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Conditionally stationary measures for random diffeomorphisms

Fritz Colonius[a]

[a] *Institut für Mathematik, Universität Augsburg, 86159 Augsburg, Germany*

Abstract

For random diffeomorphisms the relation between conditionally stationary measures and controllability properties of an associated deterministic control system is analyzed.

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Keywords: random diffeomorphisms; control sets; conditional stationarity

1. Introduction

For systems with bounded noise it is well known, that the supports of stationary measures can be described by using control theoretic methods. This an old subject, going back to [1, 2] in the 1980's; see [3] for a more recent contribution and [4] for an application to ship stability problems and relations to Melnikov's method. Here we provide a new angle to this subject, by extending the approach for stationary measures to conditionally stationary measures (or quasi-stationary measures). Open dynamical systems, sometimes called systems with holes in the state space, are widely considered in the literature on deterministic dynamical systems, cf. [5] for a survey. In the present paper we will analyze a class of open random diffeomorphisms. We aim at transferring the relation between supports of stationary measures and invariant control sets (cf. [6, 7]) to conditionally stationary measures for open systems. The present paper also uses techniques from [8], where random diffeomorphisms and conditionally stationary measures were analyzed, mainly with a view towards bifurcation theory.

We formally replace the noise by a deterministic control term. Then the invariant subsets of complete controllability, called the invariant control sets, determine the supports of the stationary measures. If, under small perturbations, invariance is lost, one may expect that the perturbed system still shows similar, although transient behavior. In particular, exit from the formerly invariant subset occurs only on a much longer time scale. This is also well documented in numerical studies, and a standard method for the analysis of related phenomena is the theory of large deviations.

The translation mechanism between stochastic systems and control systems alluded to above applies to random differential equations as well as to random diffeomorphisms with special features in the latter context. For example, the invariant control sets need not be connected. In the continuous time setting, these results are due to Stroock

* Fritz Colonius. Tel.: +49 821 598-2246 ; fax: +49 821 598-2193 .
E-mail address: fritz.colonius@math.uni-augsburg.de

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and Varadhan, Kunita, Arnold and Kliemann using appropriate hypoellipticity assumptions. On the other hand, the transient behavior of random systems can be described by conditionally stationary measures, i.e., measures which are conditioned with respect to a transient set. In particular, for simulation studies it is of interest to determine the relevant regions in the state space, which are given by the supports of the conditionally stationary measures. If the stationary measures have densities with respect to Lebesgue measure, a perturbation study of the associated conditioned transfer operator and its spectrum and its eigenvectors shows that associated conditionally stationary measures exist and have densities depending continuously on parameters. For the associated (deterministic) control system one can perform an analogous perturbation analysis. Here the invariant control sets turn into control sets which are no more invariant and one can show that they lose their invariance only if they change discontinuously in the Hausdorff metric. However, if one changes the analysis by conditioning to an appropriate subset W of the state space, they turn into relatively invariant control sets and we show that they are the supports of ergodic conditionally stationary measures.

2. Random diffeomorphisms and conditional stationarity

In this section, the considered class of random diffeomorphisms will be specified and some basic assumptions are made.

We consider random diffeomorphisms $f: M\times\Delta\to M$ determined by the following data. On the domain Δ in $\mathbb{R}^d$ a probability measure ν is given which has a continuous density g with $g(\omega)>0$ for Lebesgue almost all $\omega\in\Delta$, and $M\subset\mathbb{R}^d$ (or a d-dimensional manifold). The map $f: M\times\Delta\to M$ is continuous and for all $\omega\in\Delta$ the map $f_\omega := f(\cdot,\omega)$ is a diffeomorphism on a neighborhood of M. Also for all $x\in M$ the maps $\omega\mapsto f(x,\omega)$ are diffeomorphisms on their range $f(x,\Delta)=\{f(x,\omega)\in M \mid \omega\in\Delta\}$, hence the inverse $f_x^{-1}: f(x,\Delta)\to\Delta$ is continuous and bijective. Integration with respect to Lebesgue measure is denoted by $d\omega$ and dy, respectively.

