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Linearizing Equations with State-Dependent Delays*

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Abstract. We investigate the differentiability of the composition operator and apply the result to equations with state-dependent delays.

1. Introduction

In this note we show how to linearize delay equations of the form

$$\dot{x}(t) = f(t, x(r(t, x(t)))), \tag{1}$$

$t \in [a, b] \subset \mathbf{R}$, where the velocity $\dot{x}(t)$ at time t depends on the "instantaneous state" $x(\tau)$ at time $\tau = r(t, x(t))$. More specifically, we consider the operator $T_{f,r}$, which maps a function x to the function defined by the right-hand side of (1), and we prove that $T_{f,r}$: $W^{1,\infty}[a, b] \to L^p[a, b]$, $1 \leq p < \infty$, is continuously Fréchet-differentiable at functions x whose associated "delay" $r(t, x(t))$ is strictly increasing. Such a result is of interest, if we want to apply general optimization theory in Banach spaces in order to prove the existence of Lagrange multipliers for control systems of the form

$$\dot{x}(t) = f(u(t), x(r(t, x(t)))),$$

here Fréchet-differentiability of the function at the right-hand side with respect to the function x often is an indispensable prerequisite (see, e.g., [3], [7], [10], [2] and [6]). For examples of such control systems with state dependent delay see, e.g., [5]. Since the delay property $r(t, x(t)) \leq t$ has no bearing on the question

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of whether $T_{f,r}$ is differentiable or not, we do not assume this property here. Furthermore, we want to include the situation where $f(t, x)$ is discontinuous in t (since f may depend on the discontinuous control $u(t)$), so we cannot assume that the solution of (1) is more regular than $W^{1,\infty}$.

It is natural to consider $T_{f,r}$ as the composition of three operators F, K, R in the form

$$(T_{f,r}x)(t) = F(K(x, Rx))(t), \tag{2}$$

where

$$(Fx)(t) = f(t, x(t)),$$
$$K(x, y)(t) = x(y(t)),$$
$$(Rx)(t) = r(t, x(t)).$$

The operators F and R, which are called superposition or Nemitskij operators, have been studied by many authors (we refer to [1] for a recent extensive survey), and their differentiability properties relevant for this paper are known. We therefore concentrate on the differentiability of the composition operator

$$K(x, y) = x \circ y.$$

The arguments of this paper may presumably also be used for computing second derivatives. This would be useful in optimal periodic control in order to give a rigorous proof of a so-called Π-test for equations with state-dependent delays (see [9] and [2]).

Notations. Throughout this paper the norm of $x \in L^p$ for $1 \leq p \leq \infty$ is denoted by $\|x\|_p$.

2. Assumptions and Results

To simplify the exposition, we assume f and x to be scalar valued; generalizations to the vector case and to the case where finitely many r_i are involved in the right-hand side of (1), are immediate.

Assumption A. Let $f, r\colon [a, b] \times \mathbf{R} \to \mathbf{R}$. We assume that:

(i) $f(\cdot, x)$ is measurable for all x, $f(t, \cdot)$ is continuously differentiable for almost all t, and f and f_x are bounded for bounded arguments.
(ii) Function r is twice continuously differentiable.
(iii) $r(t, x) \in [a, b]$ for all $t \in [a, b]$, $x \in \mathbf{R}$.

We formulate the main result of this paper.

Theorem 2.1. *Let Assumption A hold, and let $x^0 \in W^{1,\infty}[a, b]$ with*

$$\frac{d}{dt} r(t, x^0(t)) \geq \varepsilon_0 > 0 \qquad \text{a.e. in } [a, b].$$

Then $T_{f,r}$: $W^{1,\infty}[a,b] \to L^p[a,b]$, $1 \le p < \infty$, *defined by*

$$(T_{f,r}x)(t) = f(t, x(r(t, x(t)))), \qquad t \in [a,b],$$

is continuously Fréchet-differentiable at x^0 *with derivative*

$$[DT_{f,r}(x^0)x](t) = f_x(t, x^0(r(t, x^0(t))))x(r(t, x^0(t))) \\ + f_x(t, x^0(r(t, x^0(t))))\dot{x}^0(r(t, x^0(t)))r_x(t, x^0(t))x(t).$$

The proof is given at the end of this section.

