Behaviors and Controllability

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Abstract

This note proposes a topological framework for the analysis of the time shift on behaviors and its asymptotics as time tends to infinity. The relations to controllability properties are explored.

Keywords: nonlinear control systems, controllability, behaviors

1 Introduction

Following a proposal by Jan Willems we consider dynamical systems defined by a set $\mathcal{B}$ of functions on a time set T and with values in a signal alphabet W. We restrict attention to time-invariant systems where T is an (additive) subgroup of $\mathbb{R}$ and $\mathcal{B}$ is invariant under the time shift. There are various ways to add more structure to this hopelessly general definition (which) nevertheless ... captures the crucial features of the notion of a dynamical system' [4]. In the present paper we intend to complement the algebraic theory of behaviors by a topological framework which allows us to use concepts from (classical) topological dynamical systems theory (see, e.g., Robinson [3]) in order to analyze controllability. This approach may be viewed as an extension of the theory of control flows and control sets as presented in [1]. In fact, also many proofs are analogous.

Our controllability notion is based-as put forward by Jan Willems-on concatenability. This will be related to the shift dynamical system on the behavior and its asymptotics as time tends to infinity. For simplicity, we write down everything for the continuous time case $T = \mathbb{R}$ only. Section 2 introduces the basic definitions and some fundamental properties. Section 3 discusses the relation between behavioral control sets and topological transitivity of the shift dynamical system. Furthermore, behavioral chain control sets are characterized as the maximal chain transitive sets. Section 4 presents a simple example.

2 Behaviors and Time Shifts

In this section behaviors are defined in a topological context and the shift dynamical system and behavioral control sets are introduced. As an example, control-affi ne nonlinear systems with input constraints and outputs are discussed, and it is shown how behavioral control sets here arise from control sets in the state space.

We will only consider bounded behaviors, i.e., subsets of $L_\infty(\mathbb{R}, \mathbb{R}^d)$. Recall (see, e.g., [1]) that the weak* topology on $L_\infty(\mathbb{R}, \mathbb{R}^d)$ is the weakest topology such that for all $\alpha \in L_1(\mathbb{R}, \mathbb{R}^d)$ the maps

$$L_\infty(\mathbb{R}, \mathbb{R}^d) \to \mathbb{R},\ w \mapsto \int_{\mathbb{R}} w(t)^T \alpha(t)\ dt$$

are continuous. Then for every bounded set in $L_\infty(\mathbb{R}, \mathbb{R}^d)$ the induced topology is metrizable. Fixing a countable dense subset $(\alpha_i) \subset L_1(\mathbb{R}, \mathbb{R}^d)$, such a metric is given by

$$\mathrm{d}(v, w) = \sum_{i=1}^{\infty} 2^{-i} \frac{\left|\int_{\mathbb{R}} [v(t) - w(t)]^T \alpha_i(t)\ dt\right|}{1 + \left|\int_{\mathbb{R}} [v(t) - w(t)]^T \alpha_i(t)\ dt\right|}. \tag{1}$$

Define the time shift Θ by

$$\Theta : \mathbb{R} \times L_\infty(\mathbb{R}, \mathbb{R}^d) \to L_\infty(\mathbb{R}, \mathbb{R}^d),\ (t, w) \mapsto (\Theta_t w)(s) = w(t + s),\ s \in \mathbb{R}.$$

Now we are ready to give the following definition of topological behaviors.

Definition 1 *A behavior is a weak* compact and $\Theta-$invariant subset $\mathcal{B} \subset L_\infty(\mathbb{R}, \mathbb{R}^d)$.*

Here $\Theta-$invariance means that $w \in \mathcal{B}$ implies $\Theta_t w \in \mathcal{B}$ for every $t \in \mathbb{R}$. Note that a weak* compact subset of $L_\infty(\mathbb{R}, \mathbb{R}^d)$ is bounded, hence a metric on $\mathcal{B}$ is given by (1). Define the behavior fbw as the restriction of the time shift Θ to the behavior $\mathcal{B}$.

The following proposition follows by a minor modification of the proof of Lemma 4.2.4 in [1].

Proposition 2 *The behavior fbw Θ is a continuous dynamical system on $\mathcal{B}$, i.e., $\Theta : \mathbb{R} \times \mathcal{B} \to \mathcal{B}$ is continuous and $\Theta_{t+s} = \Theta_t \circ \Theta_s$ for all $s, t \in \mathbb{R}$ and* $\Theta_0 = \mathrm{id}$.

