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Controllability for Systems with Almost Periodic Excitations

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***Abstract*— For control systems described by ordinary differential equations subject to almost periodic excitations the controllability properties depend on the specific excitation. Here these properties and, in particular, control sets and chain control sets are discussed for all excitations in the closure of all time shifts of a given almost periodic function. Then relations between heteroclinic orbits of an uncontrolled and unperturbed system and controllability for small control ranges and small perturbations are studied using Melnikov's method. Finally, a system with two-well potential is studied in detail.**

I. Introduction

This paper analyzes controllability properties of control systems which are subject to almost periodic excitations. The main topic are the relations between hetero- or homoclinic orbits of an uncontrolled and unperturbed system and controllability for small control ranges. Here Melnikov's method plays an important role. Furthermore, we apply our results to a second order system modeling ship dynamics and capsizing under wave excitations. See [5] for references, proofs and further details.

The paper is organized as follows: After preliminaries in Section II, we analyze chain control sets in Section III. Section IV introduces control sets and presents relations to chain control sets and to almost periodic solutions of the uncontrolled system. Section V presents relevant results on almost periodic perturbations of hyperbolic equilibria and Melnikov's method. This is used in Section VI to study the relation between heteroclinic orbits of an unperturbed system and controllability for small control ranges. Finally, in Section VII we discuss a second order system with M-potential modeling ship roll motion.

II. Preliminaries

Consider the control system

$$\dot{x}(t) = f\big(x(t), z(t), u(t)\big),\ u \in \mathcal{U}, \qquad (1)$$

in an open set $M \subset \mathbb{R}^d$ with admissible controls in $\mathcal{U}$, and assume that z is an almost periodic function (compare e. g., Scheurle [13, Definition 2.6]). Define θ as the time shift $(\theta_t z)(s) := z(t+s)$, $s, t \in \mathbb{R}$. Let $\mathcal{Z}$ be the closure in the space $C_b(\mathbb{R}, \mathbb{R}^k)$ of bounded continuous functions of the shifts of an almost periodic function. Then $\mathcal{Z}$ is a minimal set; i. e., every trajectory is dense in $\mathcal{Z}$. Observe that for $z \in \mathcal{Z}$ it holds that $z(t) = (\theta_t z)(0)$. Assuming global existence and uniqueness, we denote by $\varphi(t, t_0, x, z, u)$ the solution of the initial value problem $\dot{x}(t) = f\big(x(t), z(t), u(t)\big)$, $x(t_0) = x$. If $t_0 = 0$, we often omit this argument. The solution map of the coupled system is denoted by $\psi(t, x, z, u) = \big(\varphi(t, x, z, u), \theta_t z\big)$. We assume that the set of admissible controls is given by $\mathcal{U} = \{u \in L_\infty(\mathbb{R}, \mathbb{R}^m),\ u(t) \in U$ for almost all $t\}$, where $U \subset \mathbb{R}^m$. If we denote also the time shift on $\mathcal{U}$ by θ_t, we obtain the cocycle property $\varphi(t+s, x, z, u) = \varphi\big(s, \varphi(t, x, z, u), \theta_t z, \theta_t u\big), t, s \in \mathbb{R}$. Finally, the maps $\Phi_t : M \times \mathcal{Z} \times \mathcal{U} \to M \times \mathcal{Z} \times \mathcal{U},\ \Phi_t(x, z, u) = \big(\psi(t, x, z, u), \theta_t u\big), t \in \mathbb{R}$, define a continuous flow, the *control flow*, provided that $U \subset \mathbb{R}^m$ is convex and compact and $f(x, z, u) = f_0(x, z) + \sum_{i=1}^m u_i f_i(x, z)$ with C^1-functions $f_i : \mathbb{R}^d \times \mathbb{R}^k \to \mathbb{R}^d$; here $\mathcal{U} \subset L_\infty(\mathbb{R}, \mathbb{R}^m)$ is endowed with the weak* topology. This follows by a minor extension of Proposition 4.1.1 in [3].

The weak* topology on $\mathcal{U}$ is compact and metrizable. Throughout this paper, we endow $\mathcal{U}$ with a corresponding metric and assume that the conditions above guaranteeing continuity of the control flow are satisfied. Note that the space $\mathcal{Z}$ of almost periodic excitations is considered in the norm topology of $C_b(\mathbb{R}, \mathbb{R}^k)$. The shifts on each of these spaces are continuous.

For convenience, we also assume that $0 \in U$, and we call the corresponding differential equation with $u \equiv 0$ the *uncontrolled* system. For periodic and for quasi-periodic excitations we may be able to replace $\mathcal{Z}$ by a finite dimensional state space Z.

III. Chain Control Sets

In this section we define and characterize chain control sets relative to a subset of the state space working in the general almost periodic case. It will be convenient to write for a subset $A \subset M \times \mathcal{Z}$ the intersection with a fiber over $z \in \mathcal{Z}$ as $A_z := A \cap (M \times \{z\})$. Hence $A = \bigcup_{z \in \mathcal{Z}} A_z$. Where convenient, we identify A_z and $\{x \in M,\ (x, z) \in A_z\}$.

A controlled (ε, T)-chain along $z \in \mathcal{Z}$ is given by $T_0, \ldots, T_{n-1} \geq T$, controls $u_0, \ldots, u_{n-1} \in \mathcal{U}$, and points $x_0, \ldots, x_n \in M$ with $\mathrm{d}\big(\varphi(T_j, x_j, \theta_{T_0+\cdots+T_{j-1}} z, u_j), x_{j+1}\big) < \varepsilon$ for all $j = 0, \ldots, n-1$.

