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Christopher Heiberger, Torben Klarl, Alfred Maußner

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Christopher Heiberger[a], Torben Klarl[b], Alfred Maußner[c]

[a]University of Augsburg, Department of Economics, Universitätsstraße 16, 86159 Augsburg, Germany, christopher.heiberger@wiwi.uni-augsburg.de

[b]University of Augsburg, Department of Economics, Universitätsstraße 16, 86159 Augsburg, Germany, torben.alexander.klarl@wiwi.uni-augsburg.de

[c]University of Augsburg, Department of Economics, Universitätsstraße 16, 86159 Augsburg, Germany, alfred.maussner@wiwi.uni-augsburg.de

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Abstract

Many algorithms that provide approximate solutions for dynamic stochastic general equilibrium (DSGE) models employ the generalized Schur factorization since it allows for a flexible formulation of the model and exempts the researcher from identifying equations that give raise to infinite eigenvalues. We show, by means of an example, that the policy functions obtained by this approach may differ from those obtained from the solution of a properly reduced system. As a consequence, simulation results may depend on the numeric values of parameters that are theoretically irrelevant. The source of this inaccuracy are ill-conditioned matrices as they emerge, e.g., in models with strong habits. Therefore, researchers should always cross-check their results and test the accuracy of the solution.

1 Introduction

Dynamic stochastic general equilibrium (DSGE) models have become the workhorse of macroeconomic research. While the early proponents of this approch had to write their own computer code, their contemporaneous successors can resort to a variety of freely available toolkits, among which DYNARE is probably the most well-known and most versatile one.[1] The user-friendly toolkits have spurred the further development and prevalence of DSGE models, since they have reduced the barriers for potential users considerably. One does not have to unterstand the details of a particular algorithm, the pitfalls of numerical mathematics, and the subtleties of different programming languages in order to solve, simulate, and even estimate a particular model.

In this note we argue for a careful use. In particular, we show by means of an example that a specific algorithm, namely the generalized Schur factorization (GSF), which automates the system reduction, may provide inaccurate solutions.

The GSF is often employed in perturbation methods to compute the linear part of the solution. Theoretically, this approach delivers a unique solution (see Heiberger et al. (2012)). In numerical applications, however, the solution depends on the condition of the involved matrices. Errors that occur at this stage affect the computation of higher order terms of the solution.

It is well-known that higher-order perturbation methods and non-local methods provide more accuracy than the linear solution. However, to recognize the increased precision, one must compute certain accuracy measures, as, e.g., Euler equation residuals (Judd and Guu (1997)) or the Den Haan-Marcet statistic (Den Haan and Marcet (1994)), since it hardly surfaces in the second moments of simulated data.[2] In our example, however, we observe that the second moments depend on the stationary level of working hours, a parameter that, theoretically, has no effect on the linear solution. Interestingly, when we solve a reduced form of the model with the (simple) Schur factorization, we do not observe this effect.

Our example is by no means specific. Rather, versions of this model are routinely employed in studies of the equity premium puzzle.[3]

[1] Others are the toolkit of Harald Uhlig (1999), and the programs of Sims (2002).

[2] See Aruoba, Fernández-Villaverde, and Rubio-Ramírez (2006) and Heer and Maußner (2008).

[3] See Heer and Maußner (2012) for an overview of those models.

The researcher who relies on the use of DSGE toolkits, thus, should be aware of those strange effects and check his results by either reducing his system adequately or by computing accuracy measures.

From here we proceed with a brief description of the canonical DSGE model, the linearized form of this model, and the GSF in the next section. Section 3 presents our example. Section 4 concludes. The Appendix covers the details of the model presented in Section 3.

2 Analytical Framework

2.1 Canonical DSGE Model

Our framework closely follows Schmitt-Grohé and Uribe (2004). The solution is based on Klein (2000) and the presentation follows Heiberger et al. (2012). Let $\mathbf{x}_t \in \mathbb{R}^{n(x)}$, $\mathbf{z}_t \in \mathbb{R}^{n(z)}$, and $\mathbf{y}_t \in \mathbb{R}^{n(y)}$ denote a vector of endogenous state variables, exogenous state variables, and not predetermined (jump) variables, respectively. The equilibrium conditions of a dynamic, stochastic general equilibrium (DSGE) model are

$$\mathbf{0}_{n(x)+n(y)} = \mathbb{E}_t \mathbf{g}(\mathbf{x}_t, \mathbf{z}_t, \mathbf{y}_t, \mathbf{x}_{t+1}, \mathbf{z}_{t+1}, \mathbf{y}_{t+1}), \tag{2.1a}$$

$$\mathbf{z}_{t+1} = \Pi \mathbf{z}_t + \sigma \Omega \boldsymbol{\epsilon}_{t+1}, \quad \boldsymbol{\epsilon}_{t+1} \sim \mathcal{N}(\mathbf{0}_{n(z)}, I_{n(z)}), \tag{2.1b}$$