Next we illustrate behaviors by showing how they arise from control-affi ne systems with outputs. Consider a smooth control system on a Riemannian manifold M

$$\dot{x}(t) = f_0(x(t)) + \sum_{i=1}^{m} u_i(t) f_i(x(t)), \tag{2}$$

$$y(t) = h(x(t), u(t)) = h_0(x(t)) + \sum_{i=1}^{m} u_i(t) h_i(x(t)),$$

with inputs (u_i) taking values in a compact convex subset $U \subset \mathbb{R}^m$; furthermore, the f_i are smooth vector fields on M and the output functions $h_i : M \to \mathbb{R}^k$ are also smooth. Assume that for every $x \in M$ and input function u in

$$\mathcal{U} = \{u \in L_\infty(\mathbb{R}, \mathbb{R}^m),\ u(t) \in U \text{ for almost all } t \in \mathbb{R}\}$$

there exists an absolutely continuous global solution $\varphi(t, x, u),\ t \in \mathbb{R}$.

Proposition 3 *Consider system (3) and let $K \subset M$ be compact. Then*

$$\mathcal{B} = \left\{ (u, y) \in \mathcal{U} \times L_\infty(\mathbb{R}, \mathbb{R}^k), \begin{array}{c} \text{there is } x \in M \text{ such that for all } t \in \mathbb{R} \\ \varphi(t, x, u) \in K \text{ and } y(t) = h(\varphi(t, x, u), u(t)) \end{array} \right\}$$

defines a behavior.

Proof. We have to show that $\mathcal{B} \subset L_\infty(\mathbb{R}, \mathbb{R}^{m+k})$ is compact and $\Theta-$invariant. Invariance is obvious by definition. The set $\mathcal{U} \subset L_\infty(\mathbb{R}, \mathbb{R}^m\}$ is compact by [1, Lemma 4.2.1] and the map

$$(x, u) \mapsto (\varphi(t, x, u), u(t + \cdot)) : K \times \mathcal{U} \to K \times \mathcal{U}$$

is continuous by [1, Lemma 4.3.2]. Now compactness of $\mathcal{B}$ follows, since K is compact and h is control affi ne. ■

Remark 4 *The restriction to behaviors in $L_\infty(\mathbb{R}, \mathbb{R}^d)$ may appear as a very restrictive a-priori boundedness assumption. However, as is easily seen in the preceding example, the existence of unbounded outputs is not excluded. We just restrict attention to bounded behaviors. Furthermore, if one uses weighted Lebesgue measures, also unbounded behaviors can be admitted.*

In the rest of the paper, we suppose that a behavior $\mathcal{B}$ is given. A definition for controllability for behaviors has been proposed by Jan Willems. We adapt it to our situation in the following way.

Definition 5 *A $\Theta-$invariant subset $\mathcal{D} \subset \mathcal{B}$ is a behavioral control set if it is a maximal set with the following property:*
For all $v, w \in \mathcal{D}$ there are $w_1 \in \mathcal{D}$ and a time $T_1 > 0$ satisfying

$$w_1(t) = \begin{cases} v(t) & \text{for} \quad t \le 0 \\ w(t - T_1) & \text{for} \quad t \ge T_1 \end{cases} .$$

Remark 6 *For control-affi ne systems as considered in Proposition 3, a control set D in the state space M with nonvoid interior is a maximal set of approximate controllability. If the system is locally accessible, one easily sees that the closure of the following set contains a behavioral control set $\mathcal{D}$:*

$$\left\{(u,y)\in\mathcal{B},\ \begin{array}{c}\text{there is } x\in\operatorname{int}D \text{ with } \varphi(t,x,u)\in\operatorname{int}D\\ \text{and } y(t)=h(\varphi(t,x,u),u(t)) \text{ for all } t\in\mathbb{R}\end{array}\right\}.$$

Definition 5 is not 'dynamic' in the sense that it is defined in terms of the natural associated dynamical system which is the time shift Θ on the set $\mathcal{B}$ of behaviors. In the next section we will establish such a connection between controllability and the time shift.

3 Analysis of the Shift

Here we discuss some topological properties of the time shift on the behaviors.

