Nonlinear Iwasawa Decomposition of Control Flows

Fritz Colonius
Institut für Mathematematik, Universität Augsburg,
86135 Augsburg, Germany
fritz.colonius@math.uni-augsburg.de

Paulo R.C. Ruffino
Departamento de Matematica,
Universidade Estadual de Campinas,
13081-970 Campinas, SP, Brasil

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Abstract

Let $\varphi(t, \cdot, u)$ be the flow of a control system on a Riemannian manifold M of constant curvature. For a given initial orthonormal frame k in the tangent space $T_{x_0}M$ for some $x_0 \in M$, there exists a unique decomposition $\varphi_t = \Theta_t \circ \rho_t$ where Θ_t is a control flow in the group of isometries of M and the remainder component ρ_t fixes x_0 with derivative $D\rho_t(k) = k \cdot s_t$ where s_t are upper triangular matrices. Moreover, if M is flat, an affine component can be extracted from the remainder.

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1 Introduction

Dynamical systems in a finite dimensional differentiable manifold M (including deterministic, random, stochastic and control systems) are globally described by the corresponding trajectories in the group $\mathrm{Dif}(M)$ of global diffeomorphisms of the manifold M. In most interesting examples and applications, the manifold M has a Riemannian metric endowed with the corresponding geometric structure: orthonormal frame bundle $O(M)$ over M, Levi-Civita horizontal lift, covariant derivative of tensors, geodesics, among other structures whose constructions depend intrinsically on this metric.

Once a differentiable manifold is endowed with a Riemannian metric, one can distinguish the elements in the group of diffeomorphisms $\mathrm{Dif}(M)$ which preserve this metric, the group $I(M)$ of isometries of M. In general the group $\mathrm{Dif}(M)$ is an infinite dimensional Lie group, while the group of isometries $I(M)$ is finite dimensional. This group carries geometric and topological properties of M. Roughly speaking, what we describe in this paper is a factorization of a flow φ_t (a one-parameter family of diffeomorphisms) into a component Θ_t which lies in this finite dimensional subgroup of isometries $I(M)$ and another component (the *remainder*) ρ_t which fixes a given point on M and-via its derivative-contains the long time stability behavior (Lyapunov exponents) of the system. The title of the paper is motivated by the classical Iwasawa decomposition for linear maps which is the unique factorization, via Gram-Schmidt orthonormalization, of a matrix as a product of an orthogonal and an upper triangular matrix; hence one has a decomposition into an isometry and a matrix containing the expansion/contraction terms.

A similar decomposition has appeared in Liao [13] for stochastic flows, with hypotheses on the vector fields of the systems. A geometrical condition on the manifold M (constant curvature), instead of on the vector fields was established in Ruffino [16], with some examples also in [17]. This paper intends to apply the same technique to show that this decomposition also holds in the context of control flows.

We remark that a main interest in this kind of decomposition for (random) dynamical systems is the fact that characteristic asymptotic parameters of the systems (Lyapunov exponents and rotation numbers) appear separately in each of the components of the decomposition. For details on the definitions of these asymptotic parameters we refer to the articles by Liao [12], Ruffino [17], Arnold and Imkeller [2] and the references therein. For control flows these questions need further study.

The paper deals with control flows which include a shift on time-varying vector fields. In more detail, the main result of this paper, the non-linear Iwasawa decomposition for control flows, can be described as follows: Assume certain geometrical conditions on the vector fields or that the manifold M has constant curvature (cf. Theorem 5.1), and fix an initial condition $x_0 \in M$ and an initial orthonormal frame k of the tangent space $T_{x_0}M$. Then there exists a unique factorization

$$\varphi_t = \Theta_t \circ \rho_t, \tag{1}$$

where Θ_t corresponds to a control flow in the group of isometries, ρ_t fixes the starting point x_0 for all $t \geq 0$, i.e., $\rho_t(x_0, X) \equiv x_0$, and the derivative in the space parameter satisfies $D\rho(k) = k\, s_t$ where s_t are upper triangular matrices. Adding some other restrictions on the vector fields (or assuming that M is flat, cf. Corollary 5.2) one can go further in the decomposition and factorize the remainder ρ_t of equation (1) to get a (dynamically) weaker remainder (using the same notation ρ_t):

$$\varphi_t = \Theta_t \circ \Psi_t \circ \rho_t, \tag{2}$$

where Θ_t are isometries, Ψ_t are in the group of affi ne transformations of M (hence so does $\Theta_t \circ \Psi_t$), and the new remainders ρ_t are diffeomorphisms which again fix x_0 for all $t \geq 0$, but the derivative with respect to the space parameter x is given by the identity $D\rho_t \equiv Id_{T_{x_0}M}$. In decomposition (2) we have extracted the affi ne component from the previous remainder in (1). Hence, in this second factorization, the dynamics of ρ_t is reduced locally to the identity, up to first order.

Section 2 provides an overview of control flows and Section 3 recalls geometric preliminaries for non-expert readers (these sections can be skipped by those who are familiar with these topics). Section 4 derives the nonlinear Iwasawa decomposition and proves that the isometric part is, by itself, a control flow, with appropriate vector fields. Section 5 characterizes the manifolds for which the required assumptions are always satisfied. Finally, Section 6 adapts some examples in [16] and [17] to the context of control flows in all simply connected manifolds of constant curvature: Euclidean spaces $\mathbb{R}^n$, spheres, and a hyperbolic space.

2 Control Flows

In this section we describe some basic facts on control flows. We consider a control system in a complete connected d-dimensional Riemannian manifold M given by a family F of smooth vector fields $F \subset \mathcal{X}(M)$. We assume that the linear span of F is a finite dimensional subspace $E \subset \mathcal{X}(M)$, i.e., F is contained in a finite dimensional affi ne subspace of $\mathcal{X}(M)$. The time-dependent vector fields taking values in F, i.e. measurable curves in F, are

$$\mathcal{F} = \{X \in L_\infty(\mathbb{R}, E),\ X_t \in F \text{ for } t \in \mathbb{R}\}. \tag{3}$$

Throughout we will assume that all corresponding (nonautonomous) differential equations

$$\dot{x} = X_t(x) \text{ where } X \in \mathcal{F}, \tag{4}$$

have unique (absolutely continuous) global solutions $\varphi_t(x_0, X),\ t \in \mathbb{R}$, with $\varphi_0(x_0, X) = x_0$. Then system (4) defines a flow on $\mathcal{F} \times M$

$$\Phi_t(X, x_0) = (\theta_t X, \varphi_t(x, X)),\ t \in \mathbb{R}, \tag{5}$$

here θ_t is the shift on $\mathcal{F}$ given by $(\theta_t X)(s) = X_{t+s},\ s \in \mathbb{R}$. We call this the associated (non-parametric) control flow (cp. also [5]). It is closely related to control flows as considered in [4] with the shift on the space $\mathcal{U}$ of control functions, i.e. the space of measurable curves

$$\mathcal{U} = \{u \in L_\infty(\mathbb{R}, \mathbb{R}^m),\ u(t) \in U \text{ for } t \in \mathbb{R}\},$$

where U is the control range. Here the time dependent vector fields are parametrized by the control functions and it has to be assumed that the system is control-affi ne and the control range U is compact and convex. In fact, the time-dependent vector fields in $\mathcal{F}$ (and hence the control flow (5)) can be parametrized as follows.