Definition 1: A chain control set relative to a closed set $Q \subset M \times \mathcal{Z}$ is a nonvoid maximal set $E \subset M \times \mathcal{Z}$ such that

1) for all $(x,z),(y,w) \in E$ and all $\varepsilon, T > 0$ there exists a controlled (ε, T)-chain in Q along z from x to (y,w), i.e., $x_0 = x$, $x_n = y$, and $\mathrm{d}(\theta_{T_0+\cdots+T_{n-1}}z, w) < \varepsilon$, and

$$\psi(t, x_j, \theta_{T_0+\cdots+T_{j-1}}z, u_j) \in Q \quad \text{for all } t \in [0, T_j] \text{ and for all } j; \quad (2)$$

2) for all $(x,z) \in E$ there is $u \in \mathcal{U}$ with $\psi(t,x,z,u) \in E$ for all $t \in \mathbb{R}$.

The condition (2) can be written as $\varphi(t, x_j, \theta_{T_0+\cdots+T_{j-1}}z, u_j) \in Q_{\theta_t z_j}$. Note that the three components x, z and u are treated in different ways: Jumps are allowed in x, approximate reachability is required for z, and no condition on the controls is imposed. It is easy to show that chain control sets are closed.

The following result clarifies the relations between chain control sets and their fibers.

Proposition 2: Consider system (1) in a closed subset $Q \subset M \times \mathcal{Z}$.

1) Suppose that Q is compact, and let $E^z \subset Q_z$, $z \in \mathcal{Z}$, be a maximal family of sets satisfying the following conditions:
   a) For every $z \in \mathcal{Z}$ and all $x, y \in E_z$ and all $\varepsilon, T > 0$ there exists a controlled (ε, T)-chain in Q from x along z to (y,z).
   b) For every $z \in \mathcal{Z}$ and every $x \in E_z$ there exists a control $u \in \mathcal{U}$ such that $\varphi(t,x,z,u) \in E_{\theta_t z}$ for all $t \in \mathbb{R}$.
   c) If $x_n \in E_{z_n}$ with $(x_n, z_n) \to (x,z) \in M \times \mathcal{Z}$, then $x \in E_z$.

   If $E := \bigcup_{z \in \mathcal{Z}} E^z \subset \operatorname{int} Q$, then E is a chain control set.
2) Let E be a chain control set. Then the fibers E_z, $z \in \mathcal{Z}$, are contained in a maximal family $\tilde{E}^z \subset Q_z$, $z \in \mathcal{Z}$, of sets satisfying conditions a)–c) above. If $\tilde{E} := \bigcup_{z \in \mathcal{Z}} \tilde{E}^z \subset \operatorname{int} Q$, then $E = \tilde{E}$.

Proof: See [5, Proposition 3.5]. ■

Remark 3: In condition b) of Proposition 2 one does not have that a trajectory exists which after an appropriate time comes back to E_z (as for periodic excitations, where one comes back into the same fiber after the period). In the general almost periodic case the trajectory will never come back to the same fiber. Instead, the weaker property formulated in b) holds together with condition c), which locally connects different fibers and is an upper semi-continuity property of $z \mapsto E_z$.

It is of great interest to see if the behavior in a single fiber determines chain control sets. In fact, one can reconstruct chain control sets from their intersection with a fiber.

Theorem 4: Consider system (1), and assume that $Q \subset M \times \mathcal{Z}$ is compact. For some $z_0 \in \mathcal{Z}$ let $E^{z_0} \subset Q \times \{z_0\}$ be a nonvoid maximal set such that for all $x_0, y_0 \in E^{z_0}$ and all $\varepsilon, T > 0$ there exists a controlled (ε, T)-chain in Q from x_0 along z_0 to (y_0, z_0).

Then the set

$$E := \mathrm{cl}\left\{ \begin{array}{l} (x,z) \in M \times Z, \text{ for all } \varepsilon, T > 0 \text{ there are} \\ x_0, y_0 \in E^{z_0} \text{ and controlled } (\varepsilon,T)\text{-chains} \\ \text{in } Q \text{ from } x_0 \text{ along } z_0 \text{ to } (y_0, z_0) \text{ such that} \\ (x,z) = \psi(t, x_j, \theta_{T_0+\cdots+T_{j-1}}z_0, u_j) \text{ for} \\ \text{some } j \text{ and } t \in [0, T_j] \end{array} \right\}$$

is a chain control set relative to Q.

Proof: See [5, Theorem 3.7]. ■

Remark 5: Theorem 4 shows that, up to closure, one can find chain control sets by looking at a single fiber, i.e., a single almost periodic excitation. This significantly simplifies numerical computations, since only one almost periodic excitation $z(t)$, $t \geq 0$, has to be considered. Then the resulting sets must be considered for those times T where z and $\theta_T z$ are close. In the quasi-periodic case, one has to look for (large) times t where all $\omega_i t$ are close to zero modulo 2π.

IV. Controllability and Chain Controllability

The main aim in this section is to analyze when an almost periodic solution of the uncontrolled system is contained in the interior of a subset of complete controllability. For this purpose, we ask when a reachable point is contained in the interior of the reachable set and discuss chain controllability. This leads us to control sets and their relation to chain control sets.

Again, consider control system (1). For a closed subset $Q \subset M \times \mathcal{Z}$, a point $x \in Q$, and $z \in \mathcal{Z}$ we define the positive and negative orbits along z relative to Q as $\mathcal{O}^+(x;z,Q) := \{\varphi(t,x,z,u), \text{ with } \psi(s,x,z,u) \in Q, s \in [0,t] \text{ for some } t \geq 0, u \in \mathcal{U}\}$, $\mathcal{O}^-(x;z,Q) := \{\varphi(t,x,z,u), \text{ with } \psi(s,x,z,u) \in Q, s \in [t,0] \text{ for some } t \leq 0, u \in \mathcal{U}\}$.