Basic properties of flows on compact metric spaces are encoded in the topologically transitive and mixing sets, in the chain recurrent components, the attractors, and their relations (cp. [3]). For the behavioral control flow, a set $\mathcal{D}\subset\mathcal{B}$ is topologically transitive if there is $w_0\in\mathcal{D}$ with

$$\mathcal{D}=\omega(w_0):=\{w\in\mathcal{B},\ \text{there are } t_k\to\infty \text{ with } \Theta_{t_k}(w_0)\to w \text{ for } k\to\infty\}.$$

A set $\mathcal{D}\subset\mathcal{B}$ is topologically mixing if for all open $V^1,V^2\subset\mathcal{D}$ there is $T>1$ with $\Theta_T(V^1)\cap V^2\neq\varnothing$.

Theorem 7 *Let $\mathcal{D}$ be a behavioral control set. Then the closure $\operatorname{cl}\mathcal{D}$ is a compact invariant set for Θ and the flow Θ restricted to $\operatorname{cl}\mathcal{D}$ is topologically mixing and topologically transitive.*

Proof. The closure of $\mathcal{D}$ is compact and invariant, since $\mathcal{D}$ is an invariant subset of the compact set $\mathcal{B}$. Topological transitivity follows from topological mixing; see, e.g., [1]. Hence we only have to show that for every pair V^1, V^2 of open sets in $\mathcal{D}$ there exists $T_0>1$ such that $\Theta_{T_0}(V^1)\cap V^2\neq\emptyset$. We may for $j=1,2$ assume that V^j has the form $V^j=V(w^j)\cap\mathcal{D}$ with

$$V(w^j)=\left\{w\in\mathcal{B},\ \left|\int_{\mathbb{R}}\left[w(s)-w^j(s)\right]^T\alpha_{ij}(s)\,ds\right|<\varepsilon \text{ for } i=1,...,k_j\right\};$$

here α_{ij} are elements of $L_1(\mathbb{R},\mathbb{R}^d)$. There is $T>1$ such that for all j and i

$$\int_{\mathbb{R}\setminus[-T,T]}|\alpha_{ij}(s)|\,ds<\frac{\varepsilon}{2\operatorname{diam}\mathcal{B}};\tag{3}$$

note that $\operatorname{diam}\mathcal{B}:=\sup\{\|v-w\|_\infty,\ v,w\in\mathcal{B}\}<\infty$, since weak* compact subsets of $L_\infty(\mathbb{R},\mathbb{R}^d)$ are bounded. By controllability there are $w\in\mathcal{D}$ and a time $t_0>0$ satisfying

$$w(t):=\left\{\begin{array}{ccc}w^2(t) & \text{for} & t\leq T\\ w^1(t-t_0-2T) & \text{for} & t>T+t_0\end{array}\right..$$

Then using (3) one sees that $w \in V(u^2)$ and $w(t_0 + 2T + \cdot) \in V(w^1)$. Thus the assertion follows with $T_0 := t_0 + 2T$. ■

Remark 8 *It is not clear, if (by maximality of behavioral control sets) one obtains a maximal topologically transitive set for the time shift.*

The definition of a behavioral control set requires that one can precisely hit'the function w after some time. It may appear natural to introduce the following weaker concept, in analogy to chain controllability in the state space. Hopefully, also in the present situation this will lead to sets which are better behaved. Observe that again this definition is *not* given in the fbw context; it is strictly analogous to the definition of behavioral control sets. Define a semi-distance on $\mathcal{B}$ (taking into account only the future part of behaviors) by

$$\mathrm{d}^+(v,w) = \sum_{i=1}^{\infty} 2^{-i} \frac{\left| \int_0^\infty [v(t) - w(t)]^T \alpha_i(t)\ dt \right|}{1 + \left| \int_0^\infty [v(t) - w(t)]^T \alpha_i(t)\ dt \right|}.$$

Definition 9 *For $\varepsilon, T > 0$ an $(\varepsilon, T)^+-$chain from $v \in \mathcal{B}$ to $w \in \mathcal{B}$ is given by*

$$n \in \mathbb{N}, w_0 = v, w_1, ..., w_n = w \in \mathcal{B}, T_0, ..., T_{n-1} \geq T,$$

such that

$$\mathrm{d}^+(\Theta_{T_i}(w_i), w_{i+1}) < \varepsilon \text{ for all } i.$$

If for all $\varepsilon, T > 0$ there is an $(\varepsilon, T)^+-$chain from $v \in \mathcal{B}$ to $w \in \mathcal{B}$, we say that v is chain controllable to w.

We will consider maximal sets of behaviors which are chain controllable.