This gives rise to a discrete-time Markov process through the transition functions

$$P(x,A) := \int_{\{\omega\in\Delta \mid f(x,\omega)\in A\}} \nu(d\omega) \text{ for Borel sets } A\subset M. \tag{1}$$

The measure $P(x,\cdot)$ equals $(f_x)_*\nu$ with $[(f(x,\cdot))_*\nu](A)=\nu(f(x,\cdot)^{-1}(A))$, hence the transition functions have bounded densities $k(x,y):=g(f_x^{-1}(y))$, $y\in f(x,\Delta)$ and $k(x,y):=0$ elsewhere. Note that k is continuous on its support and

$$P(x,A)=\int_A k(x,y)dy=\int_A g(f_x^{-1}(y))dy=\int_{A\cap f(x,\Delta)} g(f_x^{-1}(y))dy. \tag{2}$$

In a standard manner, we may consider a random diffeomorphism as a discrete time system Φ. With $\mathbb{N}_0=\{0,1,\dots\}$ let $\Omega:=\Delta^{\mathbb{N}_0}$, define $\vartheta:\Omega\to\Omega$ as the shift $\vartheta(\omega_0,\omega_1,\dots):=(\omega_1,\omega_2,\dots)$ and

$$\Phi:(x,\omega)\mapsto(f(x,\omega_0),\vartheta(\omega)): M\times\Omega\to M.$$

The measure ν gives rise to a shift-invariant measure ν^∞ on Ω. The stationary measures μ correspond to the invariant measures of Φ having the form $\mu\times\nu^\infty$.

We suppose that $W\subset M$ is an open, relatively compact subset and refer to W as the world in which the system lives. Let $f_W: W\times\Delta\to M$ be the restriction of f. We are interested in conditionally stationary measures which describe the transient behavior of W.

2.1 Definition: *A probability measure μ with support in $\overline{W}$ is conditionally stationary, if*

$$\mu(B)=\frac{\int_W P(x,B)\mu(dx)}{\int_W P(x,W)\mu(dx)} \text{ for all measurable } B\subset W \text{ and all } x\in B.$$

Thus, for a conditionally stationary measure and any set B in W, a constant fraction of points in W goes into B; the fraction is determined by the fraction of points going out of W (in one step). Conditionally stationary measures are absolutely continuous with respect to Lebesgue measure and hence have a density h called conditionally stationary density. It also follows that $\mu(W) = \mu(\overline{W}) = 1$. Let the n-step conditional transition function for $x \in W$ and $A \subset W$ be recursively defined by

$$P_1^W(x,A) = P(x,A),\ P_n^W(x,A) := \int_W P(x,dy)P_{n-1}^W(y,A).$$

Then for every $A \subset W$ and $n \in \mathbb{N}$ one has

$$\rho^n \mu(A) = \int_W P_n^W(y,A)\mu(dy). \tag{3}$$

We will need a continuity property of conditionally stationary densities with respect to parameters. Here the following characterization is helpful, which uses the transfer operator L on the space of integrable functions $L^1(M,\mathbb{R})$ determined by

$$\int_A (L\phi)(x)dx = \int_M P(x,A)\phi(x)dx \text{ for measurable } A \subset M \text{ and } \phi \in L^1(M,\mathbb{R}). \tag{4}$$

The conditional transfer operator associated with W is the linear operator L_W on $L^1(\overline{W},\mathbb{R})$ defined by

$$L_W(\phi)(x) := \chi_{\overline{W}}(x)L\phi(x)/\int_{\overline{W}} L\phi(y)dy, x \in \overline{W},$$

where $\chi_{\overline{W}}$ is the characteristic function of $\overline{W}$. For the following result cf. [8, Proposition 5.3].

2.2 Proposition: *The image of the conditional transfer operators L_W is contained in the space $C(\overline{W},\mathbb{R})$ of continuous functions and we may restrict L_W to a continuous operator on this space. In particular, the operator L_W is compact on $L^1(\overline{W},\mathbb{R})$.*

Next we will relate conditionally stationary densities to eigenvectors of L_W by noting the following observation showing that a conditionally stationary density is an eigenvector corresponding to a real eigenvalue $\alpha \in (0,1)$ of the linear operator L_W on $C(\overline{W},\mathbb{R})$.