where the operator $\mathbb{E}_t$ denotes expectations as of period t. Perturbation methods yield approximate solutions

$$\mathbf{x}_{t+1} = \mathbf{h}^x(\mathbf{x}_t, \mathbf{z}_t, \sigma), \tag{2.2a}$$

$$\mathbf{y}_t = \mathbf{h}^y(\mathbf{x}_t, \mathbf{z}_t, \sigma) \tag{2.2b}$$

where the parameters of the polynomial functions $\mathbf{h}^i$, $i \in \{x, y\}$ are derived from (analytic or numeric) derivatives of the system (2.1a) at the point $(\mathbf{x}, \mathbf{0}, \mathbf{y})$ solving $\mathbf{g}(\mathbf{x}, \mathbf{0}, \mathbf{y}, \mathbf{x}, \mathbf{y}, \mathbf{0}) = \mathbf{0}_{n(x)+n(y)}$.

In a first step one must solve the following system of linear stochastic difference equations:

$$A\,\mathbb{E}_t \begin{bmatrix} \bar{\mathbf{w}}_{t+1} \\ \bar{\mathbf{y}}_{t+1} \end{bmatrix} = B \begin{bmatrix} \bar{\mathbf{w}}_t \\ \bar{\mathbf{y}}_t \end{bmatrix}, \quad \bar{\mathbf{w}}_t \equiv \begin{bmatrix} \mathbf{x}_t - \mathbf{x} \\ \mathbf{z}_t \end{bmatrix}, \quad \bar{\mathbf{y}}_t \equiv \mathbf{y}_t - \mathbf{y}, \tag{2.3a}$$

$$A = \begin{bmatrix} \mathbf{g}_4 & \mathbf{g}_5 & \mathbf{g}_6 \\ \mathbf{0}_{n(z)\times n(x)} & I_{n(z)} & \mathbf{0}_{n(z)\times n(y)} \end{bmatrix}, \tag{2.3b}$$

$$B = \begin{bmatrix} -\mathbf{g}_1 & -\mathbf{g}_2 & -\mathbf{g}_3 \\ \mathbf{0}_{n(z)\times n(y)} & \Pi & \mathbf{0}_{n(z)\times n(y)} \end{bmatrix}. \tag{2.3c}$$

where $\mathbf{g}_i$ denotes the Jacobian matrix of $\mathbf{g}$ with respect to its i-th argument.

2.2 The Generalized Schur Method

Usually, the linear system (2.3) contains a number of equations that involve only variables dated at time t. These arise, e.g., from equations like the economy's resource constraint or from static first-order conditions. In this case the matrix A is singular so that $A^{-1}B$ does not exist and the procedure outlined by Blanchard and Kahn (1980), cannot be applied.[4] As pointed out by Klein (2000), the generalized Schur factorization can be used to solve the system (2.3).[5]

There are two ways to use the generalized Schur factorization to solve the model (2.3a). As shown in Heiberger et al. (2012) both provide the same solution (if it exists at all). The first way (see Klein (2000)) rests on factoring the matrix pencil $(B - \lambda A)$, the second on factoring $(A - \mu B)$ (see Heer and Maußner (2009)). The GSF of the pencil $(B - \lambda A)$ is:

$$\begin{aligned} Q^H AZ &= S, \\ Q^H BZ &= T, \end{aligned} \tag{2.4}$$

where Q and Z are unitary matrices, S and T are upper triangular matrixes, and Q^H denotes the Hermitian transpose of Q.[6] The eigenvalues of the matrix pencil are given by $\lambda_i = t_{ii}/s_{ii}$ for $s_{ii} \neq 0$.[7] Furthermore, the matrices S and T can be arranged so

[4]Heer and Maußner (2009) present an illustrative example.

[5]King and Watson (1998, 2000) present a different way to reduce a singular system of linear stochastic difference equations. The advantage of using the generalized Schur factorization, instead, is that it is implemented in the freely available LAPACK programs, and thus, easy to implement.

[6]See, e.g., Golub and van Loan (1996), Theorem 7.7.1, p. 377 who also describe the algorithm to compute the factorization of A and B. The set of all matrices of the form $B - \lambda A$, $\lambda \in \mathbb{C}$ is called a pencil. The eigenvalues of the pencil are the solutions of $|B - \lambda A| = 0$. Unitary matrices Q are complex-valued matrices whose conjugate (Hermitian) transpose equals the inverse of Q.