Definition 10 *An invariant subset $\mathcal{E} \subset \mathcal{B}$ is a behavioral chain control set if it is a maximal set such that for all $v, w \in \mathcal{E}$ and all $\varepsilon, T > 0$ there is an $(\varepsilon, T)^+-$chain in $\mathcal{E}$ from v to w.*

For these sets, contrary to behavioral control sets, we will be able to provide a complete characterization in terms of the fbw. Recall from the theory of dynamical systems (see [3]) that an $(\varepsilon, T)-$chain for a continuous fbw is defined as in Definition 9, but with the semidistance d^+ replaced by the distance d in the metric space. They give rise to chain transitive sets in analogy to Definition 10.

Theorem 11 *Let $\mathcal{B}$ be a behavior. Then a nonempty invariant set $\mathcal{E} \subset \mathcal{B}$ is a behavioral chain control set if and only if the restriction of the time shift to $\mathcal{E}$ is chain transitive and $\mathcal{E}$ is a maximal set with this property.*

Proof. Suppose that $\mathcal{E}$ is a behavioral chain control set. Let $v, w \in \mathcal{E}$ and pick ε, $T > 0$. Recall the definition of the metric d on $\mathcal{B}$ and choose $k \in \mathbb{N}$ large enough such that

$$\sum_{i=k+1}^{\infty} 2^{-i} < \varepsilon. \tag{4}$$

For the finitely many $\alpha_1, ..., \alpha_k \in L_1(\mathbb{R}, \mathbb{R}^d)$ there is $S > 0$ such that for all i

$$\int_{\mathbb{R}\setminus[-S,S]} |\alpha_i(\tau)| \ d\tau < \frac{\varepsilon}{\text{diam}\mathcal{B}}. \tag{5}$$

We may assume without loss of generality that $T > S$. Chain controllability from v to $w(-S + \cdot)$ yields the existence of $n \in \mathbb{N}$ and $v_0, ..., v_n \in \mathcal{E}$, $T_0, ..., T_{n-1} > T + S$ with $v_0 = v$, $v_n = w(-S + \cdot)$ and

$$\mathrm{d}^+(\Theta_{T_i} v_i, v_{i+1}) < \varepsilon \text{ for } j = 0, ..., n-1. \tag{6}$$

Now construct an (ε, T)-chain from v to w in the following way (we jump later). Define

$$w_0 = v, \ w_j = \Theta_S v_j \text{ for } j = 1, ..., n-1, \ w_n = \Theta_S v_n = w,$$

and let the jump times be $t_j = T_j + S$. Then

$$\begin{aligned}
&\mathrm{d}(\Theta_{t_0} w_0, w_1) = \mathrm{d}(\Theta_{T_0+S} v, \Theta_S v_1) \\
&= \sum_{i=1}^{\infty} 2^{-i} \frac{\left| \int_{\mathbb{R}} [v(t+T_0+S) - v_1(t+S)]^T \alpha_i(t) \ dt \right|}{1 + \left| \int_{\mathbb{R}} [v(t+T_0+S) - v_1(t+S)]^T \alpha_i(t) \ dt \right|} \\
&\leq \sum_{i=1}^{k} 2^{-i} \frac{\left| \int_{\mathbb{R}} [v(t+T_0+S) - v_1(t+S)]^T \alpha_i(t) \ dt \right|}{1 + \left| \int_{\mathbb{R}} [v(t+T_0+S) - v_1(t+S)]^T \alpha_i(t) \ dt \right|} + \varepsilon.
\end{aligned}$$

Now for $i = 1, ..., k$

$$\begin{aligned}
&\left| \int_{\mathbb{R}} [v(t+T_0+S) - v_1(t+S)]^T \alpha_i(t) \ dt \right| \\
&\leq \int_{\mathbb{R}\setminus[-S,S]} |\alpha_i(\tau)| \ d\tau \ 2\text{diam}\mathcal{B} + \left| \int_{-S}^{S} [v(t+T_0+S) - v_1(t+S)]^T \alpha_i(t) \ dt \right| \\
&< 2\varepsilon + \left| \int_0^{2S} [v(t+T_0) - v_1(t)]^T \alpha_i(t) \ dt \right| < 5\varepsilon,
\end{aligned}$$

since by (6) and (5)

$$\begin{aligned}
&\left| \int_0^{2S} [v(t+T_0) - v_1(t)]^T \alpha_i(t) \ dt \right| \\
&= \left| \int_0^{\infty} [v(t+T_0) - v_1(t)]^T \alpha_i(t) \ dt - \int_{2S}^{\infty} [v(t+T_0) - v_1(t)]^T \alpha_i(t) \ dt \right| \\
&\leq \left| \int_0^{\infty} [v(t+T_0) - v_1(t)]^T \alpha_i(t) \ dt \right| + 2\varepsilon < 3\varepsilon.
\end{aligned}$$