2.3 Proposition: *Suppose that $h \in L^1(\overline{W},\mathbb{R})$ is a conditionally stationary density. Then $h \in C(\overline{W},\mathbb{R})$ and it is an eigenvector of L_W corresponding to the eigenvalue $\rho \in \mathbb{R}$,*

$$L_W h = \rho h \text{ with } \rho := \int_W (Lh)(x)dx.$$

Conversely, if $\rho \in (0,1)$ is an eigenvalue of the compact linear operator L_W on $L_1(\overline{W},\mathbb{R})$ with a corresponding eigenvector $h \in L_1(\overline{W},\mathbb{R})$ satisfying $h(x) \geq 0$ for almost all $x \in W$ and normalized such that $\int_W h(x)dx = 1$, then h is a conditionally stationary density.

2.4 Remark: We emphasize that conditionally stationary measures are related to eigenvalues of the operator L_W, not to eigenvalues of the operator L which is defined for functions on the whole state space M. This is a crucial difference to the analysis of almost invariant sets for deterministic systems in [9]. In particular, [9, Remark 3.2] shows that for an eigenvalue $\rho \neq 1$ of the Frobenius-Perron operator the eigenvectors h satisfy $\int_X h(x)dx = 0$. Then, decomposing the state space X into subsets according to positive or negative values of h, one can obtain almost invariant sets (in fact, the interest there is in cyclic behavior related to eigenvalues which are roots of unity in $\mathbb{C}$.)

A consequence is the following result.

2.5 Proposition: *Let $\rho \in (0,1)$. Then the set of conditionally stationary measures μ with $\mu(\overline{W}) = \int_W P(x,W)\mu(dx) = \rho$ is a convex and weak* compact subset of the set of all probability measures on $\overline{W}$.*

Finally, we draw conclusions for the parameter dependent systems. Thus we suppose that f depends on a real parameter $\alpha \in I$, where I is an open interval in $\mathbb{R}$ and that $f : M \times \Delta \times I \to M$ is continuous. If we look at the transfer operator, [10, Chapter IV, paragraph 3, section 5] shows that every finite system of eigenvalues of bounded operators changes continuously with the operator. Furthermore, [11, Theorem 3.3] shows that on L_1 all generalized eigenvectors for eigenvalues with absolute value 1 are, in fact, eigenvectors. One obtains the following continuity property; cf. [8, Proposition 5.5].

2.6 Proposition: *Consider a parameter dependent random diffeomorphism of the form* $f_\alpha : M \times \Delta \to M$, $\alpha \in I \subset \mathbb{R}$, *on a state space* M, *and let* μ_{α_0} *be an ergodic stationary measure with density* h_{α_0} *of* f_{α_0}. *Suppose that* W *is a relatively compact, open neighborhood of the support of* μ_{α_0} *and that there is not other stationary measure of* f_{α_0} *intersecting the closure* $\overline{W}$. *Then there is* $\varepsilon_0 > 0$ *such that for all* α *with* $|\alpha - \alpha_0| < \varepsilon_0$ *the map* f_α *possesses a conditionally stationary density* h_α *on* W *with* h_{α_0} *coinciding with the density of* μ_{α_0}, *and the map* $(x,\alpha) \mapsto h_\alpha(x) : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ *is continuous in* (x,α) *as well as the map* $\alpha \mapsto h_\alpha \in C(\overline{W},\mathbb{R})$ *and*

$$L_W^\alpha h_\alpha = \rho(\alpha) h_\alpha$$

where $\rho(\alpha) = \int_W L_W^\alpha h_\alpha(x)dx$ *depends continuously on* α.

3. Control sets and relative invariance

In this section, relevant definitions and properties of control systems in discrete time are collected. Some results in the continuous time case have been given in [12], for the discrete time case considered here we rely on [13, 14, 7].