Remark 2.2. This result shows that the linearization of equation (1) at x^0 is a nonautonomous delay equation (with time—but not state—dependent delay) for x with $\dot{x}(t)$ given by the formula in Theorem 2.1.

We decompose $T_{f,r}$, as mentioned above, in the form

$$T_{f,r}x = F(K(x, Rx))$$

and consider F, R, and K separately.

Lemma 2.3. *Let, in addition to Assumption A, f and f_x be bounded on $[a,b] \times \mathbf{R}$. Then f defines a continuously Fréchet-differentiable superposition operator F: $L^p[a,b] \to L^q[a,b]$ for all $1 \le q < p \le \infty$ with derivative*

$$[DF(x^0)x](t) = f_x(t, x^0(t))x(t).$$

Proof. This is a special case of Theorem 20.2 in [4], if we recall the basic theorem on superposition operators in L^p spaces, namely that the growth condition

$$|f(t,x)| \le a_1(t) + a_2|x|^{p/q}, \qquad a_1 \in L^q, \quad a_2 \in \mathbf{R},$$

is necessary and sufficient for F being a continuous operator from L^p to L^q. □

Proposition 2.4. *Let Assumption A hold. Then r defines a continuously Fréchet-differentiable superposition operator R: $W^{1,\infty}[a,b] \to W^{1,\infty}[a,b]$ with derivative*

$$[DR(x^0)x](t) = r_x(t, x^0(t))x(t).$$

Proof. This is a tedious but straightforward extension of the standard proof in $L^\infty[a,b]$ (resp. $C[a,b]$) if we note that the chain rule yields

$$\frac{d}{dt}[r_x(t, x^0(t))x(t)] = [r_{tx}(t, x^0(t)) + r_{xx}(t, x^0(t))\dot{x}^0(t)]x(t) \\ + r_x(t, x^0(t))\dot{x}(t).$$ □

We now state the result concerning the composition operator K.

Proposition 2.5. *Let I be a compact interval and let Y_0 be the open subset of $W^{1,\infty}[a, b]$ defined by*

$$Y_0=\left\{y\in W^{1,\infty}[a,b]: \operatorname*{ess\,inf}_{t\in[a,b]} \dot{y}(t)>0,\ \text{range}\ y\subset \operatorname{int}(I)\right\}.$$

Then the composition operator K, $K(x,y)=x\circ y$, maps bounded subsets of $W^{1,\infty}[I]\times Y_0$ into bounded subsets of $W^{1,p}[a,b]$, $1\le p\le\infty$. As an operator

$$K\colon W^{1,\infty}[I]\times Y_0\to L^p[a,b],$$

K is continuously Fréchet-differentiable for $1\le p<\infty$ with derivative

$$[DK(x^0,y^0)(x,y)](t)=x(y^0(t))+\dot{x}^0(y^0(t))y(t).$$

Proof. The proof is given in Section 3. □

Remark 2.6. For continuous differentiability we need ess inf $\dot{y}>0$ for y near y^0. This entails the choice of $W^{1,\infty}$ as space for y^0, as observed by Manitius [6]. A first look at the formula for DK shows that some restriction on y^0 is needed in order that $\dot{x}^0\circ y^0$ makes sense if $\dot{x}^0$ is not well defined on sets of measure zero.

Remark 2.7. The formula for $DK(x^0, y^0)$ involves the expression $\dot{x}^0(y^0(t))$, which does not depend continuously in L^∞ on y^0 in $Y_0\subset W^{1,\infty}$. Hence, contrary to the proposal in [6], (continuous) Fréchet-differentiability of K (and hence of $T_{f,r}$) cannot be achieved if the range space of K is taken as L^∞ instead of L^p, $1\le p<\infty$.

