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A further note on a new class of solutions to dynamic programming problems arising in economic growth

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# A further note on a new class of solutions to dynamic programming problems arising in economic growth

by

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## Abstract

This note extends the finding of Benhabib and Rusticchini (1994) who provide a class of SDGE models, whose solution is characterized by a constant savings rate. We show that this class of models may be interpreted as a standard representative agent SDGE model with costly adjustment of capital and provides a solution to the traditional discrete time Ramsey problem.

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# 1 Introduction

Stochastic dynamic equilibrium (SDGE) models have become the standard tool for analyzing many questions in business cycle research, finance, growth, and monetary economics. Except for few cases the solutions of these models must be approximated by numerical methods. In these circumstances it is often very helpful to start from a model, which is known to have an analytical solution and to approach the model of interest by way of homotopy methods (see Heer and Maußner (2005), Chapter 3 and 4). From this perspective, extensions of the class of models with analytic solutions are very valuable.

Benhabib and Rusticchini (1994), henceforth BR, extend the class of SDGE models which are known to have a solution in terms of a constant savings rate. BR employ a model with two vintages of capital. In this note we firstly show that their specification is easily extended to the case of an infinite number of vintages and secondly reinterpret this specification in terms of model with frictions in the adjustment of capital.

In the next section we present the model of BR as well as our extension and reinterpretation of it. Section three concludes and the Appendix covers the technical details of our derivations.

## 2 The model

BR consider a representative agent with additively time separable preferences, who discounts future utility at the rate  $\delta \in (0, 1)$  and whose instantaneous utility function  $u$  is given by

$$u(c_t, 1 - L_t) = \frac{A(c_t^{1-\epsilon} - 1)}{1 - \epsilon} + w(1 - L_t), \quad (1)$$

where  $c$  denotes consumption and  $L$  hours worked.  $A > 0$  and  $\epsilon \geq 0$  are given parameters.  $w$  is a concave, increasing function. The agent employs labor and

two vintages of capital,  $k_1$  and  $k_2$ , respectively, to produce output according to

$$y_t = z_t \left[ (a_1 k_{1t}^{1-\epsilon} + a_2 k_{2t}^{1-\epsilon} + (1 - a_1 - a_2) L_t^{1-\epsilon}) \right]^{\frac{1}{1-\epsilon}}. \quad (2)$$

$z$  is an iid productivity shock.<sup>1</sup> The agent's resource constraint is

$$k_{1t+1} = y_t - c_t. \quad (3)$$

In addition, capital depreciates at the rate  $\mu \in [0, 1]$  so that

$$k_{2t+1} = \mu k_{1t}. \quad (4)$$

BR prove that  $c_t = \lambda y_t$  is the agents policy function for consumption, where

$$(1 - \lambda) = \left[ a_1 \delta E_t (z_{t+1}^{1-\epsilon}) + a_2 \delta^2 \mu^{1-\epsilon} E_t (z_{t+2}^{1-\epsilon}) \right]^{\frac{1}{\epsilon}}. \quad (5)$$

The crucial assumption that allows for this solution is that the agent's preference parameter  $\epsilon$  (her coefficient of relative risk aversion) equals the reciprocal of the elasticity of substitution of the production function.

As BR note, the extension to the general case is straightforward. In the case of an infinite number of vintages  $k_j$  being related to each other via

$$k_{j+1t+1} = \mu k_{jt} \quad (6)$$

the production function (2) may be written as

$$y_t = z_t \left[ \sum_{j=1}^{\infty} a_j k_{jt}^{1-\epsilon} + \left( 1 - \sum_{j=1}^{\infty} a_j \right) L_t^{1-\epsilon} \right]^{1/(1-\epsilon)}, \quad \sum_{j=1}^{\infty} a_j < 1. \quad (7)$$

The general solution for the savings rate  $1 - \lambda$  at time  $t = 0$  is then given by (see the Appendix)

$$1 - \lambda = \left[ \sum_{j=1}^{\infty} a_j \delta^j \mu^{(j-1)(1-\epsilon)} E (z_{t+j}^{1-\epsilon}) \right]^{\frac{1}{\epsilon}}. \quad (8)$$

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<sup>1</sup>BR assume that  $z$  is governed by a first-order Markov process  $z_{t+1} = z_t^\theta \gamma_t$ , where  $\theta \in (0, 1)$  and  $\gamma_t$  is log-normally distributed. Yet, as we will demonstrate in the Appendix, this contradicts the assumption of a constant savings rate.

An alternative interpretation of this framework is to use a traditional CES production function with labor  $L_t$  and capital  $K_t$  as inputs,

$$y_t = z_t [\alpha L_t^{1-\epsilon} + (1 - \alpha) K_t^{1-\epsilon}]^{\frac{1}{1-\epsilon}}, \quad (9)$$

and to assume adjustment costs of capital that give raise to the transition function

$$K_t = [\beta K_{t-1}^{1-\epsilon} + (1 - \beta) k_{t-1}^{1-\epsilon}]^{\frac{1}{1-\epsilon}}, \quad (10)$$

where  $k_t$  denotes investment from foregone consumption in period  $t$ . Inserting (10) repeatedly into (9) yields

$$y_t = \left[ (\alpha L_t^{1-\epsilon} + (1 - \alpha) \beta \sum_{j=1}^{\infty} (1 - \beta)^{j-1} k_{t-j}^{1-\epsilon}) \right]^{\frac{1}{1-\epsilon}}. \quad (11)$$

This production technique equals (7) if  $a_j \mu^{(j-1)(1-\epsilon)} = (1 - \alpha) \beta (1 - \beta)^{j-1}$ .