Suppose a discrete time control system of the form

$$x_{k+1} = f(x_k, u_k),\ u_k \in \Delta,\ k \in \mathbb{Z}, \tag{5}$$

is given, where M is a subset of $\mathbb{R}^d$ (or a manifold), $\Delta \subset \mathbb{R}^d$ is compact and connected with $\overline{\mathrm{int}\Delta} = \Delta$ and $f : M \times \Delta \to M$ is a continuous map. Throughout we also assume that $f_v := f(\cdot, v)$ is a diffeomorphism on a neighborhood of M for every $v \in \Delta$. Suppose that an open, relatively compact subset $W \subset M$ is fixed such that $f(W \times \Delta) \cap W \neq \emptyset$ and $f(W \times \Delta) \not\subset W$.

Let $f_W := f_{|W\times\Delta} : W \times \Delta \to M$, and consider, with a slight abuse of notation, the following "open" control system

$$x_{k+1} = f_W(x_k, u_k),\ u_k \in \Delta, \tag{6}$$

Note that (6) only makes sense, if $x_k \in W$. Thus this system may enter $M \setminus W$, but it cannot leave $M \setminus W$. For $x \in M$ and a control function $u : \mathbb{Z} \to \Delta$ we often abbreviate $f^0(x,u) := x$ and

$$f^n(x,u) := f_{u_n}(f^{n-1}(x,u)) \text{ and } f^{-n}(x,u) := f_{u_{-n}}(f^{-n+1}(x,u)),\ n \geq 1.$$

Let for $x \in W$ and every $n \in \mathbb{N}$ the positive and negative orbits relative to W be

$$\mathscr{O}_W^{+,n}(x) = \{f_W^n(x,u) \mid u : \mathbb{Z} \to \Delta\}, \qquad \mathscr{O}_W^{-,n}(x) := \left\{f_W^{-n}(x,u) \mid u : \mathbb{Z} \to \Delta\right\},$$

resp. Throughout we assume that control system (5) is accessible in W meaning that all $\mathscr{O}_W^{+,n}(x)$ have nonvoid interiors and all sets $\mathscr{O}_W^{-,n}(x), x \in W$ are either empty or have nonvoid interior. We also write

$$\mathscr{O}_W^+(x) := \bigcup_{n\in\mathbb{N}} \mathscr{O}_W^{+,n}(x) \quad \text{and} \quad \mathscr{O}_W^-(x) := \bigcup_{n\in\mathbb{N}} \mathscr{O}_W^{-,n}(x).$$

Restricting attention to W, we obtain the following notion.

3.1 Definition: *A subset* $D_W \subset W$ *with nonvoid interior is a* W*-control set (or relative control set with respect to* W*) if for all* $x, y \in D_W$ *one has* $y \in \overline{\mathscr{O}_W^+(x)}$ *and* D_W *is maximal with this property, i.e., if* $D'_W \supset D_W$ *is a set such that* $y \in \overline{\mathscr{O}_W^+(x)}$ *for all* $x, y \in D'_W$, *then* $D_W = D'_W$. *A* W*-control set is called relatively invariant, if* $x \in D_W$ *and* $f^k(x,u) \notin D_W$ *for some control* u *and some* $k \in \mathbb{N}$ *implies* $f^k(x,u) \notin W$.

For the sake of brevity, we call relatively invariant W-control sets just relatively invariant control sets, if it is clear from the context, which world W is considered. If $W = M$, we omit the index W and just speak of control sets and invariant control sets. One has $D = \overline{\mathcal{O}^+(x)} \cap \mathrm{int}\mathcal{O}^-(x)$ for every x in the core (or transitivity set as it is called in [13, Section 4.2]) defined by $\mathrm{core}D := \{y \in D \mid \text{there is } z \in D \text{ with } z \in \mathrm{int}\mathcal{O}^+(y) \text{ and } y \in \mathrm{int}\mathcal{O}^+(z)\}$; the set $\mathrm{core}D$ is open and it is dense in D. Relative control sets are, in general, properly contained in control sets, since they need not be maximal with respect to the whole state space.

