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Nonlinear Systems with Multiplicative and Additive Perturbation under State Space Constraints*

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Abstract

Realistic models of mechanical systems often depend on various parameters, such as controlled inputs, material constants, tunable parameters, and uncertainties. Uncertain parameters can be (time varying) deterministic perturbations or stochastic excitations whose influence on the system depends on the perturbation dynamics (multiplicative or additive), the perturbation range, and its statistics in the stochastic case. For a given operating region of the system, i.e. for a set of state space constraints, the behavior of the system within this region depends strongly on the type of perturbation dynamics and on its range. We present some basic theory for additively and multiplicatively perturbed systems, where the uncertainty can be a family of time varying functions, or a Markov diffusion process. The uncertainty range plays the role of a bifurcation parameter and determines concepts like discontinuities of control sets and supports of invariant measures, stability radii, and invariance radii with respect to the constraint set. It turns out that in many instances the stochastic and the deterministic bifurcation scenarios agree, and the cases in which they differ are related to a nonuniform behavior of the stochastically perturbed system. The example of a model of ship roll motion is treated in detail, revealing some of the fundamental agreements and disagreements of the two bifurcation scenarios.

1. Introduction

Uncertain linear and nonlinear systems are currently studied extensively from different points of view, such as the theory of dynamical systems, perturbation theory, stochastic systems, and control theory. Starting from a nominal system $\dot{x} = X_0(x)$, a typical model for this analysis can be written in the form

$$\dot{x} = X_0(x) + \sum_{i=1}^{m} u_i(t) X_i(x) \qquad \text{in } \mathbf{R}^d \tag{1.1}$$

where $(u_i)_{i=1,\dots m} = u$ is the (time varying) deterministic or stochastic excitation, which through its own dynamics (the vector fields $X_1, ..., X_m$) acts upon the nominal system. If the perturbation vector fields are constant, one talks about additive perturbations, otherwise the term multiplicative excitation (noise, uncertainty, perturbation) is used.

In this paper we study the behavior of the system (1.1) under state space constraints, i.e. in a compact set $L \subset \mathbf{R}^d$. In particular, we analyze invariant sets in L, characterize the regions from which the system may exit from L, and describe the convergence and stability behavior inside L. There are many practical problems that lead to considering systems under state space constraints. one of these is the modeling of L as an operating region, and reaching the boundary ∂L of L means that the system enters a different operating region. E.g. in the context of reliability theory reaching ∂L is interpreted as abrupt failure. Therefore the qualitative behavior of the system inside L exhibits the dynamics of the (normally) operating system.

Usually, operating systems depend on a variety of parameters, such as tunable control parameters, material parameters, feedback or feedforward controls, or excitation parameters like the range and/or statistics of the perturbation. We concentrate here on varying excitation ranges and note that other parameter variations require similar theories, except for the case of feedback design which needs additional considerations. Hence we attempt a study of the bifurcation behavior of the system (1.1) under state space constraints, where the bifurcation parameter is the excitation range, and the uncertainty can be deterministic or a stochastic (Markov) process.

For such a study one has to distinguish the various ways in which the excitation can affect the nominal

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system. Basically, one considers two cases: In the regular situation the uncertainty changes the limit sets of the nominal system, such as e.g., fixed points of the vector field X_0 are not fixed points anymore for all perturbations in (1.1). This effect occurs, e.g., for additive perturbations. In the singular case, there exists a limit set of X_0 which is a common limit set for all vector fields $X_0, ..., X_m$. Different techniques are required for these two cases. We will discuss both situations (under additional assumptions) and study an example for which a combination of regular and singular excitation occurs.

Sction 2. presents the mathematical model in detail and describes some of the theory for regular systems. Section 3. contains results on the singular case. Both sections are written with regards to the example of a model of ship roll motion, which is discussed in Sections 4. (additive perturbation) and 5. (multiplicative perturbation). In each section, we first deal with the case of time varying deterministic uncertainties and then discuss its implications for stochastic excitations.

2. Regularly Perturbed Systems under State Space Constraints

Consider the uncertain system

$$\dot{x} = X_0(x) + \sum_{i=1}^{m} u_i(t) X_i(x) \qquad \text{in } \mathbb{R}^d \tag{2.1}$$

where $X_0, ..., X_m$ are smooth vectorfields. The excitation range is given as follows: Let $U \subset \mathbb{R}^m$ be convex, compact with $0 \in int\, U$ (the interior of U) and define a parametrized family of subsets of $\mathbb{R}^m$ via

$$U^\rho = \rho \cdot U, \qquad \rho \geq 0$$

For $\rho = 0$ we obtain the nominal system

$$\dot{x} = X_0(x) \tag{2.2}$$

For the deterministic perturbation model we consider $(u_i)_{i=1...m} = u$ as

$$u \in \mathcal{U}^\rho = \{u : \mathbb{R} \to U^\rho, \text{ measurable}\} \tag{2.3}$$

The stochastic perturbation is assumed to be a function of an underlying stochastic differential equation

$$\begin{aligned} d\eta &= Y_0(\eta)\, dt + \sum_{j=1}^{n} Y_j(\eta) \circ dW_j \text{ on } N \quad (2.4) \\ u(t) &= f^\rho(\eta(t)), \qquad f^\rho : N \to U^\rho \text{ onto} \end{aligned}$$

Here $Y_0, ..., Y_n$ are smooth vector fields on the smooth compact manifold N, $(W_j)_{j=1,...n}$ is a standard n-dimensional Wiener process and '$\circ$' denotes the symmetric (Stratonovič) stochastic differential. We assume throughout that $\{\eta(t),\ t \geq 0\}$ is a nondegenerate Markov diffusion process, which hence has a unique stationary and ergodic (Markov) solution in N with invariant measure ν which satisfies $supp\, \nu = N$. (The notation '*supp*' stands for the support of a measure). In the following, $\eta(t)$ always denotes this unique solution. The measure on the probability space that supports $\eta(t)$ (e.g., via the Kolmogorov construction) is called P.

The state space constraints for the system (2.1) are given by a compact, connected set $L \subset \mathbf{R}^d$ with $int\, L \neq \emptyset$ and $cl(int\, L) = L$, where 'cl' stands for the closure of a set. In general, the set L will not be forward invariant for (2.1). Since we are interested in the behavior of the system in L, we stop the trajectories at the boundary ∂L, i.e., let $\varphi(t, x, u)$ be the solution of (2.1) at time t with initial value $\varphi(0, x, u) = x$ under the perturbation $u \in \mathcal{U}^\rho$. For $x \in L$ let $\tau(x, u) = min\{t \geq 0,\ \varphi(t, x, u) \notin L\}$ and $\sigma(x, u) = max\{t \leq 0,\ \varphi(t, x, u) \notin L\}$, then $\varphi(\tau(x, u), x, u) \in \partial L$ and $\varphi(\sigma(x, u), x, u) \in \partial L$. We set

$$\varphi_L(t, x, u) = \begin{cases} \varphi(\sigma(x,u), x, u) & \text{for } t < \sigma(x, u) \\ \varphi(t, x, u) & \text{for } t \in [\sigma(x, u), \tau(x, u)] \\ \varphi(\tau(x,u), x, u) & \text{for } t > \tau(x, u) \end{cases} \tag{2.5}$$

With this notation the equation (2.1) has a unique solution $\varphi_L(\cdot, x, u)$ for all $(u, x) \in \mathcal{U}^\rho \times L$ and all $t \in \mathbb{R}$. Similarly, we denote by a subscript 'L' all quantities concerning the constrained trajectories φ_L.