Proposition 2.1 *(i) Let $F \subset \mathcal{X}(M)$ be a compact and convex subset of the m-dimensional subspace $E \subset \mathcal{X}(M)$ spanned by F. Then there exist a convex and compact subset $U \subset \mathbb{R}^m$, $0 \in U$ and $m + 1$ vector fields $X_0, \ldots, X_m \in \mathcal{X}(M)$ such that*

$$\mathcal{F} = \{X_0 + \sum_{i=1}^{m} u_i(\cdot) X_i,\ u \in L_\infty(\mathbb{R}, \mathbb{R}^m),\ u(t) \in U \textit{ for } t \in \mathbb{R}\}. \tag{6}$$

(ii) Conversely, consider a control-affi ne system onM of the form

$$\dot{x} = X_0(x) + \sum_{i=1}^{m} u_i(t) X_i(x),$$

where $m \in \mathbb{N},\ X_0, ..., X_m \in \mathcal{X}(M)$*,* $u \in \mathcal{U}$ *with control range* $U \subset \mathbb{R}^m$ *convex and compact. Then*

$$F = \{X_0 + \sum_{i=1}^{m} u_i X_i,\ u \in U\} \tag{7}$$

is a convex and compact subset of a finite dimensional space $E \subset \mathcal{X}(M)$ *of vector fields and*

$$\{X \in L_\infty(\mathbb{R}, E),\ X_t \in F \text{ for } t \in \mathbb{R}\} = \{X_0 + \sum_{i=1}^{m} u_i(\cdot) X_i,\ u \in \mathcal{U}\}.$$

Proof: Starting from (ii): Clearly, for a compact and convex set $U \subset \mathbb{R}^m$, the set F in (7) is a convex and compact subset of a finite dimensional vector space in $\mathcal{X}(M)$. The vector space E spanned by the vector fields $X_0, X_1, \ldots, X_m$ has dimension bounded by $m+1$. Conversely, let F be a convex and compact set generating an $m-$dimensional space $E \subset \mathcal{X}(M)$. Fixing $X_0 \in F$ and a base $X_1, ..., X_m$ of E one finds that every element $X \in F$ can uniquely be written as

$$X_0 + \sum_{i=1}^{m} u_i X_i$$

with coeffi cients$u_i \in \mathbb{R}$. We may assume that $X_1, \ldots, X_m \in F$, since E is generated by F. Clearly, the corresponding set U of coeffi cients forms a convex and compact subset of $\mathbb{R}^m$ (with $0 \in U$). It remains to show that for every $X \in \mathcal{F}$ one can find a measurable selection u with

$$X_t = X_0 + \sum_{i=1}^{m} u_i(t) X_i \text{ for almost all } t \in \mathbb{R}.$$

This follows from Filippov's Theorem, see e.g. Aubin and Frankowska [3], Theorem 8.2.10.

$\square$

Remark: If F is contained in an $(m-1)$-dimensional affi ne subspace of$E$, then, in the second part of the proof, one can restrict the linear combination to m vector fields, instead of $(m+1)$: just take e.g. $X_0 = X_1$ in the arguments above.

This proposition shows that the nonparametric control fbws are just a concise way of writing the control fbws corresponding to control-affi ne systems as considered, e.g., in [4]; here one uses the shift on the space $\mathcal{U}$ of admissible control functions instead of the shift on the space of time dependent vector fields. Nonparametric control fbws inherit all properties of control fbws; in fact they can also be considered as the special case

$$\dot{x} = u(t)(x), \; u \in \mathcal{F} = \{u \in L_\infty(\mathbb{R}, E), \; u(t) \in F \text{ for } t \in \mathbb{R}\}$$

(here the right hand side of the differential equation denotes the vector field $u(t)$ evaluated at x.) For a fixed control function $u(\cdot)$, these equations reduce to ordinary differential equations, hence one can apply all the techniques of existence and uniqueness of solution and differential dependence on parameters. The family Φ_t is a continuous skew-product fbw on $\mathcal{F} \times M$ when $\mathcal{F} \subset L_\infty(\mathbb{R}, E)$ is endowed with the weak* topology. Note that the $M-$ component of Φ satisfies the cocycle property

$$\varphi_{t+s}(x, X) = \varphi_t(\varphi_s(x, X), \theta_s X).$$

When the time-dependent control vector field X is implicit in the context, for sake of simplicity in the notation, we shall write simply φ_t instead of $\varphi_t(\cdot, X)$.

3 Geometric Preliminaries

In this section, we recall some geometric constructions; a general reference is Kobayashi and Nomizu [9].

We shall denote the linear frame bundle over a d-dimensional smooth manifold M by $GL(M)$. It is a principal bundle over M with structural group $Gl(d, \mathbb{R})$. A Riemannian structure on M is determined by a choice of a subbundle of orthonormal frames $O(M)$ with structural subgroup $O(d, \mathbb{R})$. We shall denote by $\pi : GL(M) \to M$ and by $\pi_o : O(M) \to M$ the projections of these frame bundles onto M. The canonical Iwasawa decomposition given by the Gram-Schmidt orthonormalization in the elements of a frame $k = (k^1, \ldots, k^d)$ defines a projection $\perp$: $k \mapsto k^\perp : GL(M) \to O(M)$ such that $GL(M)$ is again a principal bundle over $O(M)$ with structural group $S \subset Gl(d, \mathbb{R})$, the subgroup of upper triangular matrices with positive elements in the diagonal. The principal bundles described above factorize as $\pi = \pi_o \circ \perp$.

We recall that for a frame k in $GL(M)$ a connection Γ determines a direct sum decomposition of the tangent space at k into horizontal and vertical subspaces which will be denoted by $T_kGL(M) = HT_kGL(M) \oplus VT_kGL(M)$. An analogous decomposition holds in the tangent bundle $TO(M) \subset T\,GL(M)$. For $k \in O(M)$, we have that $HT_kO(M) = HT_kGL(M)$. Given a vector field X on M, we denote its horizontal lift to $GL(M)$ by $HX(k) \in T_kGL(M)$. Throughout this paper we restrict attention to the Levi-Civita connection.