[7]If $s_{ii} = 0$ and $t_{ii} \neq 0$, the eigenvalue $\mu_{ii} = s_{ii}/t_{ii}$ of the pencil $|\mu A - B| = 0$ is defined and equal to zero. Therefore, we can regard λ_{ii} as 'infinite eigenvalue'.

that the eigenvalues appear in ascending order with respect to their absolute value. Assume that $n(w) = n(x) + n(z)$ eigenvalues have modulus less than one and $n(y)$ have modulus greater than one. Let Z_{11} denote the upper left $n(w) \times n(w)$ block of Z, Z_{12} the upper right $n(w) \times n(y)$ block, etc., and define new variables:

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{w}}_t \\ \tilde{\mathbf{y}}_t \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{w}}_t \\ \bar{\mathbf{y}}_t \end{bmatrix}, \tag{2.5}$$

so that we can write (2.3) as

$$\begin{bmatrix} S_{11} & S_{12} \\ 0_{n(y)\times n(w)} & S_{22} \end{bmatrix} \mathbb{E}_t \begin{bmatrix} \tilde{\mathbf{w}}_{t+1} \\ \tilde{\mathbf{y}}_{t+1} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ 0_{n(y)\times n(w)} & T_{22} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{w}}_t \\ \tilde{\mathbf{y}}_t \end{bmatrix}. \tag{2.6}$$

S_{11} is a $n(w) \times n(w)$ upper triangular matrix, S_{22} is a $n(y) \times n(y)$ upper triangular matrix, and S_{12} is a $n(w) \times n(y)$ matrix. The matrices T_{11}, T_{22}, and T_{12} have corresponding dimensions.

Given these assumptions and definitions, the system

$$S_{22}\mathbb{E}_t\tilde{\mathbf{y}}_{t+1} = T_{22}\tilde{\mathbf{y}}_t$$

is unstable,[8] and to obtain a definite solution, we must set $\tilde{\mathbf{y}}_t = \mathbf{0}_{n(y)}$ for all t. Thus, from the first line of (2.6)

$$\tilde{\mathbf{w}}_{t+1} = S_{11}^{-1} T_{11} \tilde{\mathbf{w}}_t.$$

Since

$$\tilde{\mathbf{w}}_t = Z_{11}^{-1}\bar{\mathbf{w}}_t \tag{2.7}$$

from the first line of (2.5), we get

$$\bar{\mathbf{w}}_{t+1} = \underbrace{Z_{11} S_{11}^{-1} T_{11} Z_{11}^{-1}}_{L_w^w} \bar{\mathbf{w}}_t.$$

The second line of (2.5) together with (2.7) implies

$$\bar{\mathbf{y}}_t = \underbrace{Z_{21} Z_{11}^{-1}}_{L_w^y} \bar{\mathbf{w}}_t.$$

[8] To see this, consider the last line of this system, which may be written

$$\mathbb{E}_t \tilde{y}_{n(y),t+1} = \lambda_{n(y),n(y)} \tilde{y}_{n(y),t}, \quad |\lambda_{n(y),n(y)}| = |(t_{n(y),n(y)}/s_{n(y),n(y)})| > 1,$$

where $s_{n(y),n(y)}$ and $t_{n(y),n(y)}$ denote the last element on the main diagonal of S_{22} and T_{22}, respectively.

The dynamics of the solved linear model can be represented by

$$\bar{\mathbf{x}}_{t+1} = L_x^x \bar{\mathbf{x}}_t + L_z^x \mathbf{z}_t, \tag{2.8a}$$

$$\bar{\mathbf{y}}_{t+1} = L_x^y \bar{\mathbf{x}}_t + L_z^y \mathbf{z}_t, \tag{2.8b}$$

$$\mathbf{z}_{t+1} = \Pi \mathbf{z}_t + \sigma \Omega \boldsymbol{\epsilon}_{t+1}. \tag{2.8c}$$

where the matrices of the linear approximation of the policy functions (2.2) are related to L_w^w and L_w^y according to

$$L_w^w = \begin{bmatrix} L_x^x & L_z^x \\ \mathbf{0}_{n(z)\times n(x)} & \Pi \end{bmatrix}, \quad L_w^y = \begin{bmatrix} L_x^y & L_z^y \end{bmatrix}.$$

2.3 Model Reduction

In this section we assume that the researcher is able to sort the equations in $\mathbf{g}(\cdot)$ so that the first $n(u)$ equations involve only period t variables. This allows us to partition $\mathbf{y}_t = [\mathbf{u}_t', \mathbf{v}_t']'$ and to write the linearized system (2.1) as:

$$C_u \bar{\mathbf{u}}_t = C_{wv} \begin{bmatrix} \bar{\mathbf{w}}_t \\ \bar{\mathbf{v}}_t \end{bmatrix}, \tag{2.9a}$$

$$D_{wv} \mathbb{E}_t \begin{bmatrix} \bar{\mathbf{w}}_{t+1} \\ \bar{\mathbf{v}}_{t+1} \end{bmatrix} + F_{wv} \begin{bmatrix} \bar{\mathbf{w}}_t \\ \bar{\mathbf{v}}_t \end{bmatrix} = \tilde{D}_u \mathbb{E}_t \bar{\mathbf{u}}_{t+1} + \tilde{F}_u \bar{\mathbf{u}}_t, \tag{2.9b}$$

where the matrices are related to the Jacobian matrix of $\mathbf{g}$ according to:

$$D\mathbf{g} = \begin{bmatrix} C_x & C_z & C_u & C_v & 0 & 0 & 0 & 0 \\ F_x & F_z & F_u & F_v & D_x & D_z & D_u & D_v \end{bmatrix},$$

$$C_{wv} = \begin{bmatrix} -C_x & -C_z - C_v \end{bmatrix}, \; D_{wv} = \begin{bmatrix} D_x & D_z & D_v \\ 0 & I_{n(z)} & 0 \end{bmatrix}, \; F_{wv} = \begin{bmatrix} F_x & F_z & F_v \\ 0 & -\Pi & 0 \end{bmatrix},$$

$$\tilde{D}_u = \begin{bmatrix} D_u \\ 0 \end{bmatrix}, \; \tilde{F}_u = \begin{bmatrix} F_u \\ 0 \end{bmatrix}.$$

Solving (2.9a) for $\mathbf{u}_t$ and using the result in (2.9b) yields:

$$\mathbb{E}_t \begin{bmatrix} \bar{\mathbf{w}}_{t+1} \\ \bar{\mathbf{v}}_{t+1} \end{bmatrix} = W \begin{bmatrix} \bar{\mathbf{w}}_t \\ \bar{\mathbf{v}}_t \end{bmatrix}, \; W = \begin{bmatrix} D_{wv} - \tilde{D}_u C_u^{-1} C_{wv} \end{bmatrix}^{-1} \begin{bmatrix} F_{wv} - \tilde{F}_u C_u^{-1} C_{wv} \end{bmatrix}. \tag{2.10}$$

This system can be solved along the same lines as system (2.3a). The (simple) Schur factorization of the matrix W is given by

$$S = Z^H W Z, \tag{2.11}$$

where S is an upper triangular matrix with the eigenvalues of W on the main diagonal. Assume that $n(w) = n(x) + n(z)$ eigenvalues are within and $n(v)$ eigenvalues outside the unit circle. S and Z can be chosen so that the first $n(w)$ eigenvalues appear first on the main diagonal of S. In the new variables

$$\begin{bmatrix} \bar{\mathbf{w}}_{t+1} \\ \bar{\mathbf{v}}_{t+1} \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{w}}_t \\ \tilde{\mathbf{v}}_t \end{bmatrix} \tag{2.12}$$

the transformed system reads

$$\mathbb{E}_t \begin{bmatrix} \tilde{\mathbf{w}}_{t+1} \\ \tilde{\mathbf{v}}_{t+1} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ 0_{n(y)\times n(w)} & S_{22} \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{w}}_t \\ \tilde{\mathbf{v}}_t \end{bmatrix}. \tag{2.13}$$

Accordingly, the system $\mathbb{E}_t\tilde{\mathbf{v}}_{t+1} = S_{22}\tilde{\mathbf{v}}_t$ is unstable and we must set $\tilde{\mathbf{v}}_t = \mathbf{0}_{n(v)} \forall t$ so that the solution of the linear model (2.10) is

$$L_w^w = \begin{bmatrix} L_x^x & L_z^x \\ 0_{n(z)\times n(x)} & \Pi \end{bmatrix} = Z_{11}S_{11}Z_{11}^{-1}, \tag{2.14a}$$

$$L_v^w = \begin{bmatrix} L_w^v & L_z^v \end{bmatrix} = Z_{21}Z_{11}^{-1}. \tag{2.14b}$$

Using (2.14b) in (2.9a) yields

$$L_w^u = \begin{bmatrix} L_x^u & L_z^u \end{bmatrix} = C_u^{-1}C_{wv} \begin{bmatrix} I_{n(w)} \\ Z_{21}Z_{11}^{-1} \end{bmatrix} \tag{2.14c}$$

so that the matrices from (2.8b) are given by

$$L_x^y = \begin{bmatrix} L_x^u \\ L_x^v \end{bmatrix}, \; L_z^y = \begin{bmatrix} L_z^u \\ L_z^v \end{bmatrix}. \tag{2.14d}$$

3 An Example

We consider a real business cycle model taken from Heer and Maußner (2012) that features habits in consumption and hours as well as adjustment costs of capital. The model introduces endogenous labor supply in the model of Jermann (1998) who studied the equity premium implications of a production economy.

3.1 The Model

Households. Households enter the current period t with a given amount of firm shares S_t and real bonds B_t. Bonds have a maturity of one period and can be purchased at the current price v_t^b and pay one unit of consumption in period $t+1$. The real share price is v_t^e and real dividend payments per share are d_t. Firms pay the real wage w_t per unit of working hours N_t. Thus,

$$v_t^e(S_{t+1} - S_t) + v_t^b B_{t+1} \leq w_t N_t + d_t S_t + B_t - C_t \tag{3.1}$$

is the household's budget constraint, where C_t denotes consumption. The household chooses contingency plans for consumption C_t, hours N_t, and next-period stocks S_{t+1} that maximize

$$U_t = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \frac{(C_{t+s} - \chi^C C_{t+s-1})^{1-\eta} - 1}{1-\eta} - \nu_0 \frac{(N_{t+s} - \chi^N N_{t+s-1})^{1+\nu_1} - 1}{1+\nu_1} \tag{3.2}$$

subject to (3.1). The first-order conditions for this problem and any further mathematical details of this model are prensented in the Appendix.