Observe that $\varphi(t,x,z,u) \in Q_{\theta_t z}$. Analogously the orbits $\mathcal{O}_t^+(x;z,Q), \mathcal{O}_t^-(x;z,Q)$, etc. are defined, if we restrict the times accordingly. If $Q = M \times \mathcal{Z}$, we omit the argument Q.

In addition to chain control sets it is also of interest to discuss control sets, i.e., maximal subsets of approximate controllability.

Definition 6: For a closed subset $Q \subset M \times \mathcal{Z}$ a subset $D \subset Q$ is a control set relative to Q if it is maximal with the following properties:

1) For all $(x,z),(y,w) \in D$ there are $T_n \geq 0, u_n \in \mathcal{U}$ with $\psi(T_n, x, z, u_n) \to (y,w)$ and $\psi(t,x,z,u_n) \in Q$ for $t \in [0, T_n]$.
2) For every $z \in \mathcal{Z}$ and every $x \in D_z$ there exists a control $u \in \mathcal{U}$ such that $\psi(t,x,z,u) \in D$ for all $t \geq 0$.

In condition 1), it is clear that $T_n \to \infty$, unless the excitation is periodic. Condition 2) immediately implies that the projection of the control set is dense in $\mathcal{Z}$; the inclusion may be rewritten as $\varphi(t,x,z,u) \in D_{z(t+\cdot)}$ for all $t \geq 0$.

For periodic excitations, one can characterize control sets by looking at the discrete time system defined by the Poincaré map (Gayer [9]). We will show that also, in the almost periodic case, it is possible to characterize control sets fiberwise.

The following result clarifies the relations between control sets and their fibers.

Theorem 7: Consider system (1) in a closed subset $Q \subset M \times \mathcal{Z}$.

1) Let $D^z \subset Q_z$, $z \in \mathcal{Z}$, be a maximal family of sets satisfying the following conditions:
   a) For every $z \in \mathcal{Z}$ and all $x, y \in D_z$ there are $T_n \to \infty$ and $u_n \in \mathcal{U}$ with $\psi(T_n, x, z, u_n) \to (y, z)$ and $\psi(t, x, z, u_n) \in Q$ for all $t \in [0, T_n]$.
   b) For every $z \in \mathcal{Z}$ and every $x \in D_z$ there exists a control $u \in \mathcal{U}$ such that $\varphi(t, x, z, u) \in D_{\theta_t z}$ for all $t \geq 0$.
   c) For every $(x, z) \in D^z$ and all $T_n > 0$ with $\theta_{T_n} z \to w \in \mathcal{Z}$ there are $y \in M$ and $u_n \in \mathcal{U}$ such that $\psi(T_n, x, z, u_n) \to (y, w) \in D^w$ and $\psi(t, x, z, u_n) \in Q$ for all $t \in [0, T_n]$.

   Then $D := \bigcup_{z \in \mathcal{Z}} D^z$ is a control set.
2) Let D be a control set. Then the fibers D_z form a maximal family of sets satisfying conditions a) and b) above.

Proof: See [5, Theorem 4.4]. ■

Our next aim is to prove that under an inner-pair condition every almost periodic solution of the uncontrolled equation is contained in the interior of a control set. For a periodic excitation, the state space $Z = \mathbb{S}^1$ is (trivially) completely controllable. However, already for a quasi-periodic excitation with two noncommensurable (i. e., rationally independent) frequencies ω_1, ω_2, this is no longer true. Hence it does not make sense to consider exact controllability properties in the z-component. This is different in the x-component as shown by the following proposition.

Proposition 8: Let $\psi(t, x^0, z^0, 0) \in Q$, $t \in \mathbb{R}$, be an almost periodic solution of the uncontrolled system, and define $A := \text{cl}\{\psi(t, x^0, z^0, 0), t \in \mathbb{R}\}$. Assume that there are $\varepsilon, T > 0$ such that for every $(x, z) \in A$ it holds that $\mathbf{B}_\varepsilon\big(\varphi(T, x, z, 0)\big) \subset \mathcal{O}_T^+(x; z, Q)$. Then for all $(x, z), (y, w) \in A$ there is $\tau > 0$ such that $\mathbf{B}_{\varepsilon/2}(y) \subset \mathcal{O}_\tau^+(x; z, Q)$, and for every $y_0 \in \mathbf{B}_{\varepsilon/2}(y)$ there are $\tau_n \geq 0$ and $u_n \in \mathcal{U}$ with $\varphi(\tau_n, x, z, u_n) = y_0$ in Q and $\theta_{\tau_n} z \to w$.

Proof: See [5, Proposition 4.6]. ■

This proposition allows us to show that almost periodic solutions of the uncontrolled system are contained in the interior of control sets. In other words, around an almost periodic solution we have complete controllability along the almost periodic excitations.

Theorem 9: Let $\psi(t, x^0, z^0, 0) \in Q$, $t \in \mathbb{R}$, be an almost periodic solution of the uncontrolled system, and let $A := \text{cl}\{\psi(t, x^0, z^0, 0), t \in \mathbb{R}\}$. Assume that there are $\varepsilon, T > 0$ such that for every $(x, z) \in A$

$$\begin{aligned} &\mathbf{B}_\varepsilon(\varphi(T, x, z, 0)) \subset \mathcal{O}_T^+(x; z, Q) \text{ and} \\ &\mathbf{B}_\varepsilon(\varphi(-T, x, z, 0)) \subset \mathcal{O}_T^-(x; z, Q). \end{aligned} \tag{3}$$

Then there exists a control set D such that for every $(x, z) \in A$ one has $x \in \text{int}\, D^z$.