The covariant derivative of a vector field X at x is a linear map denoted by $\nabla X(x) : T_xM \to T_xM$, we write $\nabla X(Y)$ or $\nabla_Y X$ for a vector $Y \in T_xM$. In terms of fibre bundles, the covariant derivative is defined as a derivative along horizontal lift of trajectories, hence it has a purely vertical component. Considering the right action of the structural group in the frame bundle $GL(M)$, via adjoint, we can associate to ∇X an element in the structural group $Gl(d, \mathbb{R})$ of the principal bundle $GL(M)$ given by the matrix

$$\tilde{X}(k) = \mathrm{ad}(k^{-1})\nabla X, \tag{8}$$

which acts on the right such that $\nabla X(k) = k\tilde{X}(k)$. Note that, different from ∇X, the right action of the matrix $\tilde{X}(k)$ does depend on k.

The natural lift of X to $GL(M)$ is the unique vector field δX in $GL(M)$ such that $L_{\delta X(k)}\theta = 0$, where θ is the canonical $\mathbb{R}^d$-valued 1-form on $GL(M)$ defined by $\theta(Hk(\zeta)) = \zeta$ for all $\zeta \in \mathbb{R}^d$. This natural lift is given by:

$$\delta X(k) = \frac{d}{dt}[D\,\eta_t(k)]|_{t=0}. \tag{9}$$

where $D\eta_t : T_{x_0}M \to T_{\eta_t(x_0)}M$ is the derivative of the local 1-parameter group of diffeomorphisms η_t associated to the vector field X. Note that it describes the infinitesimal behavior of the linearized flow of X in a basis k of the space $T_{x_0}M$. Naturally, δX is equivariant by the right action of $Gl(d, \mathbb{R})$ in the fibres.

The next lemma guarantees that the left action of the linearized flow is also well defined in the subbundle $O(M)$. In fact, this is a well expected result since the left action of $Gl(d, \mathbb{R})$ is well defined even in smaller quotient space, e.g. in the associated flag manifolds, see e.g. [18]. In any case, for the readers' convenience we shall present a proof of this simpler version which is all that we need here.

Lemma 3.1 *The projection* $\perp: GL(M) \to O(M)$ *is invariant for the linearized flow in the sense that, for all* $k \in GL(M)$,

$$(D\eta_t(k))^\perp = (D\eta_t(k^\perp))^\perp. \tag{10}$$

Proof: This is a consequence of the commutativity of the right action of $Gl(d, \mathbb{R})$ (in particular, in this case, the action of the subgroup S of upper triangular matrices) on $GL(M)$ with any other linear left actions (in particular, in this case, the linearized flow). In fact, consider the Iwasawa decomposition $k = k^\perp \cdot s_{(k)}$ for some $s_{(k)} \in S$. Hence,

$$D\eta_t(k^\perp \cdot s_{(k)}) = (D\eta_t k^\perp) \cdot s_{(k)} = (D\eta_t(k))^\perp \cdot s_{(D\eta_t(k))}.$$

Equality (10) follows by the uniqueness of the Iwasawa decomposition.

□

The vertical component $V\delta X(k)$ at $k \in \pi^{-1}(x_0)$ is given by the covariant derivative $\nabla X(k)$ (see e.g. Elworthy [6, Chap. II, §2], or Kobayashi and Nomizu [9, Chap. III, §1]). In terms of Lie algebra, consider the canonical Iwasawa decomposition of the Lie algebra of matrices $gl(d, \mathbb{R}) = \mathcal{G} = \mathcal{K} \oplus \mathcal{S}$ into a skew-symmetric and upper triangular component, respectively. By projecting in each of these two components, we write (recall (8)) $\tilde{X}(k) = [\tilde{X}(k)]_{\mathcal{K}} + [\tilde{X}(k)]_{\mathcal{S}}$. With this notation, we have the decomposition:

$$\delta X(k) = H(X) + k[\tilde{X}(k)]_{\mathcal{K}} + k[\tilde{X}(k)]_{\mathcal{S}}, \tag{11}$$

where $H(X)$ is the horizontal lift of X to $T_k O(M)$,

The natural lift of X to the subbundle $O(M)$, denoted by $(\delta X)^\perp$ is the projection of δX onto $O(M)$, i.e. for $k \in O(M)$,

$$(\delta X)^\perp(k) := \frac{d}{dt}\,[D\,\eta_t(k)]^\perp\,|_{t=0}.$$

Again, we have the decomposition of $(\delta X)^\perp(k)$ into horizontal and vertical components: $(\delta X)^\perp(k) = H\delta X(k) + V(\delta X)^\perp(k)$. In terms of the right action of $\tilde{X}(k)$, the vertical component is simply $V(\delta X)^\perp(k) = k[\tilde{X}(k)]_{\mathcal{K}}$. In terms of the left action of (∇X) we shall denote $V(\delta X)^\perp(k) = (\nabla X(k))^\perp k$, where

$(\nabla X(k))^{\perp}$ is a skew-symmetric map: $T_xM \to T_xM$. The characterization of $(\nabla X(k))^{\perp}$ in terms of its left action on $O(M)$ is the content of the following lemma. Although the formula looks quite intricate, it helps to understand the corresponding right action of $\tilde{X}(k)$.

Lemma 3.2 *Let $k = (k^1, \ldots, k^d) \in O(M)$ with $\pi_o(k) = x$. The image of the j-th component k^j under the matrix $(\nabla X(k))^{\perp}$ is given by*

$$\begin{aligned}&(\nabla X(k))^{\perp} k^j \\ &= \nabla X(k^j) - \langle \nabla X(k^j), k^j \rangle k^j - \sum_{0<r<j} \left(\langle \nabla X(k^r), k^j \rangle + \langle \nabla X(k^j), k^r \rangle \right) k^r.\end{aligned}$$

Proof: For a differentiable function $t \longmapsto V_t : \mathbb{R} \in \mathbb{R}^d$ with $V_t \neq 0$ for all $t \in (-\epsilon, \epsilon)$ and derivative $\dot{V}_t$ one has

$$\frac{d}{dt} \left(\frac{V_t}{\|V_t\|} \right) \Bigg|_{t=0} = \frac{\dot{V}_t}{\|V_t\|} - \frac{\langle \dot{V}_t, V_t \rangle}{\|V_t\|^3} V_t \ . \tag{12}$$

For the sake of simplicity, fix a basis in T_xM and denote by A the matrix which represents the linear transformation $\nabla X(x)$. Formula (12) with $t = 0$ will be used in each coordinate of

$$(e^{At}(k))^{\perp} = \left(\frac{V_t^1}{\|V_t^1\|}, \ldots, \frac{V_t^d}{\|V_t^d\|} \right) \ ,$$

where each component comes from the orthogonalization process:

$$V_t^j = e^{At}(k^j) - \sum_{0<r<j} \frac{\langle e^{At}(k^j),\ V_t^r \rangle}{\langle V_t^r,\ V_t^r \rangle}\ V_t^r \ .$$