Firms. The representative firm uses labor N_t and capital K_t to produce output Y_t according to the production function

$$Y_t = Z_t N_t^{1-\alpha} K_t^{\alpha}, \quad \alpha \in (0,1). \tag{3.3}$$

The level of total factor productivity Z_t is governed by the AR(1)-Process

$$\ln Z_t = \rho \ln Z_{t-1} + \sigma \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0,1). \tag{3.4}$$

The firm finances part of its investment I_t from retained earnings RE_t and issues new shares to cover the remaining part:

$$I_t = v_t(S_{t+1} - S_t) + RE_t. \tag{3.5}$$

It distributes the excess of its profits over retained earnings to the household sector:

$$d_t S_t = Y_t - w_t N_t - RE_t. \tag{3.6}$$

Investment increases the firm's future stock of capital according to:

$$K_{t+1} = \Phi(I_t/K_t)K_t + (1-\delta)K_t, \quad \delta \in [0,1], \tag{3.7}$$

where we parameterize the function Φ as

$$\Phi(I_t/K_t) := \frac{a_1}{1-\zeta}\left(\frac{I_t}{K_t}\right)^{1-\zeta} + a_2, \quad \zeta > 0. \tag{3.8}$$

The firm maximizes its beginning-of-period value

$$V_t = \mathbb{E}_t \sum_{s=0}^{\infty} \varrho_{t+s}(Y_{t+s} - w_{t+s}N_{t+s} - I_{t+s}) \tag{3.9}$$

subject to (3.3) and (3.7) where ϱ_{t+s}, with $\varrho_t = 1$, is the firm's stochastic discount factor. The respective first-order conditions can be found in the Appendix.

Calibration. We calibrate the model with respect to the US economy. Table 3.1 displays our choice of parameters. The standard parameter values for the production side, α, ρ, and σ are taken from Hansen (1985), as well as the value of the discount factor β. The habit parameter χ^C, η, and the parameters of the capital accumulation equation (3.7) are taken from Jermann (1998). The parameter $\nu_1 = 2.5$ is from de Paoli, Scott, and Weeken (2010). As these authors, we assume $\chi^N = \chi^C$,[9] and choose ν_0 so that in the stationary equilibrium N equals 0.33.

Table 3.1

Parameter Choice

Preferences	β=0.99	η=5	ν_1=2.5	N=0.33
	χ^C=0.82	χ^N=0.82		
Production	α=0.36	ρ=0.95	σ=0.00712	
Capital Accumulation	δ=0.025	ζ=1/0.23		

3.2 Accuray of the Solution

Policy Functions. Theoretically, all three ways to compute the coefficients of the linear part of the policy functions (2.8) should deliver the same solution.[10] Indeed, if we

[9]Alternatively, following Heer and Maußner (2012), we could have chosen the unobserved parameters so that the model replicates certain empirical facts. Yet, since we use the model just as an example, the precise calibration does not matter.

[10]See Heiberger et al. (2012) for a proof with respect to the GSF. Since the GSF can be reduced to the SSF, if A (or B) is invertible, this also implies the uniqueness of the solution based on the SSF.

use $N = 0.33$ the coefficients are virtually identical: the maximum relative difference between the coefficients is less than 0.005 percent. This changes considerably, if we use $N = 0.13$, a value used by Heer and Maußner (2012) for the German economy.

Table 3.2 presents results regarding the coefficients of the policy functions for the variables displayed in the left column. The table entries are relative differences between three different solutions. The first block refers to the solution based on the GSF of the pencil $(B - \lambda A)$ and the solution of the reduced system. The second block compares solutions from factoring $(A - \mu B)$ to those from the reduced system, while the third block shows differences between the GSF of $(A - \mu B)$ and $(B - \lambda A)$.

There are no remarkable differences between the reduced system and the GSF of $(B - \lambda A)$ except for the coefficient that relates Tobin's q_t to the capital stock K_t. The respective coefficients differ by more than 65 percent. This difference increases to over 150 percent between the GSF of $(A - \mu B)$ and the SSF. Other noticeable differences concern the coefficients of the policy function for next period's capital K_{t+1}. Both the coefficients with respect to previous period's consumption C_{t-1} and hours N_{t-1} differ by about 100 percent. The same is true for the coefficient of the technology shock $\ln Z_t$. Further, the coefficient that relates consumption C_t to the capital stock K_t is about 10 percent larger in SSF solution.

The differences between the two generalized Schur factorizations displayed in the third block are the mirror images of the differences between the GSF and the reduced system. Most remarkable is the coefficient on capital in the policy function for Tobin's q, which is positive and equal to 0.0137 in the GSF of $(A - \mu B)$ and equal to -0.0084 in the GSF of $(B - \lambda A)$, implying a relative difference of over 260 percent.