Regularly perturbed systems are those systems for which the excitation dynamics changes the limit structure of the nominal system. We impose the following, slightly stronger condition

$$\dim \mathcal{LA}\{X_0, ..., X_m\}(x) = d \tag{H}$$

for all x in an open neighborhood of $L \subset \mathbb{R}^d$, where $\mathcal{LA}\{X_0, ..., X_m\}$ stands for the Lie algebra generated by the vector fields $X_0, ..., X_m$ and $\dim \mathcal{LA}\{X_0, ..., X_m\}$ is the dimension of the distribution of this Lie algebra in the tangent space at $x \in \mathbb{R}^d$. To study the dependence of the system (2.1), (2.3) on the parameter ρ we will assume that for all $0 < \rho < \rho'$ for all chain control sets E^ρ of (2.1), (2.3) the following inner pair condition (I) holds:

For all $(u, x) \in \mathcal{U}^\rho \times E^\rho$ with $\varphi(\cdot, x, u) \subset E^\rho$ there exists $T > 0$ such that $\varphi(T, x, u) \in int\, \mathcal{O}^{+,\rho'}(x)$

Here $\mathcal{O}^{+,\rho'}(x)$ denotes the forward orbit of x for the range $U^{\rho'}$,

$$\mathcal{O}^{+,\rho'}(x) = \{y \in \mathbb{R}^d, \begin{array}{c}\text{there is } t \geq 0 \text{ and } u \in \mathcal{U}^{\rho'} \\ \text{with } y = \varphi(t, x, u)\end{array}\}$$

For a discussion of control sets, chain control sets, and inner pairs see [3]. While condition (I) is not needed for all of the following results, it is convenient to assume it throughout this section, as it simplifies the formulation of results.

We start with the study of the deterministic uncertain system (2.1), (2.3). All the information about this system is contained in the family of control flows

$$\begin{array}{rcl}\Phi^\rho & : & \mathbb{R} \times \mathcal{U}^\rho \times L \longrightarrow \mathcal{U}^\rho \times L, \\ \Phi^\rho(t, u(\cdot), x) & = & (u(t + \cdot), \varphi_L(t, x, u))\end{array}$$

Since we are interested in the behavior of the system in $L \subset \mathbb{R}^d$, we will only introduce concepts with respect to the state space $\mathbb{R}^d$ alone. But the reader should keep in mind that most of the proofs require working with the (skew product) flow Φ^ρ.

Our philosophy is to start from the analysis of the nominal system (2.2) and study the behavior of the perturbed system for increasing $\rho \geq 0$. The limit sets (more precisely, the Morse decompositions) of the vector field X_0 together with the order between them gives a picture of the global behavior of (2.2) in the set L. For simplicity of notation we assume that the flow of X_0 has a (unique) finest Morse decomposition with Morse sets E_i^0, $i = 1, ..., l$, in $int\, L$, i.e., X_0 has only finitely many attractors. Then the Morse sets coincide with the chain recurrent components. The order on the Morse sets is induced by

$$E_i^0 \prec E_j^0 \quad \begin{array}{c}\text{if there exists } x \in \mathbb{R}^d \text{ with } \omega^*(0, x) \subset E_i^0 \\ \text{and } \omega(0, x) \subset E_j^0\end{array}$$

where $\omega^*(u, x)$ and $\omega(u, x)$ denote the limit sets of $\varphi(\cdot, x, u)$ for $t \to -\infty$ and for $t \to \infty$, respectively.

For varying perturbation range U^ρ, $\rho \geq 0$, the following maps are well defined (under Assumptions (H) and (I)) for $i = 1, ..., l$:

$$\begin{array}{rcl} E_i & : & [0, \infty) \longrightarrow \mathcal{C}(L), \\ & & E_i^\rho \text{ is the closure of a control set } D_i(\rho) \text{ with} \\ E_i^0 & \subset & int\, D_i(\rho) \end{array} \qquad (2.6)$$

Here $\mathcal{C}(L)$ denotes the compact subsets of L (with the Hausdorff topology) and the control sets are formed with respect to the constraint trajectories (2.5). The maps E_i defined above are right continuous and strictly increasing. For ρ small, the reachability order of the control sets $D_i(\rho)$ agrees with the order on the Morse sets E_i^ρ, $i = 1, ..., l$. In particular, maximal (i.e. invariant or closed) control sets correspond to maximal attractors, while minimal (i.e. open) control sets correspond to minimal repellers of X_0. At the continuity points of the maps E_i, the control sets (and the collection of limit sets) vary continuously in ρ, i.e. at these points there is no bifurcation of control sets. So we direct our attention to the discontinuity points for which there may exist limit sets of the perturbed system (2.1), (2.3) outside of the closures of control sets with nonvoid interior.

The discontinuities in (2.6) can be induced either by the global time varying dynamics of the system (see [5] or Section 5. below), for which one needs a case by case study of the system. Or they result from the bifurcation behavior of the family $X_0 + \sum_{i=1}^m u_i X_i$ of vector fields with $u \in \mathbb{R}^m$ as a bifurcation parameter. The following theorem gives a result in this direction, which will be useful in Section 4 .

Theorem 1. *Assume that there exist two Morse sets $E_1^0 \neq E_2^0$ of the vector field X_0 and a continuous path $\alpha : [0, 1] \to int\, U$ with $\alpha(0) = 0$, $\alpha(1) = \bar{u}$ such that $\{E_j(\alpha(s)),\ s \in [0, 1]\}$, $j = 1, 2$, are continuous families of Morse sets of $X_0 + \sum_{i=1}^m \alpha_i(s) X_i$ with $E_j(\alpha(0)) = E_j^0$. If $E_1(\alpha(1)) = E_2(\alpha(1))$ then there exists $\rho^* \in (0, 1)$ which is a discontinuity point of the maps E_1 and E_2 defined in (2.6).*

Proof.

(idea) For $s \in [0, 1]$ denote $\rho(s) = min\{\rho \geq 0,\ \alpha(\sigma) \in U^\rho \text{ for all } \sigma \in [0, s]\}$. Then there exist, for all $s \in [0, 1]$, control sets $D_j(s)$ for the control range $U^{\rho(s)}$ such that $E_j(\alpha(s)) \subset int\, D_j(s)$. In particular, we have $D_1(1) = D_2(1)$ with range $\rho(1)$. On the other hand, since $E_1^0 \neq E_2^0$, there exists $\bar{\rho} > 0$ such that $E_1(\rho) \neq E_2(\rho)$ for all $\rho \in [0, \bar{\rho})$, where E_i are the maps defined in (2.6). Since the family $\{E_j(\alpha(s)),\ s \in [0, 1]\}$ is continuous, we have for all $s \in [0, 1]$ that $E_j(\alpha(t)) \subset int\, D_j(s)$ for $t < s$. Continuity of α implies that $\rho(s)$ is continuous in s. Hence there exists a discontinuity point $\rho^* \in (\bar{\rho}, \rho(1))$ of the maps E_1 and E_2. ■

The second important concept, besides the control sets described in (2.6), is the set of multistable points (see [?]). Since we are working with the constraint set

L, we need to modify slightly the definition of these points.

