control set for this system, since we only have $D = D_1 \subset D_0$.

**Corollary 21.** *Consider an inner control set $D$ of control system (1). Let $(\omega_0(\cdot), \varphi(\cdot, x_0, \omega_0))$ be a regular $T$-periodic control-trajectory pair with $x_0 \in \overline{D}$ and $\omega_0 \in \mathcal{U}_1$. Then*

$$P_{out}(f, \overline{D}) \leq \sum_{j=1}^{r} \max\{0, d_j \operatorname{Re} \lambda_j\} + \frac{1}{T}\int_0^T f(\omega_0(s))ds$$

*holds, where $\lambda_1, \ldots, \lambda_r$ are the Lyapunov exponents at $(x_0, \omega_0)$ with corresponding multiplicities $d_1, \ldots, d_r$.*

**Proof.** Note that the definition of inner control sets implies that for every $\rho \in [0,1)$ the set $\overline{D}$ is a compact subset of $D_\rho$ and the pair $(\overline{D}, D_\rho)$ is admissible. By Theorem 11 it follows that the outer invariance pressure $P^\rho_{out}(f, \overline{D}, \overline{D_\rho})$ for system $(1)_\rho$ satisfies

$$P^\rho_{out}(f, \overline{D}, \overline{D_\rho}) \leq \sum_{j=1}^{r} \max\{0, d_j \rho_j\} + \frac{1}{T}\int_0^T f(\omega_0(s))ds \text{ for all } \rho \in [0,1).$$

Now for given $\varepsilon > 0$ we may choose $\rho \in [0,1)$ such that $\overline{D_\rho} \subset N_\varepsilon(\overline{D})$. Then

$$\begin{aligned} P_{out}(f, \overline{D}, N_\varepsilon(\overline{D})) &\leq P^\rho_{out}(f, \overline{D}, N_\varepsilon(\overline{D})) \leq P^\rho_{out}(f, \overline{D_\rho}, N_\varepsilon(\overline{D})) \\ &\leq \sum_{j=1}^{r} \max\{0, d_j \rho_j\} + \frac{1}{T}\int_0^T f(\omega_0(s))ds. \end{aligned}$$

The first two inequalities follow from $U_\rho \subset U_0$ and $\overline{D_\rho} \subset N_\varepsilon(\overline{D})$. Since $P_{out}(f, \overline{D}) = \lim_{\varepsilon\to 0} P_{out}(f, \overline{D}, N_\varepsilon(\overline{D}))$, the assertion follows. ■

### 6.4. Example

The following example illustrates Theorem 16. Consider the following linear control system in $\mathbb{R}^d$,

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{=:A} \begin{bmatrix} x \\ y \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{=:B} \omega(t)$$

and assume that $\omega(t) \in U := [-1,1] + u_0$ for some $u_0 \in (-1,1)$. In this case, $0 \in \mathrm{int}U$, the pair $(A,B)$ is controllable, and $A$ is hyperbolic with eigenvalues given by $\lambda_{\pm} = 1 \pm i$. There exists a unique control set $D \subset \mathbb{R}^2$ such that $(0,0) \in \mathrm{int}D$, and $\overline{D}$ is compact.

We may interpret the control functions $\omega(t)$ and also $u_0$ as external forces acting on the system. Take $f \in C(U,\mathbb{R})$ as $f(u) := |u - u_0|$, then $(S_\tau f)(\omega)$ represents the impulse of $\omega - u_0$ until time $\tau$. For a subset $K \subset D$ a $(\tau, K, D)$-spanning set $\mathcal{S}$ represents a set of external forces $\omega$ that cause the system to remain in $D$ when it starts in $K$. By Theorem 16 we obtain for a compact subset $K \subset D$ with nonempty interior that

$$P_{inv}(f,K,Q) = 2 + \min_{u \in U} f(u) = 2 + \min_{u \in [-1,1]+u_0} |u - u_0| \,.$$

Here $P_{inv}(f,K,Q)$ represent the exponential growth rate of the amount of total impulse required of the external forces $\omega - u_0$ acting on the system to remain in $D$ as time tends to infinity. The minimum of $f$ is attained in $u = u_0$, which does not correspond to an equilibrium if $u_0 \neq 0$. Hence [6, Corollary 21] (cf. Remark 18) could not be applied in this case.

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