Second Moments. Table 3.3 presents results from four different simulations of the model.[11] The moments in the first panel rest on the solution obtained from factoring $(A - \mu B)$ while the moments in the second panel are from simulations that use the policy function obtained from solving the reduced model. The single difference between the panels labeled $N = 0.33$ and $N = 0.13$ are two different values for the stationary level of hours N. The second moments refer to HP-filtered logs of simulated time series. It can be shown that the coefficient matrices of the log-linearized system do not

[11] The simulations were performed with a Fortran program. There is also a Gauss program using the `CoRRAM` toolkit written by Alfred Maußner (2011). Both programs can be obtained from the authors upon request.

Table 3.2

Policy Functions

Dependent Variables	Independent Variables K_t	C_{t-1}	N_{t-1}	$\ln Z_t$
		$(B-\lambda A)$ and reduced system		
K_{t+1}	0.00045	0.00502	0.00449	0.01746
Y_t	0.00965	−0.00399	0.00193	0.00735
C_t	−0.03604	0.00105	−0.00365	−0.01336
I_t	0.01832	0.00502	0.00449	0.01746
N_t	−0.04525	−0.00399	0.00193	−0.01520
w_t	−0.00407	−0.00399	0.00193	−0.00236
q_t	−0.65013	0.00502	0.00449	0.01746
Λ_t	−0.03604	−0.00399	−0.00365	−0.01336
		$(A-\mu B)$ and reduced system		
K_{t+1}	−0.00814	−1.01943	−1.00000	−1.00000
Y_t	0.02706	−0.01883	0.00949	0.02359
C_t	−0.10108	0.00495	−0.01795	−0.04287
I_t	0.04384	0.00328	0.00210	0.03534
N_t	−0.12691	−0.01883	0.00949	−0.04878
w_t	−0.01142	−0.01883	0.00949	−0.00756
q_t	−1.56602	0.00408	0.00289	0.03616
Λ_t	−0.10108	−0.01883	−0.01795	−0.04287
		$(A-\mu B)$ and $(B-\lambda A)$		
K_{t+1}	−0.00858	−1.01933	−1.00000	−1.00000
Y_t	0.01725	−0.01490	0.00755	0.01612
C_t	−0.06748	0.00390	−0.01435	−0.02991
I_t	0.02505	−0.00174	−0.00238	0.01757
N_t	−0.08554	−0.01490	0.00755	−0.03410
w_t	−0.00738	−0.01490	0.00755	−0.00521
q_t	−2.61779	−0.00094	−0.00160	0.01837
Λ_t	−0.06748	−0.01490	−0.01435	−0.02991

Notes: The entries represent relative differences between the coefficients of the policy functions of the variables in the leftmost column. Λ_t is the Lagrange multiplier of the household's budget constraint.

depend on N so that the simulations should yield identical second moments, given that the same sequence of random numbers is used. However, the second moments in the first panel reveal many obvious differences, both in the standard deviations and in the cross- as well as autocorrelations of the variables displayed. Yet, the second moments displayed in the lower left and the lower right panel are the same.

Table 3.3

Second Moments

Variable	s_x	s_x/s_Y	r_{xY}	r_x	s_x	s_x/s_Y	r_{xY}	r_x
				GSF				
		$N = 0.33$				$N = 0.13$		
Output	0.54	1.00	1.00	0.59	0.52	1.00	1.00	0.50
Consumption	0.40	0.74	0.88	0.85	0.21	0.41	1.00	0.50
Investment	1.21	2.25	0.89	0.33	1.38	2.65	1.00	0.50
Hours	0.62	1.15	−0.93	0.82	0.75	1.44	−0.67	0.88
Real Wage	1.14	2.12	0.98	0.73	1.16	2.24	0.88	0.75
Tobin's q	5.38	9.97	0.87	0.33	6.09	11.72	1.00	0.50
				SSF				
		$N = 0.33$				$N = 0.13$		
Output	0.54	1.00	1.00	0.59	0.54	1.00	1.00	0.59
Consumption	0.40	0.74	0.88	0.85	0.40	0.74	0.88	0.85
Investment	1.21	2.25	0.89	0.33	1.21	2.25	0.89	0.33
Hours	0.62	1.15	−0.93	0.82	0.62	1.15	−0.93	0.82
Real Wage	1.14	2.12	0.98	0.73	1.14	2.12	0.98	0.73
Tobin's q	5.38	9.97	0.87	0.33	5.38	9.97	0.87	0.33

Notes: s_x:=Standard deviation of HP-filtered simulated time series x, where x stands for any of the variables from column 1, based on 500 replications with 200 observations each. s_x/s_Y:=Standard deviation of variable x relative to standard deviation of output Y. r_{xY}:=Cross-correlation of variable x with output y, r_x:=First order autocorrelation of variable x.