Thus

$$\mathrm{d}(\Theta_{t_0} w_0, w_1) < 6\varepsilon.$$

Analogously, one shows that $\mathrm{d}(\Theta_{t_j} w_j, w_{j+1}) < 6\varepsilon$ for all $j = 1, ..., n-1$. This proves that the restriction of Θ to the behavioral chain control set $\mathcal{E}$ is chain transitive.

Conversely, suppose that $\mathcal{E}$ is a chain transitive set, and let $v, w \in \mathcal{E}$. By assumption one finds for all $\varepsilon, T > 0$ an $(\varepsilon, T)-$chain given by $v_0 = v, v_1, ..., v_n = \Theta_S w$ in $\mathcal{E}$ and $T_0, ..., T_{n-1} > T$ from v to w with

$$\mathrm{d}(\Theta_{T_i}(v_i, v_{i+1}) < \varepsilon. \tag{7}$$

We may assume that conditions (4) and (5) are satisfied and that $T_j - S > T$. This gives rise to an $(\varepsilon, T)^+-$chain in the following way ('we jump earlier').

Define

$$w_0 = v,\ w_j = \Theta_{-S} v_j \text{ for } j = 1, ..., n-1,\ w_n = \Theta_{-S} v_n = w,$$

and let the jump times be $t_j = T_j - S$. Then

$$\begin{aligned}
&\mathrm{d}^+(\Theta_{t_0} w_0, w_1) = \mathrm{d}^+(\Theta_{T_0 - S} v, \Theta_{-S} v_1) \\
&= \sum_{i=1}^{\infty} 2^{-i} \frac{\left| \int_0^\infty [v(t + T_0 - S) - v_1(t - S)]^T \alpha_i(t)\ dt \right|}{1 + \left| \int_0^\infty [v(t + T_0 - S) - v_1(t - S)]^T \alpha_i(t)\ dt \right|} \\
&\leq \sum_{i=1}^{k} 2^{-i} \frac{\left| \int_0^\infty [v(t + T_0 - S) - v_1(t - S)]^T \alpha_i(t)\ dt \right|}{1 + \left| \int_0^\infty [v(t + T_0 - S) - v_1(t - S)]^T \alpha_i(t)\ dt \right|} + \varepsilon.
\end{aligned}$$

Now for $i = 1, ..., k$

$$\begin{aligned}
&\left| \int_0^\infty [v(t + T_0 - S) - v_1(t - S)]^T \alpha_i(t)\ dt \right| \\
&\leq \int_{2S}^\infty |\alpha_i(\tau)|\ d\tau\ 2\mathrm{diam}\mathcal{B} + \left| \int_0^{2S} [v(t + T_0 - S) - v_1(t - S)]^T \alpha_i(t)\ dt \right| \\
&< 2\varepsilon + \left| \int_{-S}^{S} [v(t + T_0) - v_1(t)]^T \alpha_i(t)\ dt \right| < 5\varepsilon,
\end{aligned}$$

since by (7) and (5)

$$\begin{aligned}
&\left| \int_{-S}^{S} [v(t + T_0) - v_1(t)]^T \alpha_i(t)\ dt \right| \\
&= \left| \int_{\mathbb{R}} [v(t + T_0) - v_1(t)]^T \alpha_i(t)\ dt - \int_{\mathbb{R} \setminus [-S, S]} [v(t + T_0) - v_1(t)]^T \alpha_i(t)\ dt \right| \\
&\leq \left| \int_{\mathbb{R}} [v(t + T_0) - v_1(t)]^T \alpha_i(t)\ dt \right| + 2\varepsilon < 3\varepsilon.
\end{aligned}$$

Thus

$$\mathrm{d}^+(\Theta_{t_0} w_0, w_1) < 6\varepsilon.$$

Analogously, one shows that $\mathrm{d}^+(\Theta_{t_j} w_j, w_{j+1}) < 6\varepsilon$ for all $j = 1, ..., n-1$.