Proof: See [5, Theorem 4.7] ■

Remark 10: Condition (3) is analogous to the inner-pair condition (but slightly stronger) for autonomous control systems; see [3, Definition 4.1.5].

Next, we generalize Theorem 9 in order to show a relation between chain controllability and controllability.

Theorem 11: Let $0 \leq \rho_1 \leq \rho_2$, and consider a compact subset $Q \subset M \times \mathcal{Z}$. Let E^{ρ_1} be a chain control set relative to Q for system (1) with controls in $\mathcal{U}^{\rho_1}$. Assume that there are $\varepsilon, T > 0$ such that for every $(x, z) \in E^{\rho_1}$ and $u \in \mathcal{U}^{\rho_1}$

$$\begin{aligned} &\mathbf{B}_\varepsilon\big(\varphi(T, x, z, u)\big) \subset \mathcal{O}_T^{+,\rho_2}(x; z, Q) \text{ and} \\ &\mathbf{B}_\varepsilon\big(\varphi(-T, x, z, u)\big) \subset \mathcal{O}_T^{-,\rho_2}(x; z, Q). \end{aligned} \tag{4}$$

Then there exists a control set D^{ρ_2} such that for every $(x, z) \in E^{\rho_1}$ one has $x \in \text{int}\, D_z^{\rho_2}$.

Proof: See [5, Theorem 4.11] ■

Remark 12: Using this theorem we can, as in [3, Theorem 4.7.5], show that for all up to at most countably many ρ-values the closures of control sets and the chain control sets coincide. Hence, by Theorem 4 one may also determine the fibers of control sets via the fibers of the chain control sets. For this purpose, one has to consider 'long' times, since these fibers are determined only on long time intervals; cf. Remark 5. At first sight, this is different if the excitation is periodic; here only the Poincaré map and hence the period length are needed; see [5, Proposition 3.6]. Nevertheless, also in this case approximate controllability is relevant (the entrance boundary of a control set is reached from the interior only for time tending to infinity), and hence also these objects are determined only on long time intervals.

V. Almost Periodic Solutions and Heteroclinic Orbits

In this section we recall results on almost periodic perturbations of hyperbolic equilibria and Melnikov's method.

It is well-known that, under small periodic perturbations, a hyperbolic fixed point of an autonomous differential equation becomes a periodic solution; see e. g., [1, Theorem 25.2] for details on this result, which is known as the *Poincaré continuation*. This result can be generalized to almost periodic perturbations, in which case the existence of an almost periodic solution can be shown. Consider the differential equation

$$\dot{x} = g(x) + \mu h(t, x, \mu) \tag{5}$$

for $g : \mathbb{R}^d \to \mathbb{R}^d$ and $h : \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$. The parameter $\mu \in \mathbb{R}$ is interpreted as a small perturbation. Setting $\mu = 0$ in system (5) leads to the equation $\dot{x} = g(x)$, which will be referred to as the *unperturbed* system. Throughout we assume that (5) satisfies the following conditions:

The function g is C^1, h is continuous, h_x exists, and there are a bounded and open subset $V \subset \mathbb{R}^d$ containing x_0 and a constant $\bar{\mu} > 0$ such that h and h_x are almost periodic in t, uniformly with respect to $(x, \mu) \in \text{cl}\, V \times [-\bar{\mu}, \bar{\mu}]$, and solutions of (5) exist for all starting points in V, all $\mu \in [-\bar{\mu}, \bar{\mu}]$, and all times.

As noted in Scheurle [13, Remark 2.7], almost periodicity of h_x uniformly with respect to (x, μ) is equivalent with h_x being uniformly continuous on $\mathbb{R} \times \text{cl}\, V \times [-\bar{\mu}, \bar{\mu}]$.

Next recall the notion of exponential dichotomies, which generalize the idea of hyperbolicity to nonautonomous systems; cf. Coppel [6].

Definition 13: Consider the system

$$\dot{x} = A(t)x \tag{6}$$

for a piecewise continuous matrix function $A : J \to \mathbb{R}^{d\times d}$ defined on an interval $J \subset \mathbb{R}$, and let $X(t)$ be a fundamental matrix function for (6). System (6) has an *exponential dichotomy on* J if there is a projection $P : \mathbb{R}^d \to \mathbb{R}^d$ and constants $K \geq 1$, $\alpha > 0$ such that $\|X(t)PX^{-1}(s)\| \leq Ke^{-\alpha(t-s)}$ for $s \leq t$ and $\|X(t)(I-P)X^{-1}(s)\| \leq Ke^{-\alpha(s-t)}$ for $s \geq t$.

Then the following result holds (this is essentially [13, Lemma 2.8]).

Proposition 14: Suppose that the unperturbed system corresponding to (5) has a hyperbolic fixed point x_0; i. e., $g(x_0) = 0$ and the real parts of the eigenvalues of $g_x(x_0)$ are different from 0. For all (small) $\eta > 0$ there is $\mu_0 = \mu_0(\eta) > 0$ such that for $|\mu| \leq \mu_0$ there exists a unique solution $\zeta^\mu(t)$ of system (5) satisfying $\|\zeta^\mu(t) - x_0\| \leq \eta$ for all $t \in \mathbb{R}$. This solution is almost periodic.