Definition 2. *Consider the system (2.1), (2.3) and its restriction (2.5) to the compact set L. Denote by C_α, $\alpha \in I$, the invariant control sets of the constrained system in $int\, L$, and let $\mathcal{C} = \{C_\alpha, \alpha \in I\} \cup \{\partial L\}$. A point $x \in int\, L$ is called multistable if there exist for $i = 1, 2$, $C_i \in \mathcal{C}$, $u_i \in \mathcal{U}$, and $t_i \geq 0$ such that $\varphi(t_i, x, u_i) \in C_i$ with $C_1 \neq C_2$. We denote the set of multistable points by MS.*

Using this definition, we can prove the following statements analogously to [?]: $MS \neq \emptyset$ if and only if either (2.1) is forward invariant in L and $card(I) \geq 2$, or (2.1) is not forward invariant in L and $card(I) \geq 1$. Furthermore, if $MS \neq \emptyset$, then $MS = \bigcup \mathcal{O}^-(D)$, where D runs over the (finitely many) relatively invariant control sets in L. As a consequence, we can characterize the invariant sets in $int\, L$:

Theorem 3. *Consider the system (2.1), (2.3) with its trajectories (2.5) constrained to L. Denote by MS_1 the set of multistable points defined only via invariant control sets in $int\, L$, and let $MS_2 = MS \setminus MS_1$.*
(i) The interior $int\, L$ is positively invariant if and only if there exists at least one invariant control set in $int\, L$ and $MS = MS_1$.
(ii) For a point $x \in int\, L$ there exists a trajectory $\varphi(\cdot, x, u)$ that reaches the boundary ∂L in finite time if and only if there is no invariant control set in $int\, L$ or $MS_2 \neq \emptyset$ and $x \in \mathcal{O}^-(\partial L)$.

Proof.

(idea) (i) If there exists an invariant control set in $int\, L$ and $int\, L$ is not positively invariant, then $A = \{x \in int\, L,$ *there is* $u \in \mathcal{U}$ *with* $\varphi(t, x, u) \in \partial L$ *for some* $t > 0\} \neq \emptyset$. Hence $\bigcup_{\alpha \in I} \mathcal{O}^-(C_\alpha) \cap \mathcal{O}^-(A) \neq \emptyset$ (since $L = cl(int\, L)$ is compact and connected), and therefore $MS_2 \neq \emptyset$, which contradicts $MS = MS_1$. This proves one direction.

Definition 4. *(ii) The direction '$\Rightarrow$' and the second part of '$\Leftarrow$' are obvious. If there is no invariant control set in $int\, L$, then for all $x \in int\, L$ there is $u \in \mathcal{U}$ with $\omega_L(u, x) \subset \partial L$. Hence by (I) we have $x \in \mathcal{O}^-(\partial L)$, which also proves the second part of (i).* ■

Theorem 3 says, in particular, that there exist forward invariant sets in $int\, L$ if and only if there exists an invariant control set in $int\, L$. For an invariant control set $C \subset int\, L$ we define its strict domain of attraction as

$$\mathbf{A}^s(C) := \mathcal{O}^-(C) \setminus MS \qquad (2.7)$$

Then we obtain from Theorem 3 the maximal forward invariant set $J \subset int\, L$ as $J = \bigcup \mathbf{A}^s(C) \cup MS_1$, where the union is taken over all invariant control sets C in $int\, L$.

The long term behavior of points in $int\, L$ can now be characterized.

Corollary 5. *The set*

$$\left\{(u, x) \in \mathcal{U} \times int\, L, \begin{array}{c}\text{there exists } t \geq 0 \text{ such} \\ \text{that } \varphi(t, x, u) \in \bigcup\{C,\ C \in \mathcal{C}\}\end{array}\right\}$$

is open and dense in $\mathcal{U} \times int\, L$ (with the weak* topology in $\mathcal{U}$), and the set

$$\left\{(u, x) \in \mathcal{U} \times int\, L, \begin{array}{c}\text{there exists } t \geq 0 \text{ such} \\ \text{that } \varphi(t, x, u) \in \bigcup_{\alpha \in I} C_\alpha\end{array}\right\}$$

is open and dense in $\mathcal{U} \times J$, with J and C_α from Definition 2.

The proof of this corollary is completely analogous to the proof of [3, Theorem 6.2].

The results above allow us to develop a concept for regular systems that is analogous to the stability radius for singular systems. For a maximal Morse set E^0 of the nominal system (2.2) and for a family $\{\mathcal{U}^\rho,\ \rho \geq 0\}$ of perturbations the idea is to find the mallest ρ such that a point in E^0 is not L-invariant for the system (2.1), (2.3).

Definition 6. *Consider the perturbed system (2.1), (2.3), and let $E^0 \subset int\, L$ be a maximal Morse set (i.e. an attractor) of the nominal system (2.2). The invariance radius $r_{inv}(E^0, L)$ of E^0 with respect to the constraint set $L \subset \mathbb{R}^d$ is defined as*

$$r_{inv}(E^0, L) = inf\left\{\rho \geq 0, \begin{array}{c}\text{there exist } (u, x) \in \mathcal{U}^\rho \times E^0 \text{ such} \\ \text{that } \varphi(t, x, u) \in \partial L \text{ for some } t \geq 0\end{array}\right\}$$

Note that the non-invariant points in L were characterized in Theorem 3(ii) in terms of control sets and multistability regions. Under Assumptions (H) and (I) we know that each Morse set E_i^0 of (2.2) is contained in the interior of some control set D_1^ρ of (2.1), (2.3) for each $\rho > 0$. Hence the invariance radius $r_{inv}(E^0, L)$ is the infimum of the ρ's such that for $x \in E^0$ we have $x \in int\, \mathcal{O}^{+,\rho}(MS_2)$ or the system $(2.1)^\rho$ has no invariant control set in $int\, L$. If one $x \in E^0$ is in $\mathcal{O}^{+,\rho}(MS_2)$ then, of course, all $x \in E^0$ are in this set.

Remark 1. *Recall the maps E_i defined in (2.6). These maps are right continuous and strictly increasing. Hence for any attractor $E^0 \subset int\, L$ of (2.2) the invariance radius $r_{inv}(E^0, L)$ is strictly positive.*

Remark 2. *There are two possible scenarios for the system behavior around $r_{inv}(E^0, L)$, according to the characterization above. One is that the invariant control set D^ρ with $E^0 \subset int\, D^\rho$ crosses the boundary ∂L, i.e., for all $\rho < r_{inv}(E^0, L)$ we have $D^\rho \subset int\, L$, while $D^\rho \cap \partial L \neq \emptyset$ for all $\rho > r_{inv}(E^0, L)$. In this case it is possible that the invariance radius of E^0 is strictly smaller than the first discontinuity point of the corresponding map from (2.6).*

Remark 3. *In the second scenario we have $E^0 \subset \mathcal{O}^{+,\rho}(MS_2)$ if and only if $\rho > r_{inv}(E^0, L)$, which means that the invariance radius is greater or equal than the first discontinuity point of the corresponding map from (2.6). Hence the discontinuity of the maps (2.6) are not directly related to the invariance radius with respect to the constraint set L, but they (and hence the bifurcation behavior of the control sets) give a first important impression of the behavior of the system.*

We now turn to the stochastic system (2.1), (2.4) and draw some consequences for its behavior in the constraint set L from the results above. In addition to the Assumptions (H) and (I) for the deterministic systems, we need a condition on the interplay between the vector fields of (2.1) and of the background noise (2.4):

$$\dim \mathcal{LA}\{X_0 + \sum_{i=1}^m u_i X_i + Y_0, Y_1, ..., Y_n, u \in U^\rho\}(x, y) = \dim N + d \tag{H'}$$

for all $(x, y) \in N \times M(L)$, where $M(L)$ in an open neighborhood of $L \subset \mathbb{R}^d$. Note that this condition implies (H). It allows us to use the support theorem for the pair process $(\eta(t), x(t))$ as a Markov diffusion solution of the equations (2.1), (2.4). We continue to use the notation (2.5) for the stopped process at the boundary ∂L of the constraint set.

