Near invariance and local transience for stochastic systems

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Summary. For stochastic systems in continuous and discrete time with bounded background noise, nearly invariant and locally transient sets are characterized via associated deterministic control systems.

We consider stochastic dynamical systems in continuous and discrete time of the form

$$\dot{x}(t) = X_0(x(t)) + \sum_{i=1}^{m} \xi_i(t) X_i(x(t)) \text{ and } x_{k+1} = f(x_k, \xi_k), \text{ respectively.}$$

Here $\xi(\cdot) = F(\eta(\cdot))$ is a stochastic perturbation with values in a bounded set $U \subset \mathbb{R}^m$ with $0 \in \text{int}U$ coming from a background noise $\eta(\cdot)$. The background noise $\eta(\cdot)$ lives on a compact state space N and we assume that $\eta(\cdot)$ has a unique ergodic invariant measure ν with supp $\nu = N$. In particular, in the continuous time case, $\eta(\cdot)$ may be determined by a stochastic differential equation on a compact manifold N,

$$\mathrm{d}\eta = Y_0(\eta)dt + \sum_{j=1}^{l} Y_j(\eta) \circ dW_j.$$

It is well known, that the supports of the invariant measures of the pair process $(x(\cdot), \eta(\cdot))$ can be described by controllability properties of an associated control system of the form

$$\dot{x}(t) = X_0(x(t)) + \sum_{i=1}^{m} u_i(t) X_i(x(t)),\ u \in \mathcal{U} = \{u \in L_\infty(\mathbb{R}, \mathbb{R}^m),\ u(t) \in U \text{ for } t \in \mathbb{R}\},$$
$$x_{k+1} = f(x_k, u_k),\ u \in \mathcal{U} = \{u \in l_\infty(\mathbb{Z}, \mathbb{R}^m),\ u_k \in U \text{ for } k \in \mathbb{Z}\}$$

with trajectories $\varphi(\cdot, x_0, u)$; see Arnold/Kliemann [1]. See also Colonius et al. [4] for a recent contribution elucidating the relations between Melnikovṡ method for stochastic systems and the analysis based on associated control systems. Under appropriate assumptions, the supports of the invariant measures can be characterized in the following way. Assume local accessibility of the control system, i.e., for all $x_0 \in \mathbb{R}^d$ and all $T > 0$ the reachable sets up to time T

$$\mathcal{O}^+_{\leq T}(x_0) = \{\varphi(t, x_0, u),\ t \in [0, T] \text{ and } u \in \mathcal{U}^\rho\}$$

have nonvoid interiors. A subset $D \subset \mathbb{R}^d$ is a control set if it is a maximal set contained in the closure $\text{cl}\mathcal{O}(x_0)$ of $\mathcal{O}(x_0) = \bigcup_{T>0} \mathcal{O}^+_{\leq T}(x_0)$ for all $x_0 \in D$; an invariant control set satisfies $\text{cl}D = \text{cl}\mathcal{O}^+(x_0)$ for all $x_0 \in D$. Then the supports of the invariant measures for the pair process are given by $C \times N$, where C is an invariant control set of the associated control system.
Furthermore, for the control system define the domain of attraction of D by

$$\mathcal{A}(D) = \{x_0 \in M,\ \exists u \in \mathcal{U} \text{ with } \varphi(t, x_0, u) \in D \text{ for some } t \geq 0\}$$

and let the invariant domain of attraction $\mathcal{A}^{inv}(D)$ be the largest positively invariant set in $\mathcal{A}(D)$.