One easily checks, by induction in j and using that $\|V_0^j\| = 1$ for all j, that the derivatives satisfy:

$$\frac{dV_t^j}{dt} \Bigg|_{t=0} = A(k^j) - \sum_{0<r<j} \left(\langle A(k^j), k^r \rangle + \langle A(k^r), k^j \rangle \right) k^r \ ,$$

which gives, by formula (12)

$$\frac{d}{dt} \left(\frac{V_t^j}{\|V_t^j\|} \right) \Bigg|_{t=0} = A(k^j) - \langle A(k^j), k^j \rangle k^j - \sum_{0<r<j} \left(\langle A(k^r), k^j \rangle + \langle A(k^j), k^r \rangle \right) k^r \ .$$

□

One sees the skew-symmetry of $(\nabla X(k))^{\perp}$ by checking that

$$\langle (\nabla X(k))^{\perp} k^i, k^j \rangle = -\langle (k^i, \nabla X(k))^{\perp} k^j \rangle.$$

We shall consider the connected Lie group of diffeomorphisms $\mathrm{Dif}(M)$ which is generated by the exponential of the Lie algebra of smooth, bounded vector fields $\mathcal{X}(M)$. The exponential of vector fields here means the associated flow. We shall denote by $A(M)$ the Lie subgroup of affi ne transformations of M whose elements are given by maps $\Psi \in \mathrm{Diff}(M)$ such that their derivatives $D\Psi$ preserve horizontal trajectories in TM. This is equivalent to saying that affi ne maps are those which preserve geodesics. Its Lie algebra $a(M)$ is the set of infinitesimal affi ne transformations characterized by vector fields X such that the Lie derivative of the connection form ω on $GL(M)$ satisfies $L_{\delta X}\omega = 0$. Thus X is an infinitesimal affi ne transformation if for all vectors fields Y:

$$\nabla A_X(Y) = R(X, Y),$$

where the tensor $A_X = L_X - \nabla_X$ and R is the curvature (see e.g. Kobayashi and Nomizu [9, Chap. VI, Prop. 2.6]).

For a fixed $k \in GL(M)$, the linear map

$$i_1 : a(M) \to T_k GL(M),\ X \mapsto \delta X(k), \tag{13}$$

is injective, see e.g. [9, Theorem VI.2.3]. We denote by $\delta a(k)$ its image in $T_k GL(M)$.

By $I(M) \subset A(M)$ we denote the Lie group of isometries of M. Its Lie algebra $i(M)$ is the space of Killing vector fields or infinitesimal isometries, characterized by the skew-symmetry of the covariant derivative, i.e., a vector field X is Killing if and only if

$$\langle \nabla X(Z), W \rangle = -\langle Z, \nabla X(W) \rangle,$$

for all vectors Z, W in a tangent space $T_x M$. Then, by Lemma 3.2, for any orthonormal frame k we have that $(\nabla X(k))^{\perp} = \nabla X$ and $(\delta X)^{\perp}(k) = \delta X(k)$.

For a fixed $k \in O(M)$, the linear map

$$i_2 : i(M) \to T_k O(M),\ X \mapsto \delta X(k), \tag{14}$$

is a restriction of the map i_1 defined above, hence it is also injective. We denote by $\delta i(k)$ its image in $T_k O(M)$.

Since the dynamics can be described as trajectories in Lie groups (of diffeomorphisms, isometries, affi ne transformations, etc.), whenever convenient, we shall change from the usual dynamical terminology into the Lie group terminology. For example, vector fields are identified with Lie algebra elements which will generate right invariant vector fields in the Lie group $\mathrm{Dif}(M)$; furthermore, if ϕ belongs to $\mathrm{Dif}(M)$, one identifies the derivative $D\phi : TM \to TM$ (which sends vector fields into vector fields in M) with the derivative of the left action $L_\phi : T\mathrm{Dif}(M) \to T\mathrm{Dif}(M)$. In fact, if $\phi = e^X$, then, given another vector field Y, $D\phi(Y) = De^X(Y) = L_\phi(Y)$.

4 Decompositions of Control Flows

This section describes conditions on the vector fields of the control system for the existence of the decomposition into isometric or affi ne transformations.

We start with a theorem which, under certain conditions on the vector fields $X \in F$, factorizes the control flow φ_t of equation (4) in the form $\varphi_t = \Psi_t \circ \rho_t$ such that Ψ_t is a control flow in the affi ne transformations group, and the remainder ρ_t fixes the initial point and has trivial derivatives (identity).

Let k be an element in $GL(M)$ which is a basis for $T_{x_0}M$, i.e. $\pi(k) = x_0$. We shall assume the following hypothesis on the vector fields $X \in F$ determining the control system (4) (recall formula (9)):

(H1) For every $X \in F$, the lifted vector field $\delta(X)$ is tangent to the orbit of the frame k under the group of affi ne maps (acting on the bundle $GL(M)$).

Since the operation δ of lifting vector fields commutes with the translation by a diffeomorphism, an affi ne transformation Ψ maps tangent spaces to the orbit onto tangent spaces. Hence hypothesis (H1) is equivalent to

(H1') $\delta[D\Psi(X)](k) \in \delta a(k)$, for all affi ne transformations $\Psi \in A(M)$.

Observe that in the finite dimensional case (classical affi ne control system), this condition holds if it holds for the vector fields $X_0, ..., X_m$ in the representation (6). Intuitively, a vector field X satisfies hypothesis (H1') (hence (H1)) if the associated flow carries x_0 and its 'infinitesimal neighborhood' (i.e., a basis in $T_{x_0}M$) along trajectories which 'instantaneously' coincide with the trajectories of an infinitesimally affi ne transformation.

Theorem 4.1 *Suppose that every vector field $X \in F$ of the control system (4) satisfies the hypothesis (H1) (or equivalently (H1')) for a fixed frame $k \in GL(M)$, and let $x_0 = \pi(k)$. Then the associated control flow φ_t factorizes uniquely as*

$$\varphi_t = \Psi_t \circ \rho_t,$$

where Ψ_t is a control flow in the group of affi ne transformations $A(M)$, and the remainder ρ_t satisfies $\rho_t(x_0) \equiv x_0$ and $D\rho_t = Id_{(T_{x_0}M)}$ for all $t \geq 0$.

Proof: Since the linear map i_1 of equation (13) is injective, by hypothesis (H1), for each $X \in F$ we can uniquely define the infinitesimal affi ne transformation X^a which satisfies $\delta X^a(k) = \delta X(k)$. Hence, by the comments after Lemma 3.1, one easily sees that

$$X^a(x_0) = X(x_0) \quad \text{and} \quad \nabla X^a(x_0) = \nabla X(x_0). \tag{15}$$

Let Ψ_t be the solution of the following equation in the Lie group $A(M)$, with $\Psi_0 = Id_M$:

$$\dot{\Psi}_t = \Psi_t[D\Psi_t^{-1}(X_t)]^a, \; t \in \mathbb{R} \text{ with } X \in \mathcal{F}, \tag{16}$$

where the elements $[\ \cdot\ ]^a$ in the Lie algebra $a(M)$ act on the right in $A(M)$. We recall that, in the Lie algebra terminology, X_t here means $X_t(\Psi_t)$, the right invariant vector field evaluated at Ψ_t.