We also computed various measures of the accuracy of the solutions that we used for the simulations reported in Table 3.3. They indicate that all solutions are rather imprecise but they give no clear cut advice as to what solution performs better. The interested reader can consult the Appendix where we present the details of this exercise.

3.3 Source of the Problem

The odd results reported in the previous subsection origin in the first-order condition for consumption

$$\Lambda_t = (C_t - \chi^C C_{t-1})^{-\eta}.$$

The steady-state value of consumption is small and increases with the stationary value of working hours N. Therefore, a strong habit (χ^C close to one) and a large coefficient of relative risk aversion η imply a huge value of Λ, the multiplier of the budget constraint (3.1). This gives rise to very large coefficients in the Jacobian matrix of $\mathbf{g}$, and, accordingly, in the matrix B of (2.3a) and the W matrix of (2.10). Yet, due to the reduction of the model, W is less unbalanced as is A: the condition number (computed in the L_1 norm) of the matrix B for $N = 0.33$ ($N = 0.13$) is 0.21×10^{12} (0.169×10^{17}) while the condition number of the matrix W is 0.978×10^9 (0.787×10^{14}).

4 Conclusion

The availability of easy to use toolkits for solving dynamic stochastic general equilibrium (DSGE) models has enhanced the widespread application of these models in macroeconomic research. The researcher supplies the equations of his model to programs as, e.g., Dynare, which solve and simulate the model.

We demonstrate by means of an example that an uninformed use of DSGE solution software may produce strange results. We consider a model that has been employed in studies of the equity premium puzzle. The second moments implied by simulations of this model are theoretically independent from the steady state value of working hours. If we reduce the model adequately, the simulations confirm this prediction. However, when we automate the system reduction by use of the general Schur factorization, we find noticeable differences in the simulation results, though theoretically the solution should be invariant to the specific factorization employed. The reason for this result are more or less ill-conditioned matrices.

Researchers should be aware of those effects and check the accuracy of solutions by trying different ways to solve their model.

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Appendix (not for publication)

In this Appendix, we provide the formal details of our example model and compute several measures of the accuracy of the linear solutions.

A.1 The Model

Equilibrium Conditions. The first-order conditions of maximizing (3.2) subject to (3.1) and given initial values of S_t and B_t are:

$$\Lambda_t = (C_t - \chi^C C_{t-1})^{-\eta}, \tag{A.1.1a}$$

$$\Lambda_t w_t = (N_t - \chi^N N_{t-1})^{\nu_1}, \tag{A.1.1b}$$

$$v_t^e = \beta \mathbb{E}_t \frac{\lambda_{t+1}}{\Lambda_t}(d_{t+1} + v_{t+1}^e), \tag{A.1.1c}$$

$$v_t^b = \beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda_t}, \tag{A.1.1d}$$

where Λ_t is the Lagrange multiplier of the budget constraint (3.1).

The first-order conditions of the firm's problem – maximizing (3.9) subject to (3.3) and (3.7) and a given initial stock of capital K_t – are

$$w_t = (1-\alpha) Z_t N_t^{-\alpha} K_t^{\alpha}, \tag{A.1.2a}$$

$$q_t = \frac{1}{\Phi'(I_t/K_t)}, \tag{A.1.2b}$$

$$q_t \varrho_t = \mathbb{E}_t \varrho_{t+1} \Big\{ \alpha Z_{t+1} N_{t+1}^{1-\alpha} K_{t+1}^{\alpha-1} - (I_{t+1}/K_{t+1}) + q_{t+1} \big[\Phi(I_{t+1}/K_{t+1}) + 1 - \delta \big] \Big\}. \tag{A.1.2c}$$

In equilibrium all markets clear. We assume that bonds are in zero supply, $B_t = 0 \forall t$, and that the firm finances investment entirely from retained earnings. Using equations (3.5) and (3.6) the household's budget constraint (3.1) reduces to the economy's resource restriction $Y_t = C_t + I_t$. Equilibrium in the market for shares requires

$$\varrho_{t+s} = \beta^s \frac{\Lambda_{t+s}}{\Lambda_t}.$$

Let $\mathbf{x}_t = [K_t, C_{t-1}, N_{t-1}]'$, $\mathbf{y}_t := [Y_t, C_t, I_t, N_t, w_t, d_t, q_t, \Lambda_t, v_t^e]'$, $\mathbf{z}_t := \ln Z_t$. Then, the system (2.1) is given by:

$$\Lambda_t = (C_t - \chi^C C_{t-1})^{-\eta}, \tag{A.1.3a}$$

$$\Lambda_t w_t = (N_t - \chi^N N_{t-1})^{\nu_1}, \tag{A.1.3b}$$

$$w_t = (1-\alpha) Z_t N_t^{-\alpha} K_t^{\alpha}, \tag{A.1.3c}$$

$$d_t = Y_t - w_t N_t - I_t, \tag{A.1.3d}$$

$$q_t = \frac{1}{\Phi'(I_t/K_t)}, \tag{A.1.3e}$$

$$Y_t = Z_t N_t^{1-\alpha} K_t^{\alpha}, \tag{A.1.3f}$$

$$Y_t = C_t + I_t, \tag{A.1.3g}$$

$$v_t^e = \beta \mathbb{E}_t \frac{\lambda_{t+1}}{\Lambda_t} (d_{t+1} + v_{t+1}^e), \tag{A.1.3h}$$