Proof: See [5, Proposition 5.4]. ■

If we suppose that in our setting there exist two hyperbolic fixed points $x_\pm \in \mathbb{R}^d$ of the unperturbed system, Proposition 14 implies the existence of almost periodic solutions $\zeta^\mu_\pm$ near $x_\pm$ for sufficiently small μ. If there is a heteroclinic orbit ζ from x_- to x_+, the question arises how the system behaves near ζ for small perturbations μ.

For time-periodic perturbations Melnikov's method gives a handy criterion for the existence of transversal heteroclinic points. Palmer has developed a generalization of Melnikov's method in [12] which, in our setting, yields the following theorem.

Theorem 15: Consider the system $\dot{x} = g(x) + \mu h(t, x, \mu)$, and let the following assumptions be satisfied:

1) There are a bounded and open subset $V \subset \mathbb{R}^d$ and a constant $\bar{\mu} > 0$ such that $g : V \to \mathbb{R}^d$ is C^2 and $h : \mathbb{R} \times V \times [-\bar{\mu}, \bar{\mu}] \to \mathbb{R}^d$ is continuous. The partial derivatives h_t, h_x, h_μ, h_{xx}, $h_{x\mu}$, $h_{\mu x}$ and $h_{\mu\mu}$ exist and are bounded, continuous in t for each fixed x, μ, and continuous in x, μ uniformly with respect to t, x and μ.
2) The functions h and h_x are almost periodic in t, uniformly with respect to $(x, \mu) \in \operatorname{cl} V \times [-\bar{\mu}, \bar{\mu}]$.
3) The unperturbed equation $\dot{x} = g(x)$ has hyperbolic fixed points $x_\pm \in V$ with stable and unstable manifolds of the same dimensions.
4) There is a heteroclinic orbit ζ from x_- to x_+ contained in V.
5) The function $\Delta(t_0) := \int_{-\infty}^{\infty} \varphi(t) \cdot h\big(t + t_0, \zeta(t), 0\big)\, \mathrm{d}t$ has a simple zero at some $t_0 \in \mathbb{R}$, where $\varphi(t)$ is the unique (up to a scalar multiple) bounded solution of the adjoint system $\dot{x} = g_x\big(\zeta(t)\big)^T x$ and "$\cdot$" denotes the inner product in $\mathbb{R}^d$.

Then there exists $\delta_0 > 0$ such that for sufficiently small μ the perturbed system (5) has a unique solution $x(t, \mu)$ satisfying $\|x(t, \mu) - \zeta(t - t_0)\| \leq \delta_0$ for all $t \in \mathbb{R}$. Furthermore $\sup_{t\in\mathbb{R}} \|x(t, \mu) - \zeta(t - t_0)\| = O(\mu)$ for $\mu \to 0$ holds, and $\dot{x} = \big[g_x\big(x(t, \mu)\big) + \mu h_x\big(t, x(t, \mu), \mu\big)\big]x$ has an exponential dichotomy on $\mathbb{R}$.

Finally, it holds that $\lim_{t\to\pm\infty} \|x(t, \mu) - \zeta^\mu_\pm(t)\| = 0$ for sufficiently small μ, where $\zeta^\mu_\pm$ are the almost periodic solutions near $x_\pm$.

Proof: [5, Theorem 5.5] ■

Remark 16: This theorem is also applicable to *homoclinic* orbits by letting $x_- = x_+$.

Remark 17: If in the two-dimensional case g is Hamiltonian, $\Delta(t_0)$ coincides with the Melnikov function up to a scalar multiple, Marsden [11].

VI. Heteroclinic Orbits and Controllability

In this section, we show that existence of a heteroclinic solution of the unperturbed uncontrolled equation implies a controllability condition for perturbed systems with small control influence. Conversely, if the controllability condition holds for small control influence, existence of a heteroclinic solution of the unperturbed equation follows. These results are used to relate heteroclinic cycles to the existence of control sets.

Consider the following family of control systems depending on a parameter μ:

$$\dot{x} = g(x) + \mu h(x, z(t), \mu, u(t)), u \in \mathcal{U}, \tag{7}$$

with continuous functions g and h and control range $U \subset \mathbb{R}^m$ containing the origin; the functions z are in the hull $\mathcal{Z}$ of a single almost periodic function. We refer to $\dot{x} = g(x)$ and $\dot{x} = g(x) + \mu h(t, x, \mu, 0)$ as the unperturbed uncontrolled system and the perturbed uncontrolled system, respectively. For fixed μ this is a special case of the control system (1); we use the notation introduced in Sections II, III and IV with a superfix μ to indicate dependence on this parameter. In particular, solutions (whose existence we always assume) are denoted by $\varphi^\mu(t, x_0, z, u)$, $t \in \mathbb{R}$, $x_0 \in \mathbb{R}^d$, $z \in \mathcal{Z}$ and $u \in \mathcal{U}$.

Proposition 18: Assume that system (7) with control $u = 0$ satisfies the assumptions 1)–5) of Theorem 15. Let $\zeta^\mu_\pm$ be the almost periodic solutions near the hyperbolic equilibria $x_\pm$ of the unperturbed uncontrolled system and let $x(t, \mu) := \varphi^\mu(t, x^\mu, z_0, 0)$ be the solution near the heteroclinic orbit ζ from x_- to x_+ for some $x^\mu \in \mathbb{R}^d, z_0 \in \mathcal{Z}$. Let μ be a parameter value such that the conclusions of Theorem 15 hold, and assume that there are $\varepsilon = \varepsilon(\mu), T = T(\mu) > 0$ such that for every $(x, z) \in Q := \operatorname{cl} V \times \mathcal{Z}$ it holds that $\mathbf{B}_\varepsilon(\varphi^\mu(T, x, z, 0)) \subset \mathcal{O}^{\mu,+}_T(x; z, Q)$ and $\mathbf{B}_\varepsilon(\varphi^\mu(-T, x, z, 0)) \subset \mathcal{O}^{\mu,-}_T(x; z, Q)$. Then there are a control function $u^\mu \in \mathcal{U}$ and times $t^\mu_- < t^\mu_+$ such that the corresponding solution $\varphi^\mu(t, x^\mu, z_0, u^\mu)$ of (7) satisfies

$$\varphi^\mu(t, x^\mu, z_0, u^\mu) = \begin{cases} \zeta^\mu_-(t) & \text{if } t \leq t^\mu_-, \\ \zeta^\mu_+(t) & \text{if } t \geq t^\mu_+. \end{cases}$$