Proof of Theorem 2.1. Theorem 2.1 is a direct consequence of Propositions 2.4 and 2.5 and the chain rule if we note that, by Propositions 2.4 and 2.5,

$$\|K(x,Rx)\|_\infty\le C$$

in a neighborhood of x^0, and we can therefore assume f and f_x are globally bounded. □

3. The Composition Operator

The aim of this section is to prove Proposition 2.5. For $y\in Y_0$, we have

$$\begin{aligned}\left\|\frac{d}{dt}K(x,y)\right\|_p^p&=\int_a^b|\dot{x}(y(t))\dot{y}(t)|^p\,dt\le\|\dot{y}\|_\infty^{p-1}\int_a^b|\dot{x}(y(t))|^p\dot{y}(t)\,dt\\&\le\|\dot{y}\|_\infty^{p-1}\int_I|\dot{x}(s)|^p\,ds=\|\dot{y}\|_\infty^{p-1}\|\dot{x}\|_p^p,\end{aligned}$$

so K maps bounded subsets of $W^{1,p}(I) \times Y_0$ into bounded subsets of $W^{1,p}[a, b]$. It is now sufficient to show that the partial derivatives

$$D_x K(x^0, y^0)\colon W^{1,p}(I) \to L^p[a, b],$$
$$D_y K(x^0, y^0)\colon Y_0 \to L^p[a, b]$$

exist, are continuous with respect to (x^0, y^0), and have the form

$$[D_x K(x^0, y^0)x](t) = x(y^0(t)),$$
$$[D_y K(x^0, y^0)y](t) = \dot{x}^0(y^0(t))y(t).$$

We need the following lemma.

Lemma 3.1. *Let $g \in L^p(I)$, $y \in Y_0$, and set*

$$\varepsilon(y) = \operatorname*{ess\,inf}_{t \in [a,b]} \dot{y}(t).$$

Then

$$\int_a^b |g(y(t))|^p \, dt \le \frac{1}{\varepsilon(y)} \|g\|_p^p.$$

Moreover, if $y_n \to y$ in Y_0 in the norm of $W^{1,\infty}$, then

$$\lim_{n \to \infty} \int_a^b |g(y_n(t)) - g(y(t))|^p \, dt = 0.$$

Proof. We have

$$\int_a^b |g(y(t))|^p \, dt = \int_a^b |g(y(t))|^p \dot{y}(t) \frac{1}{\dot{y}(t)} \, dt \le \frac{1}{\varepsilon(y)} \|g\|_p^p.$$

We prove the second assertion at first if g is the characteristic function of an interval $E = [\alpha, \beta]$, i.e., $g(x) = \chi_E(x) = 1$ if $x \in E$ and 0 otherwise. Obviously, $\chi_E(y_n(t)) \ne \chi_E(y(t))$ implies that either $|y(t) - \alpha| \le \|y_n - y\|_\infty$ or $|y(t) - \beta| \le \|y_n - y\|_\infty$, therefore we have

$$\operatorname{meas}\{t\colon \chi_E(y_n(t)) \ne \chi_E(y(t))\} \le \frac{4}{\varepsilon(y)} \|y_n - y\|_\infty.$$

This proves the assertion for the case $g = \chi_E$. The triangle inequality yields the result if g is a step function. Now if $g \in L^p(I)$, for any step function $s \in L^p(I)$ we know from the first part of the lemma that

$$\int_a^b |g(y(t)) - s(y(t))|^p \, dt \le \frac{1}{\varepsilon(y)} \|g - s\|_p^p,$$
$$\int_a^b |g(y_n(t)) - s(y_n(t))|^p \, dt \le \frac{1}{\varepsilon(y_n)} \|g - s\|_p^p.$$

This proves the lemma, since the step functions are dense in $L^p(I)$ and since $\varepsilon(y_n)\to\varepsilon(y)$. □

We now consider the partial derivatives separately.