The main result on existence of relatively invariant control sets is the following.

3.2 Theorem: *Consider a control system of the form (6) which is accessible in an open, relatively compact world W in a state space M. The following assertions are equivalent:*

(i) There is a closed set $Q \subset W$ such that $\overline{\mathcal{O}_W^+(x)} \cap Q \neq \emptyset$ for all $x \in W$.

(ii) For every $x \in W$ there is a relatively invariant control set D with $D \subset \overline{\mathcal{O}_W^+(x)}$.

If (i) holds, there are only finitely many relatively invariant control sets.

Now suppose that f depends on a real parameter α as in the previous section and consider a family of control systems of the form

$$x_{k+1} = f_\alpha(x_k, u_k),\ u_k \in \Delta,\ \alpha \in I. \tag{7}$$

The corresponding objects for the α-system are marked by $D^\alpha, \mathcal{O}^{\alpha,+}(x)$, etc. Then [7, Proposition 3 and Theorem 2] shows that under small perturbations one finds near an invariant control set D^{α_0} a control set for the perturbed system. When an invariant control set loses its invariance, $\overline{D^\alpha}$ changes discontinuously in the Hausdorff metric at $\alpha = \alpha_0$. The following theorem gives more information on this situation. It shows that, in addition to the control sets D^α, relative control sets are generated. While the D^α are not invariant control sets, these relative control sets are relatively invariant, if the world W is chosen small enough.

3.3 Theorem: *Consider the family (7) of control systems.*

(i) For every open neighborhood W of D^{α_0} there are $\varepsilon_0 > 0$ and a family of relative control sets D_W^α for $|\alpha - \alpha_0| < \varepsilon_0$ with the following property: For all compact subsets $K \subset \mathrm{core}D^{\alpha_0}$ there is $\varepsilon_K \in (0, \varepsilon_1)$ such that $K \subset \mathrm{core}D_W^\alpha$ for all α with $|\alpha - \alpha_0| < \varepsilon_K$. In particular, $D_W^{\alpha_0} = D^{\alpha_0}$ and the map $\alpha \mapsto \overline{D_W^\alpha}$ is lower semicontinuous in the Hausdorff metric.

(ii) There are a neighborhood W of D^{α_0}, a constant $\varepsilon_0 > 0$ and $x^0 \in \mathrm{core}D^{\alpha_0}$ such that for all α with $|\alpha - \alpha_0| < \varepsilon_0$ one has $W \subset \mathcal{O}_W^{\alpha,-}(x^0)$. Then for $|\alpha - \alpha_0|$ small enough, the relative control sets D_W^α from assertion (i) are relatively invariant.

While relatively invariant control sets share many properties with invariant control sets, this notion sheds new light on the perturbation theory of invariant control sets: If the world W around an invariant control set is chosen small enough, then an invariant control set D^{α_0} always generates a family of relatively invariant control set (this also holds if the control sets D^α have lost invariance).

4. Conditionally stationary measures and relative control sets

In this section we will derive relations between the supports of conditionally stationary measures and relative control sets. This partially generalizes the results for the stationary case in [7]. We will consider the situation of the preceding sections, where the world W is open and relatively compact in the state space M of a random diffeomorphism. If we disregard the presence of the measure ν, we obtain a control system as discussed in Section 3.

The following relation between supports of stationary measures and invariant control sets has been established in [7, Theorem 3] (under slightly stronger assumptions): For every stationary measure of a random diffeomorphism f, the support is contained in the union of the invariant control sets. If μ is an ergodic stationary measure, the support of

μ is an invariant control set of the associated control system. Conversely, for every invariant control set D there exists a unique invariant measure μ with support equal to D and μ is ergodic. Recall also that every stationary measure has a continuous density with respect to Lebesgue measure.

While existence of conditionally stationary measures remains open at this point, we can show the following results on their properties.

4.1 Proposition: *Let μ be a conditionally stationary measure.*

(i) Suppose that $y_0 \in \mathrm{int}\mathscr{O}_W^{+,n}(x_0)$ for some $x_0 \in \mathrm{supp}\mu$ and $n \in \mathbb{N}$. Then $y_0 \in \mathrm{supp}\mu$.