Proof: [5, Proposition 6.1] ■

The previous proposition shows that existence of a heteroclinic orbit for the unperturbed uncontrolled equation implies the existence of a control steering the system with almost periodic excitation from the almost periodic solution near one equilibrium to the almost periodic solution near the other equilibrium. The following result considers a converse situation where the unperturbed equation has equilibria x_+ and x_- and we want to conclude from existence of controlled trajectories of the perturbed system from points near x_- to x_+ that a heteroclinic orbit of the unperturbed equation exists.

Proposition 19: Suppose that g and $h(x, z(t), \mu, 0)$ satisfy assumptions 1) and 2) of Theorem 15 for all $z \in \mathcal{Z}$; i.e., these assumptions hold for system (7) with $u = 0$. Moreover, assume that the chain recurrent set of the unperturbed uncontrolled system $\dot{x} = g(x)$ relative to $\mathrm{cl}\, V$ is equal to $\{x_+, x_-\}$.

Suppose furthermore that the control range U is bounded and there are $\mu_n \to 0$, almost periodic excitations $z_n \in \mathcal{Z}$, control functions $u_n \in \mathcal{U}$, times $t_-^n < t_+^n$, and points $x_n \in \mathrm{cl}\, V$ such that the solution $\varphi_n(t) := \varphi^{\mu_n}(t, x_n, z_n, u_n), t \in \mathbb{R}$, of (7) is contained in $\mathrm{cl}\, V$ and satisfies $\varphi_n(t_-^n) \to x_-$ and there is $\delta > 0$ with $\|\varphi_n(t) - x_-\| \geq \delta$ for all $t \geq t_+^n$ and all n.

Then the unperturbed uncontrolled system has a heteroclinic orbit from x_- to x_+.

Proof: [5, Proposition 6.2] ■

Next we discuss consequences of these results for control sets of systems with almost periodic excitations. Roughly, the results above imply that the existence of a heteroclinic cycle of the unperturbed uncontrolled system is equivalent to the existence of a control set containing all almost periodic solutions near the equilibria for the systems with almost periodic excitation and small control ranges.

Recall that a heteroclinic cycle of the unperturbed equation is given by a finite set $x_0, x_1, \ldots, x_n = x_0$ of equilibria together with heteroclinic solutions ζ_i from x_i to x_{i+1} for $i = 0, \ldots, n-1$. Existence of heteroclinic cycles can be expected in the presence of symmetries.

Theorem 20: Let $x_0, x_1, \ldots, x_n = x_0$ be pairwise different hyperbolic equilibria of the unperturbed uncontrolled system $\dot{x} = g(x)$, and consider control system (7) with a bounded control range U containing the origin. For $|\mu| \neq 0$, small, and $z \in \mathcal{Z}$ denote the almost periodic solutions near x_i for excitation z by $\zeta_i^\mu(z)$. Assume that system (7) with $u = 0$ satisfies assumptions 1) and 2) of Theorem 15 for all $z \in \mathcal{Z}$ on an open set V containing all equilibria x_i.

1) Assume that for all i there are open subsets $V_i \subset \mathbb{R}^d$ containing the equilibria $x_- = x_i$ and $x_+ = x_{i+1}$ such that assumptions 3)–5) of Theorem 15 are satisfied for (7) with $u = 0$, and let $x_i(t, \mu, z) = \varphi^\mu(t, x_i^\mu, z, 0)$ be the solution near the heteroclinic orbit $\zeta_i(z)$ from x_i to x_{i+1} . Assume that for all sufficiently small $|\mu| \neq 0$ there are $\varepsilon_i, T_i > 0$ such that for every $(x, z) \in Q_i := \mathrm{cl}\, V_i \times \mathcal{Z}$ it holds that $\mathbf{B}_{\varepsilon_i}(\varphi^\mu(T_i, x, z, 0)) \subset \mathcal{O}_{T_i}^{\mu,+}(x; z, Q_i)$ and $\mathbf{B}_{\varepsilon_i}(\varphi^\mu(-T_i, x, z, 0)) \subset \mathcal{O}_{T_i}^{\mu,-}(x; z, Q_i)$. Then for all $|\mu| \neq 0$, small, there exists a control set D^μ such that for all $z \in \mathcal{Z}$ and all i the almost periodic solutions satisfy $\zeta_i^\mu(t) \in D_{z(t+\cdot)}^\mu$ and the heteroclinic solutions satisfy $x_i(t, \mu, z) \in D^{\mu, z(t+\cdot)}$.
2) Conversely, suppose for all i there are open subsets V_i containing x_i and x_{i+1} such that the chain recurrent set of the unperturbed uncontrolled system $\dot{x} = g(x)$ relative to $\mathrm{cl}\, V_i$ is equal to $\{x_i, x_{i+1}\}$. Furthermore, suppose that for a sequence $0 \neq \mu_n \to 0$ there are control sets D^{μ_n} containing the almost periodic solutions $\zeta_i^{\mu_n}$ near x_i for almost periodic excitations $z_n \in \mathcal{Z}$. Then the unperturbed system has a heteroclinic cycle through the x_i.