Equation (16) is obviously a control system in $A(M)$ and the solution Ψ_t generates a control flow on $A(M)$: Indeed, it is generated by the convex and compact set of vector fields on $A(M)$

$$\Psi \mapsto \Psi[D\Psi^{-1}(X)]^a, \quad X \in F,$$

which is contained in the finite dimensional vector space obtained by considering all $X \in E$. Using that $\Psi_t\Psi_t^{-1} = Id_M$ one easily finds that the control system for the inverse Ψ_t^{-1} in $A(M)$ is

$$\dot{\Psi}_t^{-1} = -[D\Psi_t^{-1}(X_t)]^a\Psi_t^{-1}, \; t \in \mathbb{R} \text{ with } X \in \mathcal{F}.$$

We define $\rho_t = \Psi_t^{-1} \circ \varphi_t$. In the Lie group of diffeomorphisms of M we have the following equation for ρ_t:

$$\begin{aligned} \dot{\rho}_t &= D\Psi_t^{-1}(\dot{\varphi}_t) + (\dot{\Psi}_t^{-1})\varphi_t \\ &= D\Psi_t^{-1}(X_t(\varphi_t)) - [D\Psi_t^{-1}(X_t)]^a\Psi_t^{-1}\,\varphi_t \\ &= \left\{D\Psi_t^{-1}(X_t) - [D\Psi_t^{-1}(X_t)]^a\right\}(\rho_t). \end{aligned} \tag{17}$$

In the last line we use the right invariance of the X and the fact that $D\Psi_t^{-1}(X_t(\varphi_t)) = L_{\Psi^{-1}}(R_{\rho_t}X_t(\Psi_t))$, which (by commutativity of right and left action) yields $D\Psi_t^{-1}(X_t(\Psi_t))\ (\rho_t)$. That is, it is a direct application of the formula $L_g(X)(h) = L_g(X(g^{-1}h))$ for right invariant vector fields in a Lie group (with $L_g = D\Psi^{-1}, h = \rho_t,\ g = \Psi^{-1}$).

By definition of X^a (equation (15)) and equation (17) we have that not only $\dot{\rho}_t(x_0) = 0$ but also that $\delta\big\{D\Psi_t^{-1}X_t - [D\Psi_t^{-1}(X_t)]^a\big\}(\rho_t) = 0$, hence the derivative of the linearization $\frac{d}{dt}D\rho_t(u) = 0$. This establishes the properties of each component of the factorization of $\varphi_t = \Psi_t \circ \rho_t$ stated in the theorem.

For uniqueness, suppose that $\Psi'_t \circ \rho'_t = \Psi_t \circ \rho_t$ where Ψ'_t and ρ'_t also satisfy the properties stated. This implies that $\Psi_t^{-1}\Psi'_t(x_0) = x_0$ for all $t \geq 0$. Besides, the derivative $D_{x_0}(\Psi_t^{-1}\Psi'_t) = Id$, hence the natural lift to $GL(M)$ satisfies the differential equation $\frac{d}{dt}D(\Psi_t^{-1}\Psi'_t) = 0$. Since the map i_1 is injective, it follows that $\Psi_t^{-1} \circ \Psi'_t = Id_M$.

□

Remark. We emphasize that the affi ne transformation system Ψ_t does depend on the choice of the initial frame k.

Remark. Observe that, in general, ρ_t is not a control system in Diff(M) since the vector fields involved in the equation do not depend exclusively on X_t and on the point ρ_t. On the other hand, the control fbw Ψ_t may be considered as a skew product fbw in $\mathcal{F} \times A(M)$. This follows at once from its definition. Then (Ψ_t, ρ_t) is a skew product fbw in the fiber bundle $\mathcal{F} \times A(M) \times M \to A(M) \times M$ with base fbw Ψ_t. In the linear case, this is well known and was used, e.g., by Johnson, Palmer and Sell [7] in their proof of the Oseledets theorem for linear fbws on vector bundles.

For the next theorem, fix an element $k \in O(M)$. We shall assume the following hypothesis on the vector fields $X \in F$ of the system (recall that $i(k)$ denotes the image of the map i_2 defined in (14)):

(H2) For every $X \in F$, the lifted vector field $\delta(X)$ is tangent to the orbit of the frame k under the group of isometries (acting on the bundle $O(M)$).

Again, the operation δ of lifting vector field (in $Gl(M)$) and its orthonormalization $\perp$ commute with the translation by an isometry Θ; hence Θ maps tangent spaces to the orbit onto tangent spaces and hypothesis (H2) is equivalent to

(H2') $[\delta(D\Theta(X))(k)]^\perp \in \delta i(k)$ for every isometry $\Theta \in I(M)$.

Intuitively, a vector field X satisfies hypothesis (H2) if the associated flow carries x_0 and its infinitesimal neighborhood' (i.e., an orthonormal basis in $T_{x_0}M$) along trajectories which instantaneously' coincide with trajectories of a Killing vector field (infinitesimal isometry). That is, a vector field X violates (H2), if there is no isometry which rotates and translates the infinitesimal neighborhood' of x_0 into the same directions as the flow induced by X does.

The nonlinear Iwasawa decomposition is described in the following theorem.

Theorem 4.2 *Suppose that for a certain frame $k \in O(M)$ with $x_0 = \pi_o(k)$, all vector fields $X \in F$ of the control system (4) satisfy hypothesis (H2) (hence (H2')). Then, the associated control flow φ_t has a unique decomposition*

$$\varphi_t = \Theta_t \circ \rho_t,$$

where Θ_t is a control flow in the group of isometries $I(M)$, $\rho_t(x_0) = x_0$ and $D_{x_0}\rho_t(k) = k\, s_t$ for all $t \geq 0$, where s_t lies in the group of upper triangular matrices.