Thus, the savings rate at time  $t = 0$  is given by

$$1 - \lambda = \left[ \sum_{j=1}^{\infty} (1 - \alpha) \beta (1 - \beta)^{j-1} \delta^j E(z_{t+j}^{1-\epsilon}) \right]^{\frac{1}{\epsilon}}. \quad (12)$$

### 3 Conclusion

SDGE models featuring an analytical solution are helpful for the applied researcher because he can use this solution as a starting point for the computation of the solution of more complicated models. In this note, we have shown that the class of SDGE models provided by BR can be interpreted as a more traditional SDGE model with adjustment costs of capital. This interpretation can also be seen as a generalization of the well known closed form solution to the Ramsey problem (see e.g. McCallum (1989)) with log utility, Cobb-Douglas production and a capital accumulation equation given by  $K_t = K_{t-1}^\beta k_{t-1}^{1-\beta}$  arising from (10) if  $\epsilon$  equals unity.

## Appendix

The Lagrangian of the agent's problem to maximize

$$E_0 \sum_{t=0}^{\infty} \delta^t u(c_t, 1 - L_t)$$

subject to (7) and (6) may be written as

$$\begin{aligned} \mathcal{L} = E_0 \left\{ \sum_{t=0}^{\infty} \left[ \frac{c_t^{1-\epsilon} - 1}{1-\epsilon} + w(1 - L_t) \right] \right. \\ \left. + \Gamma_0 \left[ z_0 (a_1 k_{10}^{1-\epsilon} + \dots + b L_0^{1-\epsilon})^{\frac{1}{1-\epsilon}} - c_0 - k_{11} \right] \right. \\ \left. + \delta \Gamma_1 \left[ z_1 (a_1 k_{11}^{1-\epsilon} + \dots + b L_1^{1-\epsilon})^{\frac{1}{1-\epsilon}} - c_1 - k_{12} \right] \right. \\ \left. + \delta^2 \Gamma_2 \left[ z_2 (a_1 k_{12}^{1-\epsilon} + a_2 \mu k_{11}^{1-\epsilon} \dots + b L_2^{1-\epsilon})^{\frac{1}{1-\epsilon}} - c_2 - k_{13} \right] \right. \\ \left. + \dots \right\}, \end{aligned}$$

where  $b := \sum_{j=1}^{\infty} a_j$ . Differentiating this expression with respect to  $c_t$  provides

$$c_t^{-\epsilon} = \Gamma_t.$$

The derivative with respect to  $k_{12}$  yields:

$$\Gamma_0 = E_0 \left\{ \delta \Gamma_1 + \frac{\partial y_1}{\partial k_{12}} + \delta^2 \Gamma_2 \frac{\partial y_2}{\partial k_{22}} \mu + \delta^3 \Gamma_3 \frac{\partial y_3}{\partial k_{33}} \mu^2 + \dots \right\}.$$

Since

$$\frac{\partial y_t}{\partial k_{it}} = z_t^{1-\epsilon} a_i \left( \frac{y_t}{k_{it}} \right)^\epsilon$$

the above two equations may be combined to yield

$$\begin{aligned} c_0^{-\epsilon} = E_0 \left\{ \delta c_1^{-\epsilon} z_1^{1-\epsilon} a_1 (y_1/k_{12})^\epsilon + \delta^2 \mu c_2^{-\epsilon} z_2^{1-\epsilon} a_2 (y_2/k_{22})^\epsilon \right. \\ \left. + \delta^3 \mu^2 c_3^{-\epsilon} z_3^{1-\epsilon} a_3 (y_3/k_{33})^\epsilon + \dots \right\}. \end{aligned} \tag{A1}$$

Assume  $c_t = \lambda y_t$  for some constant  $\lambda$  so that (6) implies

$$k_{jt} = \mu^{j-1}(1 - \lambda)y_0.$$

Inserting this into (A1) and rearranging yields

$$(1 - \lambda) = E_0 \left\{ \delta a_1 z_1^{1-\epsilon} + \delta^2 \mu^{1-\epsilon} a_2 z_2^{1-\epsilon} + \delta^3 \mu^{2(1-\epsilon)} a_3 z_3^{1-\epsilon} + \dots \right\}.$$

which reduces to (12) if  $a_j \mu^{(j-1)(1-\epsilon)} = (1 - \alpha)\beta(1 - \beta)^{j-1}$ . Note that the savings rate  $\lambda$  cannot be a constant if  $z_t$  is not iid. If  $z_t$  follows a first-order Markov process  $E_0 z_{t+j}^{1-\epsilon}$  equals  $E[(z_{t+j}|z_0)^{1-\epsilon}]$  which depends upon  $z_0$  and thus contradicts the assumption of constant savings rate.

## 4 References

Benhabib, J. and A. Rustichini (1994): A Note on a New Class of Solutions to Dynamic Programming Problems arising in Economic Growth. *Journal of Economic Dynamics and Control*, 18, 807-813.

Heer, B. and A. Maußner (2005): *Dynamic General Equilibrium Models: Computational Methods and Applications*. Springer.

McCallum, B. T. (1989): *Monetary Economics: Theory and Policy*. Prentice Hall.