It only remains to show the maximality properties. A chain control set $\mathcal{E}$ is a maximal chain transitive set: Suppose that the restriction of Θ to $\mathcal{E}' \supset \mathcal{E}$ is chain transitive. Then it follows that $\mathcal{E}' = \mathcal{E}$, since chain transitivity of $\mathcal{E}'$ implies, as just proven, that $\mathcal{E}'$ is chain controllable and $\mathcal{E}$ is a maximal chain controllable set. In the same way, one sees that a behavioral chain control set $\mathcal{E}$ is a maximal set with the property that the restriction of Θ to $\mathcal{E}$ is chain transitive. ■

4 An Example

The following simple example (taken from Gayer [2], Example 27) illustrates that in the context of system (2) one cannot find behavioral control sets just by looking at the property, if one can steer the system from points y_1 to y_2 in the output space.

Consider the single well potential

$$V(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3$$

and the corresponding oscillator (sometimes also called ‘escape equation’)

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -\gamma x_2 - x_1 + x_1^2 + u(t)$$

with $u(t) \in U = [-\rho, \rho]$ and parameter values

$$\rho = 0.041, \ \gamma = 0.1.$$

Then (see Figure 1) in the state space $\mathbb{R}^2$ there is an invariant control set C (around the stable equilibrium $(0,0)$ for $u \equiv 0$) and a variant control set D (containing the hyperbolic equilibrium $(1,0)$ for $u \equiv 0$ and a loop around C). This is only a numerical approximation; however, one can prove that there is a $\rho-$value, where two control sets of this form occur. If we consider the output

$$y = h(x_1, x_2, u) = x_1,$$

and fix a compact set $K \subset \mathbb{R}^2$ containing C and $\mathrm{cl}D$, Proposition 3 yields a behavior $\mathcal{B} \subset L_\infty(\mathbb{R}, \mathbb{R}) \times L_\infty(\mathbb{R}, \mathbb{R})$. Then each control set in $M = \mathbb{R}^2$ give rise to a behavioral control set. The projections of D and C to the output space $\mathbb{R}$ overlap. Hence looking at the corresponding points in $\mathbb{R}$ does not allow us to identify the corresponding behavioral control sets (naturally, this is different, if we look at pieces of outputs $y(t), \ t$ in some interval; this also determines $\dot{y}(t)$ and hence the control set.)

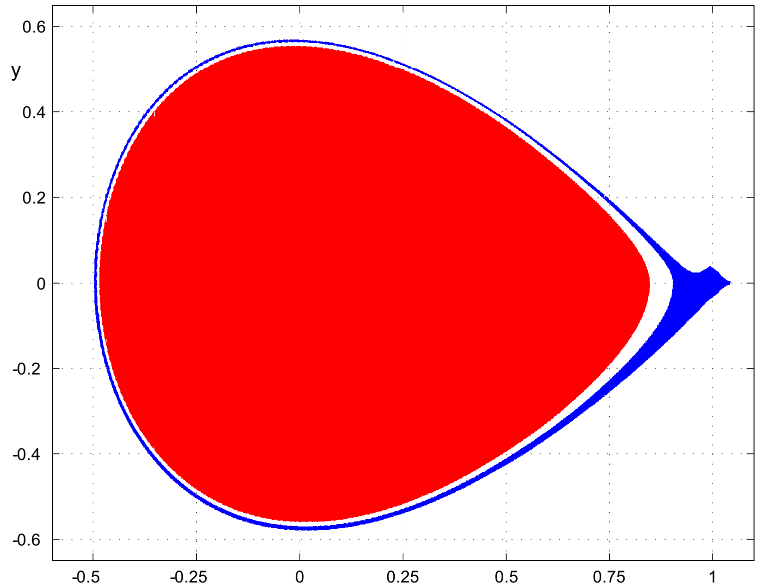

Figure 1: Two control sets for the escape equation

5 Conclusions

The paper presents a step towards the study of controllability properties of general behaviors. Controllability is related to the asymptotics of the time shift as time tends to infinity. This is analogous to the state space theory. In fact, in the behavioral context this relation may appear even more natural. In particular, this is true for the slightly weakened chain version.

Further topics include the role of geometric properties, in particular, differential flatness, and conditions implying that chain control sets coincide with control sets. Furthermore, linearized behaviors can be analyzed via Lyapunov exponents. This allows, in particular, for an analysis of asymptotic controllability properties by linearization.

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