The pair process $(\eta(t), x(t))$ has a unique invariant measure μ_α, i.e., unique stationary, ergodic Markov solutions on each set $N \times C_\alpha$, where C_α is an invariant control set of (2.1), (2.3) in $int\, L$, compare Definition 2. For each μ_α the marginal on N is the invariant measure ν on N. Hence for each $x \in \mathbf{A}^s(C_\alpha)$, the strict domain of attraction of C_α, compare (2.7), the solution of (2.1), (2.4) converges in distribution to μ_α (ergodic theorem). For the multistable points $x \in MS_1$ (compare Theorem 3), we have: There exist $p_\alpha \geq 0$ such that $\sum p_\alpha = 1$ and the solution of (2.1), (2.4) converges in distribution to the measure $\sum p_\alpha \mu_\alpha$. Here p_α is the probability of reaching C_α from x, i.e. $p_\alpha > 0$ if and only if there exist $u \in \mathcal{U}^\rho$ and $t \geq 0$ with $\varphi(t, x, u) \in C_\alpha$. In particular, we have $P\{\varphi(t, x, \omega) \in \partial L \text{ for some } t \geq 0\} > 0$ if and only if $x \in int\, L$ satisfies Theorem 3(ii).

With these observations we obtain interpretations of the discontinuity points of the maps E_i in (2.6), and of the invariance radius in Definition 6.

Remark 4. *Let $E^0 \subset int\, L$ be a maximal Morse set (attractor) of the nominal system (2.2). Consider the corresponding map $E(\rho)$ defined in (2.6). For small ρ, $E(\rho)$ is an invariant control set of (2.1), (2.3) and hence $N \times E(\rho)$ carries a unique invariant measure μ of (2.1), (2.4), which is the distribution of the unique stationary and ergodic Markov solution in $N \times E(\rho)$. Furthermore supp $\mu = N \times E(\rho)$. At the first discontinuity point ρ^* of $E(\rho)$ the support changes abruptly. If for $\rho > \rho^*$ the control set in $E(\rho)$ is not invariant, then the ergodic solution 'disappears' and the solution starting from E^0 leaves the set $E(\rho)$ w. p. 1. Their long term behavior then depends on wether $E^0 \subset \mathcal{O}^{+,\rho}(MS_2)$, as described above.*

Remark 5. *Let $E^0 \subset int\, L$ be a maximal Morse set of the nominal system (2.2), and let $r_{inv}(E^0, L)$ be its invariance radius with respect to the constraint set L. Then we have by the results above: $\rho > r_{inv}(E^0, L)$ if and only if $P\{\varphi(t, x, \omega) \in \partial L \text{ for some } t \geq 0\} > 0$ for all $x \in E^0$, where the stochastic excitation is given by $u(t, \omega) = f^\rho(\eta(t, \omega))$. Hence the stochastic and the deterministic invariance radii agree, if we define the stochastic radius as the infimum of the ρ's such that the boundary ∂L can be reached with positive probability from E^0 under the excitation $f^\rho(\eta)$ of size ρ. At this point, further statistical characterizations of the exit behavior of (2.1), (2.4) from the operating region become important, namely the exit probability, the exit time, and the exit location. For the general class of systems that is considered here one cannot expect to obtain explicit expressions for these quantities. Numerical and statistical studies in this direction are under way and will be presented elsewhere.*

3. Singularly Perturbed Systems under State Space Constraints

The theory developed in Section 2. depends on the fact that the Morse sets of the nominal system (2.2) are contained in the interior of control sets of the perturbed system (2.1), (2.3) for any perturbation range $\rho > 0$. This fact does not hold for singular systems and hence different techniques are required for the study of uncertain systems with singular limit sets. Linearization and Lyapunov exponents are standard tools for the analysis of dynamical systems, and we present an extension of these ideas to uncertain systems.

Consider the perturbed system (2.1), (2.3) and let $x^* \in int\, L$ be a singular fixed point of (2.1), i.e., $X_i(x^*) = 0$ for $i = 1,...,m$. The linearized system at x^* reads

$$\dot{y} = A_0 y + \sum_{i=1}^{m} u_i(t) A_i y \text{ in } \mathbb{R}^d,\ u \in \mathcal{U}^\rho,\ \rho \geq 0 \quad (3.1)$$

where $A_i = D_x X_i(x^*)$ is the Jacobian of the vector field X_i at x^*. In order to avoid degeneracies, we assume a Lie algebra rank condition for the system (3.1) projected onto projective space $\mathbb{P}^{d-1}$ in $\mathbb{R}^d$:

$$\dim \mathcal{LA}\{h(\cdot,u),\ u \in U^\rho\}(p) = d-1 \quad (\mathrm{H}_{lin})$$

for all $p \in \mathbb{P}^{d-1}$, where $h(p,u) = h_0(p) + \sum_{i=1}^{m} u_i h_i(p)$ and $h_i(p) = \left[A_i - p^T A_i p \cdot I\right] p$ is the projection of Ay onto $\mathbb{P}^{d-1}$.

The Lyapunov exponents of (3.1) are defined as

$$\lambda(u,y) = \limsup_{t \to \infty} \log |\psi(t,y,u)|$$

where $\psi(t,y,u)$ is the solution of (3.1) at time t with initial value $\psi(0,y,u) = y \in \mathbb{R}^d \setminus \{0\}$. If $\lambda(u,y) < 0$ then $\psi(\cdot,y,u)$ is exponentially stable. Hence we are interested in the maximal Lyapunov exponent of (3.1) which determines exponential stability of all solutions of (3.1):

$$\kappa(\rho) = \sup_{u \in \mathcal{U}^\rho} \sup_{y \in \mathbb{R}^d \setminus \{0\}} \lambda(u,y)$$

For individual time varying excitations $u \in \mathcal{U}^\rho$ negativity of $\sup_{y \in \mathbb{R}^d \setminus \{0\}} \lambda(u,y)$ may not imply (uniform) exponential stability of (3.1), but it can be shown that $\kappa(\rho) < 0$ is equivalent to exponential stability of (3.1) with perturbation range U^ρ.

The function $\kappa : [0,\infty) \to \mathbb{R}$ is continuous and leads to the definition of the (linear) stability radius of (3.1)

$$r_{lin} = inf\{\rho \geq 0,\ \kappa(\rho) \geq 0\}$$

If the matrix A_0 is stable (i.e., all eigenvalues of A_0 have negative real parts), then $r_{lin} > 0$ by continuity of $\kappa(\rho)$.

Returning to the nonlinear system (2.1), (2.3) with singular fixed point $x^* \in int\, L$ we define the nonlinear stability radius at x^* as

$$r(x^*) = inf\left\{\rho \geq 0,\ \begin{array}{l}\text{there is } u \in \mathcal{U}^\rho \text{ such that (2.1)} \\ \text{is not asymptotically stable at } x^*\end{array}\right\} \quad (3.2)$$

A uniform stable manifold theorem for nonlinear uncertain systems allows us to show that

$$r_{lin} \leq r(x^*) \leq inf\,\{\rho \geq 0,\ \kappa(\rho) \geq 0\} \quad (3.3)$$

and hence we have $r_{lin} = r(x^*)$ if $\kappa(\rho)$ is strictly monotone at $\rho = r_{lin}$.

The interpretation of these results for the behavior of the system (2.1), (2.3) around a singular point which is stable for the nominal system (2.2) is as follows: If $\rho \in [0, r(x^*))$ then there exists an open neighbohood $N(x^*)$ of x^* such that $\varphi(t,z,u) \longrightarrow x^*$ for $t \to \infty$ for all $z \in N(x^*)$ and for all $u \in \mathcal{U}^\rho$. For $\rho > r(x^*)$ there exists for all neighborhoods $N(x^*)$ a point $z \in N(x^*)$ and a $u \in \mathcal{U}^\rho$ such that $x^* \notin \omega(u,z)$. Hence the omega limit set $\omega(u,z)$ will be contained in some chain control set of (2.1), (2.3) that is not the set $\{x^*\}$. In this sense one can consider the stability radius $r(x^*)$ as the analogue (for singular points) of the first discontinuity of a map $E_i(\rho)$ as defined in (2.6). In particular, if for $\rho > r(x^*)$ the chain control sets of (2.1), (2.3) are the closures of control sets, then there exists a chain control set $D(\rho)$ such that $x^* \in cl\, \mathcal{O}^{-,\rho}(D(\rho))$ for all $\rho > r(x^*)$. This means that for $\rho = r(x^*)$ we have a collision of $cl\, \mathcal{O}^{-,\rho}(D(\rho))$ with the singular fixed point x^*.