$$q_t = \beta \mathbb{E}_t \frac{\Lambda_{t+1}}{\Lambda t} \Big\{ \alpha Z_{t+1} N_{t+1}^{1-\alpha} K_{t+1}^{\alpha-1} - (I_{t+1}/K_{t+1}) + q_{t+1} \big[\Phi(I_{t+1}/K_{t+1}) + 1 - \delta \big] \Big\} \tag{A.1.3i}$$

$$K_{t+1} = \Phi(I_t/K_t) K_t + (1-\delta) K_t. \tag{A.1.3j}$$

Note that equations (A.1.3a)-(A.1.3g) involve only variables dated at t (using the definition of $\mathbf{x}_t$ above). Therefore, the matrix A of the linearized model will be singular.

Deterministic Stationary Equilibrium. Assume $\sigma = 0$ so that $\ln Z_t$ equals its unconditional expectation 0 for all t (and, hence, $Z = 1$). In this case, we can ignore the expectations operator $\mathbb{E}_t$. Stationarity implies $x_{t+1} = x_t = x$ for any variable in our model. As usual, we specify Φ so that adjustment costs play no role in the stationary equilibrium, i.e., $\Phi(I/K)K = \delta K$ and $q = \Phi'(\delta) = 1$. This requires that we choose

$$a_1 = \delta^{\zeta},$$
$$a_2 = \frac{-\zeta\delta}{1-\zeta}.$$

These assumptions imply via equation (A.1.3i) the stationary solution for the output-capital-ratio:

$$\frac{Y}{K} = \frac{1-\beta(1-\delta)}{\alpha\beta}. \tag{A.1.4a}$$

Using the production function, we can solve for the capital-labor ratio and for labor productivity:

$$\frac{K}{N} = \left(\frac{Y}{K}\right)^{\frac{1}{\alpha-1}}, \tag{A.1.4b}$$

$$\frac{Y}{N} = \left(\frac{Y}{K}\right)^{\frac{\alpha}{\alpha-1}}. \tag{A.1.4c}$$

Given these solutions equations (A.1.3a)-(A.1.3c) and (A.1.3f) can be reduced to an equation in N:

$$(1-\alpha)\frac{Y}{N} = (1-\chi^N)^{\nu_1}(1-\chi^C)^{\eta}\left(\frac{Y}{N} - \delta\frac{K}{N}\right)^{\eta} N^{\nu_1+\eta}. \tag{A.1.4d}$$

Given the solution for N, the levels of the stock of capital K, output Y, consumption C, and investment I can be computed. In the final step, equation (A.1.3a) delivers the stationary level of the Lagrange multiplier Λ.

A.2 Accuracy Measures.

Table A.1 displays various measures that indicate the accuracy of the solutions. The Euler equation residuals refer to the maximum absolute value of the Euler equations residuals computed over a four dimensional grid over the state variables K_t, C_{t-1}, N_{t-1}, and $\ln Z_t$. The grid covers almost all points that the model visited in a simulation with 1,000,000 obserations. Each of the four subintervals was divided into 50 points so that we had to compute 50^4 residuals. The meaning of the residuals is as in Judd and Guu (1997): it is the fraction by which consumption had to be raised via its value given by the policy function so that the left and the ride hand side of the Euler equation for the optimal stock of capital (A.1.3i) are equal to each other.

Table A.1

Accuracy Measures

	GSF		SSF	
	N=0.33	N=0.13	N=0.33	N=0.13
Euler equation residuals	2.00	1.48	2.02	1.11
Equity premium	1.56	1.69	1.56	1.56
DM-Statistic	13.10	1.90	13.10	13.10
percent < 1.69				
percent >16.01	1.00	4.50	1.00	1.00

For all four solutions, the residuals reveal a serious accuracy problem which is somewhat less pronounced in the case of $N = 0.13$.

The equity premium was computed as average of the ex-post premium from a simulated time series with 1,000,000 observations. As with the second moments, there is no difference between the two simulations based on the SSF solution.

The Den Haan-Marcet (DM) statistic reports the percentage of simulations (out of 1,000) for which the Wald statistic of the null hypotheses that 3 lags of consumption and 3 lags of the productivity shock do not help to predict the ex-post residual of the Euler equation (A.1.3i) is either below the 2.5 or above the 0.975 critical value of the χ^2-distribution with 7 degrees of freedom. What we would expect from a good solution are approximately 2.5 percent below and 2.5 percent above the respective critical values. Again, the numbers displayed in Table 3.3 are quite distinct from these benchmarks and, thus, indicate inaccurate solutions.

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