For the stochastic system, the first entrance time into a set A, starting in $x_0 \in \mathbb{R}^d$, is the random number $\tau_{x_0}(A,\omega) = \inf\{t \geq 0,\ x(t,x_0,\omega) \in A\}$; the entrance probability into a set A, starting in $x_0 \in \mathbb{R}^d$, is $P_{x_0}(A) = P\{\tau_{x_0}(A) < \infty\}$. Then for an invariant control set C one has $P_{x_0}(C) > 0$ if and only if $x_0 \in \mathcal{A}(C)$ and $P_{x_0}(C) = 1$ if and only if $x_0 \in \mathcal{A}^{inv}(C)$. For the first exit times $\sigma_{x_0}(D)$ from a control set D one has that the exit probability satisfies $P_{x_0}(\sigma_{x_0}(D) < \infty) = 0$ if and only if D is invariant and $P_{x_0}(\sigma_{x_0}(D) < \infty) = 1$ if and only if D is not invariant.

In the present paper we discuss the behavior when invariance is lost due to changes in the amplitudes of the noise. For this purpose we consider perturbations of varying size:

$$\xi = F^\rho(\eta),\ \rho \geq 0,\ F^\rho[N] = U^\rho = \rho \cdot U \subset \mathbb{R}^m.$$

Then the admissible controls for the associated control system are in $\mathcal{U}^\rho = \{u \in L_\infty(\mathbb{R},\mathbb{R}^m),\ u(t) \in U^\rho$ for $t \in \mathbb{R}\}$; similarly, other objects for the ρ-systems are characterized by a superindex ρ.
In this situation, a set $A \subset \mathbb{R}^d$ is called nearly invariant in $x_0 \in \mathrm{int}A$ for $\rho > \rho_0$, if (i) the exit probability $P^\rho_{x_0}(A) > 0$ for $\rho > \rho_0$; and (ii) for all $x \in A$ one has for the exit times $\sigma^\rho_x(A) \nearrow \infty$ almost surely for $\rho \searrow \rho_0$ and $\sigma^{\rho_0}_x(A) = \infty$ almost surely. If A is a set with nonvoid interior which is nearly invariant for all $x_0 \in \mathrm{int}A$, we A a nearly invariant set. Then the following result holds.

Theorem. Let $x_0 \in \mathrm{int}A$ for a closed set A. Then A is nearly invariant in x_0 for the stochastic system if and only if for the control system A is positively invariant for ρ_0 and $\mathrm{int}(\mathcal{O}^{\rho,+}(x_0) \setminus A) \neq \varnothing$ for all $\rho > \rho_0$.

For a straightforward consequence of this result consider an invariant control set C^{ρ_0} and let D^ρ be the control set with $C^{\rho_0} \subset D^\rho$ for $\rho > \rho_0$. Suppose that there is $x_0 \in \mathrm{int}C^{\rho_0}$ with $\mathrm{int}\,(\mathcal{O}^{\rho,+}(x_0) \setminus C^{\rho_0}) \neq \varnothing$ for all $\rho > \rho_0$. Then C^{ρ_0} is nearly invariant for $\rho > \rho_0$. Furthermore, for a compact set K the intersection $\mathcal{A}_{inv}(C^{\rho_0}) \cap K$ is nearly invariant if it is positively invariant for the control systems with ρ_0.

In order to characterize the case when the support of an invariant measure changes discontinuously or the invariant measure disappears altogether for increasing ρ, we introduce the following notion.

A closed nearly invariant set $A \subset \mathbb{R}^d$ is called locally transient in $x_0 \in \mathrm{int}A$, if the first exit time from a neighborhood W of the set A satisfies

$$\sigma^\rho(x_0, W) < \infty \text{ with positive probability for all } \rho > \rho_0.$$

If the set A has nonvoid interior and there is a neighborhood W of A such that this holds for all $x_0 \in \mathrm{int}A$, the set A is called locally transient. Then the following theorem holds indicating a change in the qualitative behavior in the stochastic system.

Theorem. Let C^{ρ_0} be an invariant control set and, for $\rho > \rho_0$, denote the control sets containing C^{ρ_0} by D^ρ. If for $\rho > \rho_0$ the control sets D^ρ are not invariant, then C^{ρ_0} is a locally transient, nearly invariant set.

Several examples from stochastic mechanics illustrating these results will be discussed. In particular, for a simple one-degree-of freedom system with a two-well potential one finds that the converse of the theorem above does not hold.

References

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