The invariance radius for a singular point $x^* \in int\, L$ with respect to the constraint set $L \subset \mathbb{R}^d$ is defined (in analogy to Definition 6) as

$$r_{inv}(x^*,L) = inf\left\{\rho \geq 0,\ \begin{array}{l}\text{for all neighborhoods } N(x^*) \text{ there} \\ \text{exist } z \in N(x^*) \text{ and } u \in \mathcal{U}^\rho \text{ such} \\ \text{that } \varphi(t,x,u) \in \partial L \text{ for some } t \geq 0\end{array}\right\} \quad (3.4)$$

It follows from this definition that $r(x^*) \leq r_{inv}(x^*,L)$ (compare Remark 2 for regular systems). A characterization of the invariance radius (3.4) in terms off the qualitative behavior of the system (2.1), (2.3) in L leads to results that are analogous to the comments and remarks after Definition 6, we skip the details.

We now turn to the study of systems with singular points under stochastic excitation of the type (2.4). Let $x^* \in int\, L$ be a singular point of (2.1) and consider the linearization (3.1). We continue to assume that the Lie algebra rank condition (H_{lin}) holds for the induced system on the projective space $\mathbb{P}^{d-1}$. In this case we obtain for the Lyapunov exponents of the linearized system that they are constant with probability one, i.e., for each $\rho > 0$ there exists $\lambda(\rho) \in \mathbf{R}$ such that

$$\lambda(\omega, y) \equiv \lambda(\rho) \text{ a.s. for all } y \in \mathbf{R}^d \setminus \{0\}$$

Obviously, we have $\lambda(\rho) \leq \kappa(\rho)$, and $\lambda(\rho)$ depends on the dynamics (2.4) of the noise process $\eta(t)$ and on the maps $f^\rho : N \to U^\rho$, which map the background noise $\eta(t)$ onto the perturbation ranges U^ρ.

The maximal Lyapunov exponent $\kappa(\rho)$ can be recovered for any nondegenerate noise process (2.4) satisfying (H') and (H_{lin}) as the limit of the p-th moment Lyapunov exponents $\gamma(p, \rho) \to \gamma(\rho) = \kappa(\rho)$ as $p \to \infty$. We define the almost sure stability radius of the linearization (3.1) with respect to $\eta(t)$ and f^ρ as

$$r_{ls}(\eta, f) = inf\{\rho \geq 0,\ \lambda(\rho) \geq 0\} \tag{3.5}$$

The observations above imply that

$$r_{ls}(\eta, f) \geq r_{lin}$$

and

$$r_{lin} = inf\,\{\rho \geq 0,\ \gamma(\rho) \geq 0\}$$

for any η, f^ρ as above.

Hence we have

- for $\rho < r_{lin}$: The deterministic system (3.1), (2.3) is exponentially stable; the stochastic system (3.1), (2.4) is a.s. and p-th moment stable for all $p > 0$ and for all η, f^ρ.
- for $r_{lin} < \rho < r_{ls}(\eta, f)$: there exists a $u \in \mathcal{U}^\rho$ such that the deterministic system $\dot{y} = [A_0 + \sum_{i=1}^m u_i(t) A_i] y$ is not stable; the stochastic system is a.s. stable, but there exists $p > 0$ such that the p-th moment of the system is not stable.
- for $\rho > r_{ls}(\eta, f)$: There exists $u \in \mathcal{U}^\rho$ such that the deterministic system with this perturbation is not stable; the stochastic system is a.s. unstable and all its p-th moments are unstable for $p > 0$.

As in the deterministic case one can use stochastic stable manifold theorems to investigate the stability radius of the nonlinear system (2.1), (2.4) at the singular point $x^* \in int\, L$. We define

$$r_{stoch}(x^*, \eta, f) = inf\{\rho \geq 0, \begin{array}{l}\text{the system (2.1), (2.4) is not}\\ \text{asymptotically stable}\\ \text{at } x^* \text{ under } f^\rho(\eta(t))\end{array} \}$$

A result of Pinsky [7] implies that

$$r_{ls}(\eta, f) \leq r_{stoch}(x^*, \eta, f)$$

and equality holds if $\lambda(\rho)$ is strictly increasing at $r_{ls}(\eta, f)$. But for ρ in the interval $(r(x^*), r_{stoch}(x^*, \eta, f))$ the stochastic stable manifolds $W^s(x^*, \omega)$ are not uniform in $\omega \in \Omega$. This means that there exists a measurable subset $\Omega' \subset \Omega$ with $P(\Omega') = 1$ such that for all $\omega \in \Omega'$ the system (2.1), (2.4) admits a stable manifold $M^s(x^*, \omega)$ with $x^* \in int\, M^s(x^*, \omega)$, but for every neighborhood $N(x^*)$ there are $z \in N(x^*)$ and $\Omega_z \subset \Omega'$ with $P(\Omega_z) > 0$ and $z \notin M^s(x^*, \omega)$ for $\omega \in \Omega_z$. This fact makes simulations of the system (2.1), (2.4) with these uncertainty ranges difficult around x^*. Of course, the stable manifolds are uniform in $\omega \in \Omega$ for $\rho < r(x^*)$.

The invariance radius $r_{inv}(x^*, L)$ of the singular fixed point x^* with respect to the constraint set L, as defined in (3.4), has a similar interpretation for stochastic excitations as in the regular case, compare Remark 5. In particular, we have:

$\rho > r_{inv}(x^*, L)$ if and only if for every neighborhood $N(x^*)$ there exists $z \in N(x^*)$ such that $P\{\varphi(t, z, \omega) \in \partial L \text{ for some } t \geq 0\} > 0$.

While it holds that $r(x^*) \leq r_{inv}(x^*, L)$, there may exist a background noise $\eta(t)$ and a family of maps $f^\rho : N \to U^\rho$ such that

$$r_{inv}(x^*, L) < r_{stoch}(x^*, \eta, f)$$

compare the example in Section 5. This fact reflects again the non-uniform stable manifolds of the stochastic system for $\rho > r(x^*)$.

4. A Model for Ship Roll Motion under Additive Perturbation

Capsizing of vessels can be modelled by a one degree of freedom system which is limited to the roll motion, compare Falzanaro/Shaw/Troesch [4], Hsieh/Troesch/Shaw [6], and Thompson/Rainey/Soliman [8]. The nominal model in non-dimensionalized form is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \alpha x_1^3 - \delta_1 x_2 - \delta_2 x_2 \,|x_2| \end{aligned} \tag{4.1}$$

where $\delta_1 > 0$ and $\delta_2 > 0$ represent the linear and quadratic viscous damping coefficients, respectively, and α denotes the strength of the nonlinearity. Capsizing occurs in this model when $|x_1|$ reaches $\frac{1}{\sqrt{\alpha}}$.