Proof: See [5, Theorem 6.3] ■

VII. An Oscillator with M-Potential

In this section we will apply our results to a second order system with M-potential, which models ship roll motion.

Consider the system

$$\ddot{x} + \mu\beta_1\dot{x} + \mu\beta_3\dot{x}^3 + x - \alpha x^3 = \mu z(t) + \mu u(t) \quad (8)$$

with positive parameters α, β_1 and β_3, a small perturbation parameter $\mu \in \mathbb{R}$, almost periodic excitations $z : \mathbb{R} \to \mathbb{R}$, and control functions $u : \mathbb{R} \to [-\rho, \rho]$ for a control radius $\rho > 0$. This model, proposed in Kreuzer and Sichermann [10], has been studied in Colonius et al. [4] without time-dependent excitation z. Note that in this application the terms $u(\cdot)$ are interpreted as time-dependent perturbations (not as controls) where only the range $[-\rho, \rho]$ is known. Here the control sets give information on the global stability behavior: An invariant control set around the origin indicates stability. If (for large perturbation amplitudes) it has merged with a variant control set and itself becomes variant, stability is lost. Hence it is of interest to compute all control sets.

By [5, Proposition 4.9], assumption (4) in Theorem 11 is satisfied for all $\rho_2 > \rho_1 \geq 0$. Thus every compact chain control set E^{ρ_1} is contained in the interior of a control set D^{ρ_2}, and hence, for all up to countably many $\rho > 0$, Remark 12 shows that the compact chain control sets coincide with the closures of control sets.

Writing (8) as a first order system yields the two-dimensional perturbed Hamiltonian system

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 + \alpha x_1^3 + \mu\big(-\beta_1 x_2 - \beta_3 x_2^3 + z(t) + u(t)\big). \end{aligned} \quad (9)$$

Denote by $\varphi^\mu(t, x, z, u)$ the solution of this system, and let $\psi^\mu(t, x, z, u) := \big(\varphi^\mu(t, x, z, u), \theta_t z\big)$. In the unperturbed and uncontrolled case $\mu = 0$ system (9) has a fixed point in the origin and two hyperbolic fixed points at $(\pm 1/\sqrt{\alpha}, 0)$. The hyperbolic fixed points are connected by two heteroclinic orbits given by $x_\pm^h(t) := \pm\big(x_1(t), x_2(t)\big)$, where $x_1(t) := {}^1\!/\!\sqrt{\alpha}\tanh({}^t\!/\!\sqrt{2})$, $x_2(t) := {}^1\!/\!\sqrt{2\alpha}\,\mathrm{sech}^2({}^t\!/\!\sqrt{2})$, $t \in \mathbb{R}$; cf. Simiu [14, p. 131]. In the perturbed uncontrolled case $u \equiv 0$ denote by $\Delta_\pm$ the Melnikov functions of system (9) with respect to $x_\pm^h$ and denote by $\zeta_\pm^\mu$ the almost periodic solutions near $(\pm 1/\sqrt{\alpha}, 0)$, which exist for sufficiently small μ (see Proposition 14). Let $z_0 \in \mathcal{Z}$ be the corresponding excitation and $\xi_\pm^\mu(t) := \big(\zeta_\pm^\mu(t), \theta_t z_0\big)$.

Proposition 21: Assume that the almost periodic excitation z is continuously differentiable with bounded derivative. If the functions $\Delta_\pm$ have simple zeros and μ is small enough, then system (9) has a control set D containing $\xi^\mu_\pm(\mathbb{R})$. Then D will be called a heteroclinic control set.

Proof: See [5, Proposition 7.1] ■

First we study the periodic case and choose $z(t) := F\cos\omega t$ for positive parameters F and ω. The excitation z is C^1 and its derivative is bounded, so Proposition 21 is applicable. The Melnikov functions $\Delta_\pm$ can easily be computed using the residue theorem:

$$\Delta_\pm(t_0) = -\frac{2\sqrt{2}\beta_1}{3\alpha} - \frac{8\sqrt{2}\beta_3}{35\alpha^2} \pm \frac{\sqrt{2}\pi\omega F}{\sqrt{\alpha}\sinh\frac{\pi\omega}{\sqrt{2}}} \cdot \cos\omega t_0.$$

The Melnikov functions $\Delta_\pm$ have simple zeros if and only if F exceeds a certain *critical amplitude* F_c, i. e., if $F > F_c := A^{-1}B$ for $A := \sqrt{2}\pi\omega\big(\sqrt{\alpha}\sinh(\pi\omega/\sqrt{2})\big)^{-1}$ and $B := 2\sqrt{2}\beta_1/3\alpha + 8\sqrt{2}\beta_3/35\alpha^2$.

Corollary 22: If $F > F_c$, system (9) with $z(t) := F\cos\omega t$ has a heteroclinic control set for sufficiently small μ.

Proof: This follows from Proposition 21. ■

As the excitation is T-periodic for $T := 2\pi/\omega$, it is possible to compute fibers of control sets by looking at the discrete control system given by the time-T map. For the following computations we restrict our view to the parameter values $\alpha = 0.674$, $\beta_1 = 0.231$ and $\beta_3 = 0.375$ (see [10] for a discussion of these parameters and this choice) and choose $\omega = 2.5$ and $\rho = 1.0$. Then $F_c \approx 5.62880$, so let $F := 6 > F_c$. Fig. 1 shows the fiber in phase 0 for $\mu = 0.1$.