Proof: The first part of the proof proceeds similarly to the proof of Theorem 4.1, replacing the group $A(M)$ by $I(M)$: Since the linear map i_2 of equation (14) is injective, for every $X \in F$, we can take X^i, the unique infinitesimal isometry which satisfies $\delta X^i(u) = (\delta X)^\perp(u)$. Analogously to equation (15), we have that

$$X^i(x_0) = X(x_0) \ \text{ and } \nabla X^i(k) = (\nabla X(k))^\perp k. \tag{18}$$

We define the following system in the group $I(M)$, with initial condition $\Theta_0 = Id_M$:

$$\dot{\Theta}_t = \Theta_t [D\Theta_t^{-1}(X_t)]^i \tag{19}$$

Note that the equation above is a control system in $I(M)$ and the solution Θ_t generates a control flow on $I(M)$: Indeed, it is generated by the convex and compact set of vector fields on $I(M)$

$$\Theta \mapsto \Theta[D\Theta^{-1}(X)]^i, \quad X \in F.$$

The control system for the inverse Θ_t^{-1} in $I(M)$ is given by

$$\dot{\Theta}_t^{-1} = -[D\Theta_t^{-1}(X_t)]^i \Theta_t^{-1}, \ t \in \mathbb{R} \text{ with } X \in \mathcal{F}.$$

We define $\rho_t = \Theta_t^{-1} \circ \varphi_t$. In the Lie group of diffeomorphisms of M we have the following equation for ρ_t (by the same arguments as for equation (17)):

$$\begin{aligned}\dot{\rho}_t &= D\Theta_t^{-1}\,(\dot{\varphi}_t) + (\dot{\Theta}_t^{-1})\varphi_t \\ &= D\Theta_t^{-1}(X_t(\varphi_t)) - [D\Theta_t^{-1}(X_t)]^i\Theta_t^{-1}\ \varphi_t \\ &= \left\{D\Theta_t^{-1}(X_t) - [D\Theta_t^{-1}(X_t)]^i\right\}(\rho_t).\end{aligned} \tag{20}$$

By the first part of equation (18) and equation (20) we have that $\dot{\rho}_t(x_0) = 0$. Moreover, by the decomposition of formula (11) and the second part of equation (18) we have that, for a given $k \in O(M)$,

$$\delta\left\{D\Theta_t^{-1}(X_t) - [D\Theta_t^{-1}(X_t)]^i\right\}\ (k) = k\ \left[\widetilde{D\Theta_t^{-1}(X_t)}\right]_{\mathcal{S}},$$

where $\left[\widetilde{D\Theta_t^{-1}(X_t)}\right]_{\mathcal{S}}$ on the right hand side are upper triangular matrices. As mentioned before, the canonical lift of a vector field gives the infinitesimal behavior of the linearized flow acting on a basis, that is by the definition in (9),

$$\frac{d}{dt}D\rho_t(k) = D\rho_t(k)\ \left[\widetilde{D\Theta_t^{-1}(X_t)}\right]_{\mathcal{S}}.$$

Since the Lie algebra element on the right hand side is upper triangular and $D\rho_0(k) = k$, one can write $D\rho_t(k) = k\ s_t$ where s_t are upper triangular matrices which solve the following left invariant differential equation in the Lie group of upper triangular matrices:

$$\dot{s}_t = s_t\ [\widetilde{D\Theta_t^{-1}(X_t)}]_{\mathcal{S}},\ \ s_0 = Id.$$

This establishes the derivative property of the remainder ρ_t. Uniqueness of the decomposition follows easily from the fact that the map i_2 is injective, analogous to uniqueness in Theorem 4.1.

□

Note that in Theorem 4.2, again, the decomposition depends on the initial orthonormal frame $k \in O(M)$ and the flow Θ_t may be viewed as a skew product flow on $\mathcal{F} \times I(M)$. Now, juxtaposing the decompositions established by Theorems 4.1 and 4.2, we have the following factorization of φ_t into three components.

Corollary 4.3 *Suppose all vector fields $X \in F$ in the control system (4) satisfy conditions (H1) and (H2) for a certain frame $k \in O(M)$, with $x_0 = \pi_o(k)$. Then, for the associated control flow φ_t, one has the unique decomposition*

$$\varphi_t = \Theta_t \circ \Psi_t \circ \rho_t,$$

where each of the components Θ_t, Ψ_t, ρ_t have the properties stated in Theorems 4.1 and 4.2. Moreover $\Theta_t \circ \Psi_t$ corresponds to a control system in the group of affine transformations.

Proof: Let $\varphi_t = \Psi'_t \circ \rho_t$ be the unique decomposition according to Theorem 4.1, where Ψ'_t is a control system in the group of affine transformations $A(M)$, $\rho_t(x_0) = x_0$ and $D\rho_t = Id_{T_{x_0}M}$ for all $t \geq 0$.

Let $\varphi_t = \Theta_t \circ \rho'_t$ be the unique decomposition according to Theorem 4.2, where Θ_t is a control system in the group of isometries $I(M)$ with $\rho'_t(x_0) = x_0$ and $D_{x_0}\rho'_t(k) = k\, s'_t$ for a certain family s'_t in the group of upper triangular matrices.

Take the process Θ_t and ρ_t of the statement of this corollary as defined above and define the process $\Psi_t = \Theta_t^{-1}\Psi'_t$. These assignments define the decomposition.

It only remains to prove that there exists a family on the group of upper triangular matrices such that $D\Psi_t(k) = k\, s_t$. By the properties above, $D\Psi'_t = D\varphi_t$, hence

$$D\Psi_t(k) = D\Theta_t^{-1} \circ D\Psi'_t(k) = D\Theta_t^{-1} \circ D\varphi_t(k) = D\rho'_t(k) = k\ s'_t.$$

Thus the upper triangular matrix family s_t of the statement is given by s'_t. This confirms the expected fact that although, in general, Ψ_t is different from Ψ'_t they have the same derivative behavior (which carries the Lyapunov information of the system).

□

5 Conditions on the Manifold

This section characterizes Riemannian manifolds such that every vector field satisfies hypotheses (H1) and (H2), respectively, and hence the corresponding decompositions hold. These manifolds are precisely Riemannian manifolds with constant curvature (simply connected or quotients of them) for the isometric decomposition and flat space for the affi ne transformations decomposition. In particular, the three-factor decomposition of Corollary 4.3 exists for every control system if and only if M is a flat space. More precisely, we have the following result.

Theorem 5.1 *If M is simply connected with constant curvature (or its quotient by discrete groups), then for every control system (4) and every orthonormal frame $k_0 \in O(M)$, the control flow admits a unique non-linear Iwasawa decomposition $\varphi_t = \Theta_t \circ \rho_t$ as in Theorem 4.2. Conversely, if every control flow on M admits this decomposition, then the space M has constant curvature.*

Proof: For a simply connected manifold M of constant curvature the dimension of $\mathcal{I}(M)$ equals $d(d+1)/2$. Hence the linear map i_2 defined in equation (14) is bijective. Therefore, hypothesis (H2) is always satisfied for any set of vector fields.

Conversely, assume that for all vector fields X and for every orthonormal frame $k \in O(M)$, the corresponding flow η_t has the non-linear Iwasawa decomposition $\eta_t = \Theta_t \circ \rho_t$. Then, the trajectory k_t in $O(M)$ induced by η_t satisfies

$$k_t := [D\,\eta\,(k)]^{\perp} = [D\Theta_t \circ D\rho_t(k)]^{\perp} = D\Theta_t(k).$$

We recall that

$$\frac{d}{dt}\left(D\Theta_t(k)\right)|_{t=0} = (\delta X)^{\perp}(k). \tag{21}$$

For any fixed $k \in GL(M)$, the linear map : $\mathcal{X} \to T_k GL(M)$ given by $X \mapsto \delta X(k)$ is surjective because it concerns only the local behavior of X on M. Hence, the projection of its image by $\perp$: $T_k GL(M) \to T_{k^\perp} O(M)$ is also surjective. In other words, for $k \in O(M)$, the map $X \mapsto (\delta X)^{\perp}(k)$ is surjective. If there exists the decomposition, equality (21) shows that the dimension of $\mathcal{I}(M)$ equals $d(d+1)/2$ which implies that M has constant curvature (see, e.g. Klingenberg [8], Ratcliffe [14] or Kobayashi and Nomizu [9, Thm. VI.3.3]).