In this section we analyze the ship roll model under additive perturbation, i.e. we consider

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \alpha x_1^3 - \delta_1 x_2 - \delta_2 x_2 |x_2| + u(t) \quad (4.2) \\ u &\in U^\rho,\ U = [-1, 1],\ \rho \geq 0 \end{aligned}$$

It follows from the references above that the limit sets of (4.2) for constant $u \in U$ are fixed points, which are obtained as follows:

$x_2 = 0$ and x_1 is the solution of $-x_1 + \alpha x_1^3 + u = 0$

and hence we have with $A = \frac{2}{3\sqrt{3\alpha}}$

- for $u \in (-A, A)$: There are three fixed points with x_1-component $x_1^{(1)} < x_1^{(2)} < x_1^{(3)}$, where $(x_1^{(2)}, 0)$ is stable and $(x_1^{(i)}, 0)$ is unstable (hyperbolic) for $i = 1, 3$. For $u < 0$ it holds that $x_1^{(1)} < -1/\sqrt{\alpha}$, and $u > 0$ implies $x_1^{(3)} > 1/\sqrt{\alpha}$.
- for $u = A$: There are two fixed points with x_1-components $x_1^{(1)} = -\frac{2}{3\sqrt{\alpha}}$ and $x_1^{(2)} = \frac{1}{\sqrt{3\alpha}}$. The linearization at the point corresponding to $x_1^{(1)}$ has one negative and one zero eigenvalue, while the point $(x_1^{(2)}, 0)$ is unstable (hyperbolic).
- for $u = -A$: In this case $x_1^{(1)} = \frac{2}{3\sqrt{\alpha}}$ (hyperbolic) and $x_1^{(2)} = -\frac{1}{\sqrt{3\alpha}}$ (one negative and one zero eigenvalue).
- for $|u| > A$: there exists one (unstable, hyperbolic) fixed point with x_1-component $\left|x_1^{(1)}\right| > \frac{2}{\sqrt{3\alpha}}$.

Therefore, this system exhibits, with bifurcation parameter $u \in \mathbf{R}$, an S-bifurcation, with saddle-node points at $u_1 = -A$ and $u_2 = A$.

In this model it is natural to define the operating region as $|x_1| \leq \alpha_1$ for some $\alpha_1 > 0$. By the results above, it suffices to consider $|u| \leq 2A$, i.e. $\rho \in [0, 2A]$. For this range there exists a constant $B > 0$ such that $\dot{x}_2(x_1, x_2) < 0$ for all $(x_{1,2}) \in \mathbf{R}^2$ with $|x_1| \leq 1/\sqrt{\alpha}$ and $x_2 = B$, and $\dot{x}_2(x_1, x_2) > 0$ for all $|x_1| \leq 1/\sqrt{\alpha}$ and $x_2 = -B$. Hence we use the compact constraint set

$$L = \{(x_1, x_2) \in \mathbf{R}^2,\ |x_1| \leq 1/\sqrt{\alpha},\ |x_2| \leq B\}$$

Exit from L occurs on the boundaries $|x_1| = 1/\sqrt{\alpha}$ indicating capsizing.

We turn to the behavior of the system (4.2) in the region L under deterministic (time varying) perturbations. The system satisfies the Conditions (H) and (I) of Section 2, and hence we are in the regular case. The bifurcation analysis above and the results from Section 2. yield the following picture.

For $\rho = 0$ the phase portrait of the nominal system with

$$\alpha = 1.0,\ \delta_1 = \delta_2 = 1.0 \qquad (4.3)$$

is shown in Figure 1. As ρ increases, the control sets of (4.2) form around the fixed points of the nominal system. For small ρ the control set D^2 around $(x_1^2, 0)$ is invariant, while the control sets D^i around $(x_1^i, 0)$ are variant for $i = 1, 3$. The set of multistable points is given as $MS = \mathcal{O}^{-,\rho}(D^1) \cup \mathcal{O}^{-,\rho}(D^3)$, and we have $MS = MS_2$ in the notation of Theorem 3. Figure 2. shows, for $\rho = 0.3$, the invariant control set (in the center), and the regions of multistability. According to Theorem 3, $int\, L$ is not positively invariant and the points from which the system can reach the boundary ∂L are given by the multistability regions MS^1 and MS^3, and the points to the left of MS^1 and to the right of MS^3.

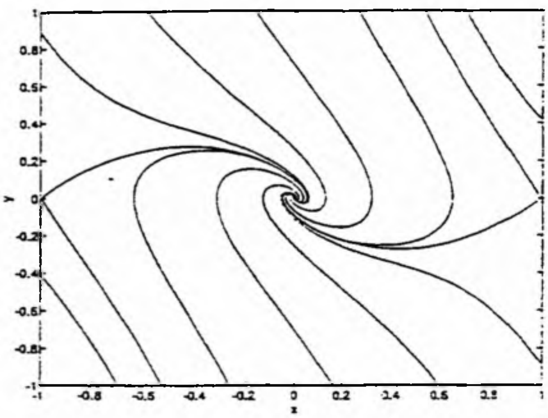

Figure 1: Phase portrait of the nominal system

By Theorem 1 we have that the first discontinuity point ρ^* of the map $D^2(\rho)$ (compare (2.6)) satisfies $0 < \rho^* \leq A$. It can be shown by looking at the vector fields in (4.2) that $\rho^* = A = \frac{2}{3\sqrt{3\alpha}}$ holds. Figure 3 shows the boundaries of $D_2(\rho)$ for various ρ-values, using the values (4.3). With these values we have $\rho^* = A = \frac{2}{3\sqrt{3}} \sim 0.3849$.

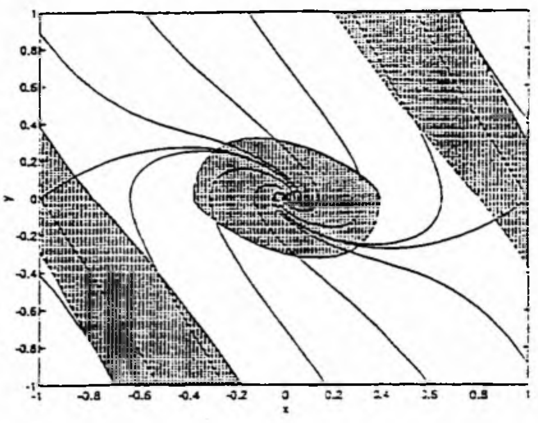

Figure 2: Invariant control set and multistability region for $\rho = 0.3$

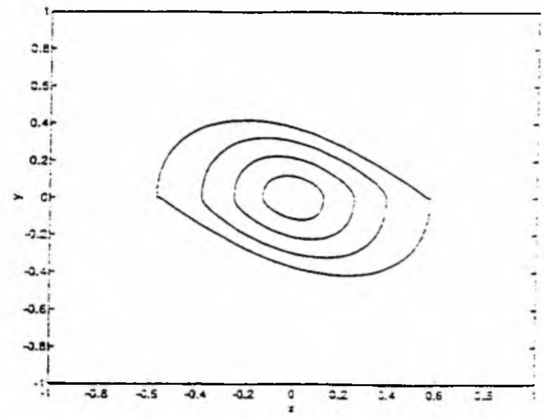

Figure 3: The invariant control sets for $\rho = 0.1, 0.2, 0.3, 0.3849$

For $\rho > \rho^*$ the invariant control set $D^2(\rho)$ merges with the two variant control sets $D^i(\rho)$, $i = 1, 3$, and it follows from Theorem 3 that there are no invariant points in $int\, L$. Hence, with Definition 6 we obtain for the invariance radius of the point $(0,0)$

$$\rho^* = r_{inv}((0,0), L) = A$$

Note that the first discontinuity of $D_2(\rho)$ as well as the invariance radius of the point $(0,0)$ depend only on α, but not on δ_1 or δ_2. The scenario of the system behavior for ρ around r_{inv} corresponds to the description in Remark 3. If one chooses the constraint set L via $|x_1| \leq C$ with $C < A$, then the scenario in Remark 2 occurs.