(ii) If $\mathrm{supp}\mu \cap D_W \neq \emptyset$ for some relative control set D_W, then $\overline{D_W} \subset \mathrm{supp}\mu$.

Proof: (i) Note that $\mathrm{supp}\mu$ equals the closure of its interior. Suppose that $y_0 \notin \mathrm{supp}\mu$. Then there is a nonvoid open set $V \subset \mathscr{O}_W^{+,n}(x_0)$ with $V \cap \mathrm{supp}\mu = \varnothing$ and we may assume that $V \subset \mathscr{O}_W^{+,n}(x)$ for all x in a nonvoid open set in the intersection $\mathrm{supp}\mu \cap V(x_0)$ with a neighborhood $V(x_0)$ of x_0 and hence $P_n^W(x,V) > 0$ for these x. This yields a contradiction to conditional stationarity, since by (3) $0 < \int_W P_n^W(x,V)\mu(dx) = \rho^n \mu(V) = 0$.

(ii) Suppose, contrary to the assertion, that there is $y_0 \in D_W \setminus \mathrm{supp}\mu$. Then there is a neighborhood $V(y_0)$ of y_0 in the open set W such that $V(y_0) \cap \mathrm{supp}\mu = \emptyset$. Since the core of D_W is dense in D_W this shows that we may take $y_0 \in \mathrm{core}D_W$ and $V(y_0) \subset \mathrm{core}D_W$. By assumption there is $x_0 \in \mathrm{supp}\mu \cap D_W$. Thus $y_0 \in \mathscr{O}_W^{+,n}(x_0)$ for some $n \in \mathbb{N}$ and $P_n^W(x,V(y_0)) > 0$ for all x in a neighborhood $V(x_0)$ of x_0. Then $V(x_0) \cap \mathrm{supp}\mu$ has nonvoid interior, since μ is absolutely continuous. This yields a contradiction to conditional stationarity, since it follows that

$$0 < \int_W P_n^W(x,V(y_0))\mu(dx) \leq \int_W P_n^W(x,D_W \setminus \mathrm{supp}\mu)\mu(dx) = \rho^n \mu(D_W \setminus \mathrm{supp}\mu) = 0.$$

□

Suppose that the assumption in Theorem 3.2(i) is satisfied. Then it follows by Proposition 4.1, that for every conditionally stationary measure μ the support will contain a relatively invariant control set.

4.2 Proposition: *For every conditionally stationary measure μ, the support of μ is contained in the closure of*

$$\mathrm{Ker}^-(W) := \left\{ x \in W \mid \mathscr{O}_W^{-,n}(x) \neq \varnothing \text{ for all } n \in \mathbb{N} \right\}.$$

A conditionally stationary measure μ is called ergodic, if

$$\mu(E \triangle f_W^{-1}(E,\Delta)) = 0 \text{ implies } \mu(E) = 0 \text{ or } \mu(E) = 1 \text{ for } E \subset W.$$

4.3 Proposition: *Let $\rho \in (0,1)$. The extremal points of the set of conditionally stationary measures for ρ are ergodic measures for ρ. In particular, if there is a conditionally stationary measure for ρ, then there exists an ergodic conditionally stationary measure μ for ρ. If there is a unique conditionally stationary measure μ for ρ, then μ is ergodic.*

Proof: One can show that the set of conditionally stationary measures for ρ is a convex and weak* compact subset of the set of all probability measures on $\overline{W}$. Hence there exist extreme points and every extreme point is ergodic: If μ is not ergodic, there exists an invariant subset E with $\mu(E) \in (0,1)$ and also the complement of E is invariant. Then the induced measures on E and its complement are conditionally stationary for ρ and one can write μ as a proper convex combination. □

We can use ergodicity to show the following result.