□

As a particular case of the theorem above, we have the following conditions on M which guarantee that every system on it will have a flow which factorizes into the three components stated in Corollary 4.3.

Corollary 5.2 *If M is flat, simply connected (or its quotient by discrete groups) then for every control system (4) and every orthonormal frame $k \in O(M)$, the associated flow φ_t has a unique decomposition $\varphi = \Theta_t \circ \Psi_t \circ \rho_t$ as described in Corollary 4.3. Conversely, if every flow φ_t has this decomposition then M is flat.*

Proof: If M is flat and simply connected, then a direct check shows that the dimensions of the groups $i(M)$ and $A(M)$ are $d(d+1)/2$ and $d(d+1)$ respectively. This implies that the injective maps i_1 and i_2 are bijective, hence hypotheses (H1) and (H2) are satisfied for any set of vector fields on M.

Conversely, assume that for all vector fields X and for every orthonormal frame $k \in O(M)$ the corresponding flow η_t has the decomposition $\eta_t = \Theta_t \circ \Psi_t \circ \rho_t$ with the properties asserted. Then, the trajectory k_t in $GL(M)$ induced by η_t satisfies

$$k_t = D\Psi'_t(k),$$

where $\Psi'_t = \Theta_t \circ \Psi_t$. We recall that

$$\frac{d}{dt} \left(D\Psi'_t(k)\right)|_{t=0} = \delta X(k). \tag{22}$$

As before, for a fixed $k \in GL(M)$, the linear map $X \mapsto \delta X(k)$ is surjective because it concerns only the local structure of X on M. Hence, equality (22) implies that the dimension of the group of affi ne transformations$A(M)$ equals $d(d+1)$, which implies that M is flat (see, e.g. Klingenberg [8] or Kobayashi and Nomizu [9, Thm. VI.2.3]).

□

6 Examples

Liao in [13] illustrates the isometric decomposition by working out one example in the sphere S^n. The results in the above section enlarge the class of examples to many well known manifolds including projective spaces, hyperbolic manifolds, flat torus and many non-compact manifolds. In this section we shall describe calculations on all the three possible simply-connected cases. We shall concentrate mainly on the isometric part Θ_t since this is the component which carries more intuitive motivation. Note that (for stochastic flows) this is the component which presents the angular behavior (matrix of rotation, see e.g. [17], [2]), while Ψ_t presents the stability behavior (see [13] or [12]).

The control system Θ_t in the group of isometries presented in Theorem 4.2 becomes well defined by equation (19). In this section we shall give a description of the calculation of the vector fields X^i involved in this equation in each one of the three possibilities of simply connected manifolds with constant curvature. In the case of flat spaces, the coefficients X^a of equation (16) for the system $\Psi'_t = \Theta_t \circ \Psi_t$ (Theorem 4.1) will also be described.

6.1 Flat spaces

We recall that the group $A(\mathbb{R}^d)$ of affine transformations in $\mathbb{R}^d$ (or any of its quotient space by discrete subgroup) can be represented as a subgroup of $Gl(d+1,\mathbb{R})$:

$$A(\mathbb{R}^d) = \left\{ \begin{pmatrix} 1 & 0 \\ v & g \end{pmatrix} \text{ with } g \in Gl(d,\mathbb{R}) \text{ and } v \text{ is a column vector} \right\}.$$

It acts on the left in $\mathbb{R}^d$ through its natural embedding on $\mathbb{R}^{d+1}$ given by $x \mapsto (1,x)$. The group of isometries is the subgroup of $A(M)$ where $g \in O(n,\mathbb{R})$. Given a vector field X, assume that the initial condition x_0 is the origin and that k is an orthonormal frame in the tangent space at x_0. One can easily compute the vector fields $X^a \in a(\mathbb{R}^d)$ and $X^i \in i(\mathbb{R}^d)$ using the properties established in equations (15) and (18):

$$X^a(x) = X(0) + (D_0X)x$$

and

$$X^i(x) = X(0) + (D_0X(k))^{\perp}x.$$

We shall fix k to be the canonical basis $\{e_1,\ldots,e_d\}$ of $\mathbb{R}^d$. Then the matrix $(D_0X(k))^{\perp}$ is simply the skew-symmetric component $(D_0X)_{\mathcal{K}}$.

In terms of the Lie algebra action of $a(\mathbb{R}^d)$, the vector fields X^a and X^i are given by the action of the elements

$$X^a = \begin{pmatrix} 1 & 0 \\ X & D_0X \end{pmatrix} \text{ and } X^i = \begin{pmatrix} 1 & 0 \\ X & (D_0X)_{\mathcal{K}} \end{pmatrix}.$$

Let φ_t be the flow associated with the vector field X. One checks by inspection and by uniqueness that the component $\Psi'_t = \Theta_t \circ \Psi_t$ in the group of affi ne transformations (Theorem 4.1) and the componentΘ_t (Theorem 4.2) which solve equations (16) and (19), respectively, are given by

$$\Psi'_t = \begin{pmatrix} 1 & 0 \\ \varphi_t & (D_0\varphi_t) \end{pmatrix}, \ \Theta_t = \begin{pmatrix} 1 & 0 \\ \varphi_t & (D_0\varphi_t)^{\perp} \end{pmatrix} \tag{23}$$

and

$$\Psi_t = \begin{pmatrix} 1 & 0 \\ 0 & (D_0\varphi_t)^{k} \end{pmatrix}, \tag{24}$$

where $D_0\varphi_t = (D_0\varphi_t)^{\perp} \cdot (D_0\varphi_t)^{k}$ is the classical Iwasawa decomposition of the derivative $D_0\varphi_t$.

We are representing both the isometries and the affi ne transformations as subgroups of the Lie group of matrices $Gl(n+1,\mathbb{R})$. Recall that in the group of matrices the differential of left or right action coincides with the product of matrices itself, i.e., $DL_g h = gh$ for $g, h \in Gl(n+1,\mathbb{R})$. Hence one sees that equation (16) is given simply by:

$$\dot{\Psi}'_t = \begin{pmatrix} 1 & 0 \\ X & D_0X \end{pmatrix}.$$

Note that, in general, though the X^a corresponds to the first two elements of the Taylor series of a vector field X, the factor Ψ_t presents a strong non-linear behavior (in time) due to the fact that the coeffi cients of equation (16) are non-autonomous.