We now consider the stochastic situation, where the system (4.2) is additively perturbed by a background noise of the type (2.4). The results above and Section 2. imply for varying excitation range:

- $\rho \leq A$: there exists a unique invariant measure μ of the pair process $(\eta(t), x(t))$ with support $supp\, \mu = N \times D^2(\rho)$. Hence the system (4.2), (2.4) has a unique stationary and ergodic solution in L and for all initial values in the strict domain of attraction $\mathbf{A}^s(D^2(\rho))$ the solution converges in distribution to the invariant measure. The set $\mathbf{A}^s(D^2(\rho))$ is shown in Figure 2. as the area between the two multistability sets MS^1 and MS^3. For all points $x \in MS^1 \cup MS^3$ we have $P\{\varphi(t,x,\omega) \in \partial L \text{ for some } t \geq 0\} > 0$; to the left of MS^1 and to the right of MS^3 the points satisfy $P\{\varphi(t,x,\omega) \in \partial L \text{ for some } t \geq 0\} = 1$.
- $\rho > A$: Since for this excitation range there exists no invariant control set in $int\, L$, we obtain for all initial vlues $x \in int\, L$ that $P\{\varphi(t,x,\omega) \in \partial L \text{ for some } t \geq 0\} = 1$. Hence in this example the invariance radius is the excitation range which separates for the point $(0,0)$ a.s. invariance in L from a.s. exit from L. This shows that exit probabilities from L need not depend continuously on the excitation range, and may jump from 0 to 1.

5. A Model of Ship Roll Motion under Multiplicative Perturbation

In this section we study the model (4.1) for ship roll motion under multiplicative perturbations

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \alpha x_1^3 - \delta_1 x_2 - \delta_2 x_2 |x_2| - u(t) x_1 \quad (5.1) \\ u &\in \mathcal{U}^\rho,\ U = [-1,1],\ \rho \geq 0 \end{aligned}$$

The fixed points of (5.1) are given by

- $u \leq -1$: one unstable fixed point at $(0,0)$
- $u > -1$: three fixed points with x_2-component equal to zero, and x_1-component $x_1^{(1)} = -\sqrt{\frac{1+u}{\alpha}}$, $x_1^{(2)} = 0$, $x_1^{(3)} = \sqrt{\frac{1+u}{\alpha}}$. The points corresponding to $x_1^{(1)}$ and $x_1^{(3)}$ are unstable, the origin (0.0) is stable.

Therefore, if we view (5.1) as an equation with bifurcation parameter $u \in \mathbf{R}$, the system undergoes a pitchfork bifurcation at $u = -1$.

The origin $(0,0)$ is a fixed point of (5.1) for all $u \in \mathbf{R}$, and hence this is a singular point. In the rest of the state space $\mathbf{R}^2 \setminus \{(0,0)\}$ the system is regular. Note that (4.1) is the nominal model for (4.2) and (5.1), and hence for $\rho = 0$ the behavior of the two systems agrees.

We again consider an operating region of (5.1) given by $|x_1| \leq \frac{1}{\sqrt{\alpha}}$. Since for $u \leq -1$ the system with constant parameter u becomes unstable, it suffices to consider perturbations of the size $\rho \leq 2$. For this range there exists a constant $B > 0$ such that $\dot{x}_2(x_1, x_2) < 0$ for all $(x_1, x_2) \in \mathbf{R}^2$ with $|x_1| \leq \frac{1}{\sqrt{\alpha}}$ and $x_2 = B$, and $\dot{x}_2(x_1, x_2) > 0$ for $|x_1| \leq \frac{1}{\sqrt{\alpha}}$ and $x_2 = -B$. Hence the constraint set

$$L = \{(x_1, x_2) \in \mathbf{R}^2,\ |x_1| \leq \frac{1}{\sqrt{\alpha}} \text{ and } |x_2| \leq B\}$$

is appropriate for our purposes, and exits from L occur on the boundaries $|x_1| = \frac{1}{\sqrt{\alpha}}$.

We first analyze the behavior of (5.1) at the singular point $(0,0)$, using the theory presented in Section 3. Linearization of (5.1) at the origin yields

$$\dot{y} = \begin{pmatrix} 0 & 1 \\ -1 & -\delta_1 \end{pmatrix} y + u(t) \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} y \qquad (5.2)$$

which is a linear oscillator with positive damping δ_1. The corresponding projected system on the projective space $\mathbf{P}^1$ satisfies the Lie algebra rank condition (H_{lin}) for all $\rho > 0$, and therefore we can use the results from Section 3. We compute the maximal Lyapunov exponent $\kappa(\rho, \delta_1)$ (depending on the damping δ_1) numerically (compare [2]) and obtain the linear stability radius of (5.2) as the zero-level sets of $\kappa(\rho, \delta_1)$ for $\rho, \delta_1 \geq 0$. Figure 4. shows the radius $r_{lin}(\delta_1)$ for (5.2). Note that for $\delta_1 < \delta_1^* \sim 0.80$ the stability radius is strictly less than 1, which is the 'bifurcation radius' of the system (5.1) with $u \in \mathbf{R}$ as bifurcation parameter. This difference is due to the time varying nature of the perturbation, and differs from the behavior of the additive uncertainty model in Section 4., where the first discontinuity of the map $D^2(\rho)$ occurs at the bifurcation value $\rho = A$.

We now turn to the nonlinear system (5.1) and consider its behavior around the origin. Since the numerical calculations show that $\kappa(\rho, \delta_1)$ is strictly monotone in ρ for each $\delta_1 \geq 0$, we obtain from (3.3) for the nonlinear stability radius $r((0,0), \delta_1)$

$$r_{lin}(\delta_1) = r((0,0), \delta_1) \text{ for all } \delta_1 \geq 0$$

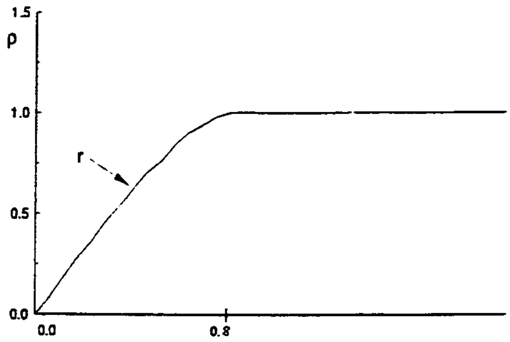

Figure 4: Stability radius of the linearized system depending on δ_1

Hence there exists for $\rho < r_{lin}(\delta_1)$ a uniform (in $u \in \mathcal{U}^\rho$) stable manifold $W^s((0,0), \rho, \delta_1)$ of the origin. The precise form of this stable manifold depends, of course, on the global dynamics of the nonlinear system (5.1). Choosing the parameter values

$$\alpha = 1.0,\ \delta_1 = 0.5,\ \delta_2 = 1.0 \qquad (5.3)$$

the stable manifold $W^s((0,0), \rho, \delta_1)$ for $\rho = 0.5$ is shown in Figure 5. Note that for individual perturbations $u \in \mathcal{U}^\rho$ the stable manifold may be larger than the uniform one, but it cannot exceed the asymptotic domain of attraction of $(0,0)$, indicated by the leftmost and the rightmost boundaries of the shaded regions in Figure 5.

The invariance properties of the system (5.1) with respect to the constraint set L depend on the global behavior of the system in L. Since (5.1) is regular outside the origin, we use the theory presented in Section 2. for this analysis.

For $\rho = 0$ the phase portrait of the nominal system is shown in Figure 1. As ρ increases, variant control sets $D^1(\rho)$ and $D^3(\rho)$ with their attached multistability regions form around the unstable fixed points $(x_1^1, 0)$ and $(x_1^3, 0)$. Figure 5. shows these multistability regions for the parameter values (5.3) and $\rho = 0.5$. The set of points in L that are L-invariant for all $u \in \mathcal{U}^\rho$ is given by the uniform stable manifold of the origin in the center of Figure 5.