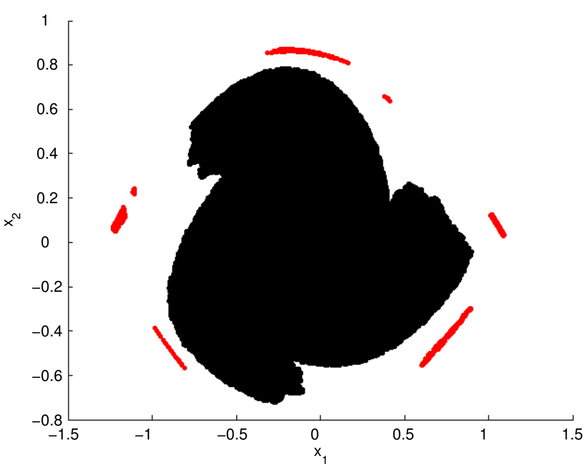

Fig. 1. Fiber of control sets for the periodically excited system (9) with $z(t) := F\cos\omega t$. Computed in phase 0 for $\alpha = 0.674$, $\beta_1 = 0.231$, $\beta_3 = 0.375$, $\omega = 2.5$, $\rho = 1.0$, $F = 6$ and $\mu = 0.1$.

The control sets were approximated with the *graph algorithm* (see Dellnitz and Junge [8] and Szolnoki [15]) using the implementation in GAIO [7]. For a spatial discretization into boxes, this algorithm computes strongly connected components of an associated graph whose nodes are given by the boxes and whose edges indicate reachability. The union of the resulting boxes is an approximation to a chain control set; as noted above, for system (8) the chain control sets typically coincide with the closures of control sets. Note that this figure shows the fiber of two control sets: an invariant control set around the origin (black) and the heteroclinic control set (red).

Remark 23: The main interest in this result comes from the relations between the deterministic system and a related stochastic system, where $u(t)$ is replaced by a stochastic perturbation. Then the invariant control sets correspond to the supports of invariant measures (see, e. g., Colonius, Gayer and Kliemann [2]). For small perturbation amplitudes, system (8) has an invariant control set around the origin and hence small random perturbations will not lead to capsizing (i. e., there are no unbounded solutions $x(t)$ starting near the origin). For large perturbation amplitudes, there is no invariant control set and capsizing will occur with probability 1. Hence it is of interest to analyze how invariance is lost. The results above indicate that this happens when the invariant control set around the origin unites with the heteroclinic control set. This shows that the picture is more complicated than indicated in [9] (where, as a simplified model, the escape equation with a single hyperbolic equilibrium was discussed).

References

[1] H. Amann, *Ordinary Differential Equations. An Introduction to Nonlinear Analysis*, ser. Stud. Math. Berlin: de Gruyter, 1990, vol. 13.

[2] F. Colonius, T. Gayer, and W. Kliemann, "Near invariance for Markov diffusion systems," *SIAM J. Appl. Dyn. Syst.*, vol. 7, pp. 79–107, 2008.

[3] F. Colonius and W. Kliemann, *The Dynamics of Control*. Boston: Birkhäuser, 2000.

[4] F. Colonius, E. Kreuzer, A. Marquardt, and W. Sichermann, "A numerical study of capsizing: Comparing control set analysis and Melnikov's method," *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, vol. 18, no. 5, pp. 1503–1514, 2008.

[5] F. Colonius and T. Wichtrey, "Control systems with almost periodic excitations," *SIAM J. Control Optim.*, vol. 48, no. 2, pp. 1055–1079, 2009.

[6] W. Coppel, *Dichotomies in Stability Theory*, ser. Lecture Notes in Math. Berlin: Springer-Verlag, 1978, vol. 629.

[7] M. Dellnitz and O. Junge. GAIO – global analysis of invariant objects. [Online]. Available: http://www-math.uni-paderborn.de/~agdellnitz/gaio/

[8] ——, "Set oriented numerical methods for dynamical systems," in *Handbook of Dynamical Systems*, B. Fiedler, Ed. Amsterdam: Elsevier, 2002, vol. 2, ch. 5, pp. 221–264.

[9] T. Gayer, "Controllability and invariance properties of time-periodic systems," *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, vol. 15, no. 4, pp. 1361–1375, 2005.

[10] E. J. Kreuzer and W. M. Sichermann, "Investigation of large amplitude roll motions and capsizing," in *Proceedings of the Ninth International Symposium on Practical Design of Ships and Other Floating Structures*, H. Keil and E. Lehmann, Eds. Lübeck-Travemünde: Seehafen Verlag, 2004, pp. 689–696.

[11] J. E. Marsden, "Chaos in dynamical systems by the Poincaré-Melnikov-Arnold method," in *Chaos in Nonlinear Dynamical Systems*, J. Chandra, Ed. Philadelphia: SIAM, 1984, pp. 19–31.

[12] K. J. Palmer, "Exponential dichotomies and transversal homoclinic points," *J. Differential Equations*, vol. 55, pp. 225–256, 1984.

[13] J. Scheurle, "Chaotic solutions of systems with almost periodic forcing," *Z. Angew. Math. Phys.*, vol. 37, no. 1, pp. 12–26, 1986.

[14] E. Simiu, *Chaotic Transitions in Deterministic and Stochastic Dynamical Systems*. Princeton, NJ: Princeton University Press, 2002.

[15] D. Szolnoki, "Set oriented methods for computing reachable sets and control sets," *Discrete Contin. Dyn. Syst. Ser. B*, vol. 3, pp. 361–382, 2003.