4.4 Theorem: *Let the assumption in Theorem 3.2(i) be satisfied and let μ be an ergodic conditionally stationary measure such that $\mathrm{supp}\mu$ intersects a relatively invariant control set D_W. Suppose that $x \in \mathrm{Ker}^-(W)$ with $\lim_{n\to\infty} f_W^{-n}(x,u) \cap \overline{D_W} \neq \varnothing$ for some $u \in \mathscr{U}$ implies that $x \in \overline{D_W}$. Then the support of μ coincides with the closure of D_W, i.e., $\mathrm{supp}\mu = \overline{D_W}$.*

Proof: The assumption implies that $\overline{D_W} \subset \operatorname{supp}\mu$ and $\mu(\overline{D_W}) > 0$. The set

$$\left\{ x \in \operatorname{Ker}^-(W) \mid \lim_{n\to\infty} f_W^{-n}(x,u) \cap \overline{D_W} = \varnothing \text{ for all } u \in \mathscr{U} \right\}. \tag{8}$$

is invariant and its complement in $\operatorname{Ker}^-(W)$ is contained in $\overline{D_W}$. Thus the set in (8) has μ-measure in $[0,1)$ and ergodicity implies that its μ-measure is zero. Hence $\mu(\overline{D_W}) = 1$. □

It remains to discuss existence of conditionally stationary measures. For this purpose we analyze systems depending on a parameter α, where for α_0 we have an ergodic stationary measure. If we look at the transfer operator, [10, Chapter IV, paragraph 3, section 5] shows that every finite system of eigenvalues of bounded operators changes continuously with the operator. The support of an ergodic stationary measure is given by an invariant control set D_{α_0}. We suppose that for α near α_0 the invariant control set D_α has lost invariance. Hence, for an appropriate neighborhood W relatively invariant control sets are generated. On the level of the stochastic system, recall the continuity property from Proposition 2.6. Combining this with our previous results one obtains the following theorem.

4.5 Theorem: *Consider a parameter dependent random diffeomorphism of the form $f_\alpha : M \times \Delta \to M, \alpha \in I \subset \mathbb{R}$, satisfying for each α the assumptions of Theorem 4.4 and let μ_{α_0} be an ergodic stationary measure with density h_{α_0} of f_{α_0} and support given by a compact invariant control set D^{α_0}. Then there are a relatively compact open neighborhood W of D^{α_0} and $\varepsilon_0 > 0$ such that for all α with $|\alpha - \alpha_0| < \varepsilon_0$ the following holds:*

(i) There is a unique eigenvalue $\rho(\alpha) = \int_W L_W^\alpha h_\alpha(x)dx$ near 1 of the transfer operator L_W^α corresponding to f_α and its eigenvector h_α is the continuous density of a unique ergodic conditionally stationary measure μ_α.

(ii) The maps $\alpha \mapsto \rho(\alpha)$ and $\alpha \mapsto h_\alpha$ are continuous and the support of μ_α coincides with closure of the relatively invariant control set D_W^α existing according to Theorem 3.3.

Proof: The assertions about the eigenvalues and eigenvectors of L_W^α follow from Proposition 2.6. By Proposition 4.3 the conditionally stationary measure μ_α is ergodic. On the other hand, Theorem 3.3 determines relatively invariant control sets D_W^α. The support of μ_α intersects the relatively invariant control set D_W^α, since the continuous density h_α depends continuously on α. Now Theorem 4.4 shows $\overline{D_W^\alpha} = \operatorname{supp}\mu_\alpha$. □

A simple example where Theorem 4.5 applies is a family f_α of random diffeomorphisms on the circle $\mathbb{R}/\mathbb{Z}$ exhibiting a random saddle node bifurcation; cf. [8].