For $\rho \sim 0.6$ a first (numerically observed) discontinuity of the maps $D^i(\rho)$, $i = 1, 3$, occurs and the two variant control sets merge into one. Figure 6. shows

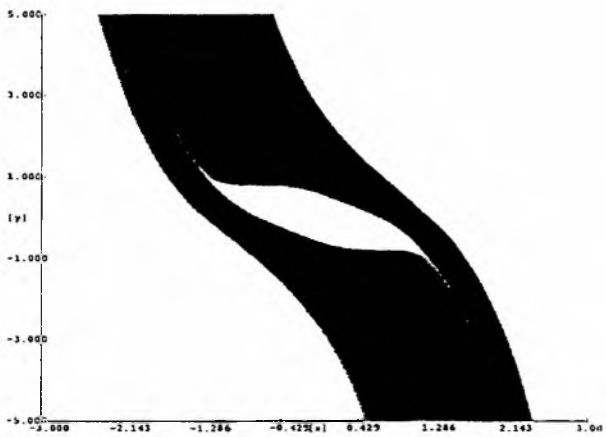

Figure 5: Uniform stable manifold $W^s((0,0),0.5,0.5)$

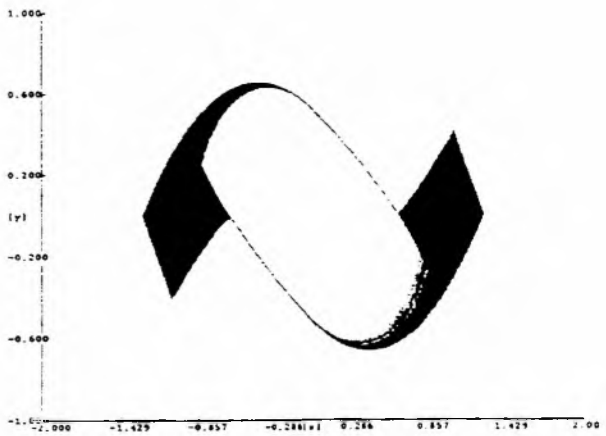

Figure 6: Variant control set with $\rho = 0.6$

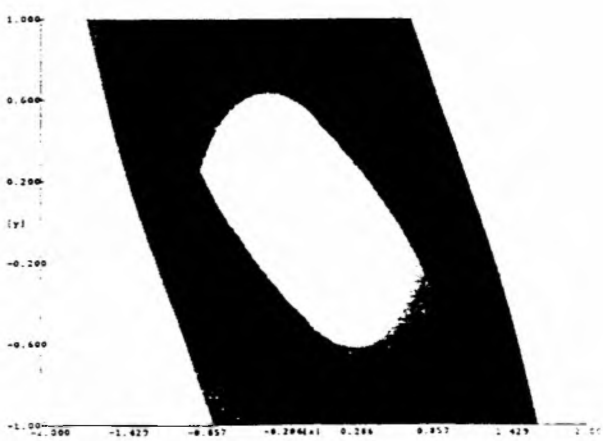

Figure 7: Multistability region with $\rho = 0.6$

the resulting control set and Figure 7. shows the attached multistability region.

The unique variant control set is formed due to the time varying perturbations, since the analysis of (5.1) with (time invariant) bifurcation parameter $u \in \mathbf{R}$ shows only three isolated fixed points for $\rho \in [0,1)$. This control set could be called a perturbation heteroclinic connection since it connects the two fixed points $(x_1^1, 0)$ and $(x_1^3, 0)$ of the nominal system. A similar effect concerning perturbation homoclinic connections was observed in [5] for the perturbed Takens-Bogdanov model. The stable manifold $W^s((0,0),\rho,\delta_1)$ for $\rho = 0.6$ consists of the 'interior' region of the variant control set, and constitutes also the L-invariant points for this perturbation range.

Increasing ρ further results in the disappearance of the uniform stable manifold of $(0,0)$ for $\rho^* = r_{lin}(\delta_1)$, see Figure 4. For $\rho > \rho^*$, the variant control set has collapsed around the fixed point $(0,0)$, as shown in Figure 8. for $\rho = 0.65$. The only L-invariant point is now the origin, and hence we have for the invariance radius defined in (3.4)

$$r_{inv}((0,0),L,\delta_1) = r_{lin}(\delta_1) = r((0,0),\delta_1) \text{ for all } \delta_1 \geq 0$$

The upper and lower boundary of the control set in Figure 8. are entrance boundaries, and hence the system exits the constraint set L through the right or left boundary for any initial value within the control set. Note that there exist, of course, $u \in \mathcal{U}^\rho$ and $z \in int\, L$ such that the trajectory $\varphi(t,z,u)$ converges exponentially to the origin.

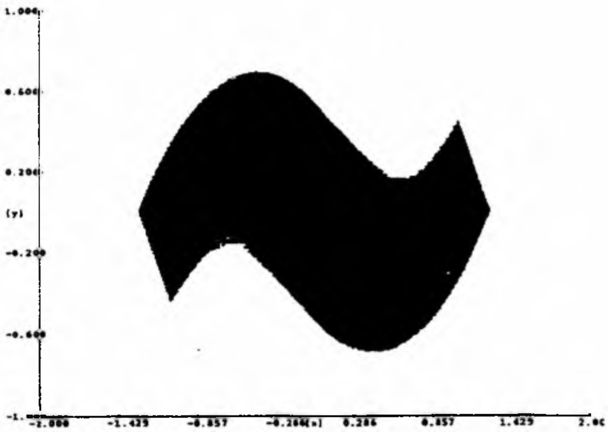

Figure 8: Variant control set with $\rho = 0.65$

Some of the main differences between the additively perturbed model (4.2) and the multiplicative model (5.1) are the different invariance radii (0.3849 for (4.2) and 0.65 for (5.1) with $\delta_1 = 0.5$); the fact that the invariance radius of (4.2) does not depend on δ_1, while this dependence is crucial for the multiplicative perturbation; and the bifurcation behavior of (4.2) with parameter $u \in \mathbb{R}$ determines the invariance behavior of the additively perturbed system, while (5.1) exhibits various changes in its qualitative behavior that are due to the time varying nature of the perturbation and that cannot be seen from the bifurcation picture with $u \in U$.

The stochastic multiplicatively perturbed system (5.1) can be studied using the theory from Section 3. (around the origin) and from Section 2. (for the global behavior in L). The results are fairly obvious and we mention briefly only a few observations. For a given noise η, f as in (2.4) the linear stability radius $r_{ls}(\eta, f)$ as defined in (3.5) satisfies

$$r_{ls}(\eta, f, \delta_1) \geq r_{lin}(\delta_1)$$

and hence

$$r_{stoch}((0,0), \eta, f, \delta_1) \geq r((0,0), \delta_1) \text{ for all } \delta_1 \geq 0$$

Since the Dirac measure $\delta_{(0,0)}$ is the only invariant Markov measure of the system (5.1), (2.4), the invariance radii of the origin with respect to the constraint set L agree for the deterministic and the stochastic perturbation model. However, the invariant measure looses its stability only at $\rho = r_{stoch}((0,0), \eta, f, \delta_1) \geq r_{inv}((0,0), L, \delta_1)$. For perturbation ranges $\rho \in (r_{inv}, r_{stoch})$ the stochastic system has again non-uniform stable manifolds. This effect occurs for the system (5.1) e.g. for η the Wiener process on the sphere $N = \mathbb{S}^1$ and $f^\rho = \rho \cdot \cos \eta$, compare [1].

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