Linear control systems

Consider the linear control system

$$\dot{x}(t) = Ax(t) + Bu(t),$$

where A is a $d \times d$-matrix, B is a $d \times m$-matrix, $x(t) \in \mathbb{R}^d$ and the controls u take values $u(t) \in U \subset \mathbb{R}^m$. Let us fix the initial condition $x_0 = 0$ and the orthonormal frame bundle $k_0 = (e_1, \ldots, e_d)$, the canonical basis. The affi ne transformation decomposition is obvious: the vector fieldsAx and the columns of B are in the affi ne transformation Lie algebra, hence the solution flow φ_t already lives in $A(\mathbb{R}^d)$.

For the Iwasawa decomposition, the projection of each vector field in the Lie algebra of isometries provides the equation for the isometric component of the flow, see equation (19). Hence the isometric component is the flow (rotations and translations) associated to the control system

$$\dot{x}(t) = A^{\perp}x(t) + Bu(t),$$

where $A^{\perp}$ is the skew-symmetric matrix such that $A^{\perp}k = \frac{d(e^{At}k)^{\perp}}{dt}|_{t=0}$.

If A is skew-symmetric, the decomposition is trivial because the original system already lives in the group of isometries of $\mathbb{R}^d$.

Bilinear control systems

Consider the bilinear control system

$$\dot{x}(t) = A_0x(t) + \sum_{i=1}^{m} u_i(t)A_ix(t),$$

where the A_i are $d \times d$-matrices, $x(t) \in \mathbb{R}^d$ and $(u_i(t)) \in U \subset \mathbb{R}^m$. Again, the affi ne transformation decomposition is obvious: the vector fields A_ix are in the affi ne transformation Lie algebra, hence the solution flow φ_t already lives in $A(\mathbb{R}^d)$.

For the Iwasawa decomposition, let us fix the initial condition $x_0 = 0$ and the orthonormal frame bundle $k_0 = (e_1, \ldots, e_d)$, the canonical basis. Then the isometric component Θ_t (pure rotations) is the flow associated to the system

$$\dot{x}(t) = A_0^{\perp}x(t) + \sum_{i=1}^{m} u_i(t)A_i^{\perp}x(t).$$

6.2 Spheres S^d

Let X be a vector field in the sphere S^d. Assume that the starting point is the north pole $N = (0, 0, \ldots, 1) \in S^d$ and that the orthonormal frame is the canonical basis $k = (e_1, \ldots, e_d)$. One way to calculate X^i is finding the element A in the Lie algebra of skew-symmetric matrices $so(d+1)$ whose vector field $\widetilde{A}$ induced in S^d satisfies equations (18), i.e.

$$\widetilde{A}(e_{d+1}) = X(N) \text{ and } \frac{d}{dt}[\ e^{At}k]_{t=0} = (\nabla X(k))^{\perp}k.$$

Hence

$$A = \begin{pmatrix} (\nabla X(N))_{\mathcal{K}} & X(N) \\ X(N)^t & 0 \end{pmatrix},$$

where $X(N)^t$ is the transpose of the column vector $X(N)$.

To complement this description of the vector X^i, we refer the reader to the calculations in Liao [13] in terms of the partial derivatives of the components of X. In that (rather analytical) description, however, one misses the geometrical insight which our description (in terms of the action of the skew-symmetric matrix A) tries to provide.

North-south flow: Let $S^2 - \{N\}$ be parametrized by the stereographic projection π from $\mathbb{R}^2$ which intersects S^2 in the equator. The north-south flow is given by the projection on S^2 of the linear exponential contraction on $\mathbb{R}^2$, precisely: $\varphi_t(p) = \pi \circ e^{-t}\pi^{-1}(p)$. It is associated to the vector field $X(x) = \pi_x(-e_3)$, where π_x is the orthogonal projection into the tangent space T_xS^d. For a point $(x, y, z) \in S^2$, one checks that the flow is given by

$$\varphi_t(x, y, z) = \frac{1}{\cosh(t) - z\sinh(t)} \left(x, y, z\cosh(t) - \sinh(t)\right).$$

Let $x_0 = e_1$ and $k = (e_2, e_3)$. For these initial conditions we have the decomposition: $\varphi_t = \Theta_t \circ \rho_t$ where

$$\Theta_t = \begin{pmatrix} \operatorname{sech}(t) & 0 & \tanh(t) \\ 0 & 1 & 0 \\ -\tanh(t) & 0 & \operatorname{sech}(t) \end{pmatrix}$$

and, using the double-angle formulas $\sinh(2t) = 2\sinh(t)\cosh(t)$ and $\cosh(2t) = 2\cosh^2(t) - 1$, we find

$$\rho_t = \left(\frac{2x - 2}{\cosh(2t) - z\sinh(2t) + 1} + 1, \frac{y}{\cosh(t) - z\sinh(t)}, \frac{2(z\cosh(t) + (x-1)\sinh(t))}{\cosh(2t) - z\sinh(2t) + 1} \right).$$

Hence, the derivative of ρ_t at $(1, 0, 0)$ is

$$D_{(1,0,0)}\rho_t = \begin{pmatrix} \operatorname{sech}^2(t) & 0 & 0 \\ 0 & \operatorname{sech}(t) & 0 \\ \tanh(t) & 0 & \operatorname{sech}(t) \end{pmatrix}.$$

One sees that

$$D_{(1,0,0)}\rho_t\ (k) = k\ s_t,$$

where s_t are the upper triangular matrices

$$s_t = \begin{pmatrix} \operatorname{sech}(t) & 0 \\ 0 & \operatorname{sech}(t) \end{pmatrix}.$$

6.3 Hyperbolic spaces

In the stochastic context this example has already been worked out in [17], where we deal with the hyperboloid H^n in $\mathbb{R}^{n+1}$ with the metric invariant under the Lorentz group $O(1,n)$. In this case, a global parametrization centered at $N = (1,0,\ldots,0) \in H^n$ is given by the graph of the map $x^1 = \sqrt{1+\sum_{j=2}^{n+1}(x^j)^2}$. We just recall the following formula for a vector field $X(x) = a_1(x)\,\partial_1 + \ldots + a_{n+1}(x)\,\partial_{n+1}$ with respect to the coordinates above, at the point $N = (1,0,\ldots,0) \in H^n$ and for an orthonormal frame k in $T_N M$

$$X^i(k) = \begin{pmatrix} 0 & a_2(N) & \cdots & a_{n+1}(N) \\ a_2(N) & & & \\ \vdots & & [\partial_j a_i](k)^{\perp} & \\ a_{n+1}(N) & & & \end{pmatrix}$$

Note that, if k is the canonical basis in $T_N M$, then $([\partial_j\, a_i](k))^{\perp}$ is simply $[(\partial_j a_i)]_{\mathcal{K}}$.

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