4.6 Example: Let $f : \mathbb{R}/\mathbb{Z} \times [-1,1] \to \mathbb{R}/\mathbb{Z}$ be given by

$$f_\alpha(x,\omega) = x + \frac{\sigma}{2\pi}\cos(2\pi x) + A\omega + \alpha \mod 1, \tag{9}$$

where $0 < \sigma < 1$. The random parameter ω is drawn from a distribution supported on $\Delta := [-1,1]$. Consider a small positive value of the noise amplitude A. For $\alpha_0 = -\frac{\sigma}{2\pi} - A$ the extremal graph $f_{\alpha_0}(\cdot,1)$ is tangent to the diagonal at a point b. Here f_{α_0} admits a stationary measure supported on an interval $D^{\alpha_0} = [b,c]$ with $b < c$ as indicated in Figure 1. For α below α_0 there is a stationary measure supported on an invariant control set D^α which is an interval that varies continuously with α. For $\alpha > \alpha_0$ the only stationary measure has support equal to $\mathbb{R}/\mathbb{Z}$.Take W as an open set containing D^{α_0} with $W \subset \mathscr{O}_W^{\alpha,-}(x^0)$ for some $x^0 \in D^{\alpha_0}$, e.g., let $W := (0.2, 0.6)$. Then for $\alpha > \alpha_0$ the only control set is the (invariant) control set $D^\alpha = \mathbb{R}/\mathbb{Z}$. There is a unique relatively invariant control set D_W^α, which has the form $D_W^\alpha = [b(\alpha), 0.6)$, where $b(\alpha)$ is given by the intersection of the lower sinusoidal curve (depending on α) with the diagonal; $b(\alpha_0) = b$ is the left boundary point of D^{α_0}. Thus this interval is closed relative to $W = (0.2, 0.6)$. For every $x \in W$ to the left of D_W^α there is $n \in \mathbb{N}$ with $\mathscr{O}_W^{-,\alpha,n}(x) = \varnothing$. Hence the conditionally stationary measure μ_α has support equal to $\overline{D_W^\alpha}$.

5. Conclusions

The results of this paper give some insight into the general mechanisms when invariance of random dynamical systems is lost. The similarities between invariant control sets and relatively control sets on one hand, and between

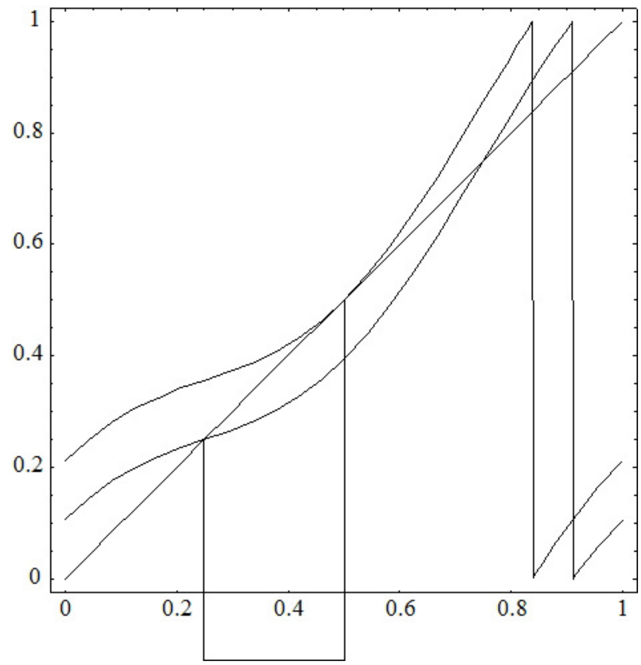

Fig. 1. Extremal graphs $f_{\alpha_0}(\cdot, \pm 1)$ for the random diffeomorphisms f_α given by (9). The interval indicates the support of the stationary measure for the unperturbed system.

stationary and conditionally stationary measures on the other hand have led us to the conjecture that for conditionally stationary measures relatively invariant control sets play a role, which is analogous to the role of invariant control sets for stationary measures. This conjecture could be confirmed for a certain class of random diffeomorphisms.

This theory is an alternative to the theory of almost invariance which amounts to a decomposition of the state space into two complementary, almost invariant subsets. Instead, a single subset is determined by a relatively invariant control set. The relations between these apparently disjoint theories certainly need further investigations. Furthermore, it is tempting to conjecture that similar results hold for models in continuous time which are closer to applications in mechanical engineering. Here an essential point will be the spectral properties of associated conditioned Fokker-Planck operators.

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