3.1. *The Partial Derivative D_xK*

Since the mapping $x\to x\circ y^0$ obviously defines a linear continuous operator from $C(I)$ to $C[a,b]$, only the continuity of D_xK remains to be shown. Let $(x_n,y_n)\to(x^0,y^0)$ in $W^{1,p}(I)\times Y_0$, then we have, for all $x\in W^{1,p}(I)$,

$$\begin{aligned}
&\|D_xK(x_n,y_n)x-D_xK(x^0,y^0)x\|_p^p\\
&\quad=\int_a^b |x(y_n(t))-x(y^0(t))|^p\,dt\\
&\quad=\int_a^b \left|\int_0^1 \dot{x}(y^0(t)+u(y_n(t)-y^0(t)))\cdot(y_n(t)-y^0(t))\,du\right|^p dt\\
&\quad\le\|y_n-y^0\|_\infty^p\int_0^1\int_a^b|\dot{x}(y^0(t)+u(y_n(t)-y^0(t)))|^p\,dt\,du\\
&\quad\le\|y_n-y^0\|_\infty^p\cdot\frac{1}{\varepsilon_0}\|\dot{x}\|_p^p
\end{aligned}$$

by Lemma 3.1, if $\varepsilon_0=\min\{\varepsilon(y^0),\min_{n\in\mathbb{N}}\varepsilon(y_n)\}$, which we may assume to be positive since $y_n\to y^0$ in the norm of $W^{1,\infty}$. This implies

$$\|D_xK(x_n,y_n)-D_xK(x^0,y^0)\|\le\frac{1}{\varepsilon_0}\|y_n-y^0\|_\infty,$$

therefore D_xK is continuous.

3.2. *The Partial Derivative D_yK*

We define $A\colon W^{1,\infty}[a,b]\to L^p[a,b]$ by

$$(Ah)(t)=\dot{x}^0(y^0(t))h(t).$$

By Lemma 3.1, we have $\dot{x}^0\circ y^0\in L^p[a,b]$, so A is well defined, linear, and continuous. We estimate the remainder as follows:

$$\begin{aligned}
&\|K(x^0,y^0+h)-K(x^0,y^0)-Ah\|_p^p\\
&\quad=\int_a^b|x^0(y^0(t)+h(t))-x^0(y^0(t))-\dot{x}^0(y^0(t))h(t)|^p\,dt\\
&\quad=\int_a^b|h(t)|^p\cdot\left|\int_0^1[\dot{x}^0(y^0(t)+uh(t))-\dot{x}^0(y^0(t))]\,du\right|^p dt\\
&\quad\le\|h\|_\infty^p\int_0^1\int_a^b|\dot{x}^0(y^0(t)+uh(t))-\dot{x}^0(y^0(t))|^p\,dt\,du.
\end{aligned}$$

By Lemma 3.1, the inner integral

$$I_h(u) = \int_a^b |\dot{x}^0(y^0(t) + uh(t)) - \dot{x}^0(y^0(t))|^p \, dt$$

converges to 0 pointwise in u, if $h \to 0$ in $W^{1,\infty}[a, b]$, and

$$|I_h(u)|^{1/p} \le \left(\frac{1}{\varepsilon(y^0 + uh)} + \frac{1}{\varepsilon(y^0)} \right) \|\dot{x}^0\|_p,$$

so $I_h \to 0$ in $L^1[0, 1]$, if $h \to 0$ in $W^{1,\infty}[a, b]$. This proves that $A = D_y K(x^0, y^0)$. For the continuity of $D_y K$, we observe that

$$\begin{aligned}
&\|D_y K(x_n, y_n)h - D_y K(x^0, y^0)h\|_p^p \\
&= \int_a^b |\dot{x}_n(y_n(t)) - \dot{x}^0(y^0(t))|^p |h(t)|^p \, dt \\
&\le \|h\|_\infty^p \left[\int_a^b |\dot{x}_n(y_n(t)) - \dot{x}^0(y_n(t))|^p \, dt + \int_a^b |\dot{x}^0(y_n(t)) - \dot{x}^0(y^0(t))|^p \, dt \right] \\
&\le \|h\|_\infty^p \left[\frac{1}{\varepsilon(y_n)} \|\dot{x}_n - \dot{x}_0\|_p^p + \int_a^b |\dot{x}^0(y_n(t)) - \dot{x}^0(y^0(t))|^p \, dt \right]
\end{aligned}$$

by Lemma 3.1, and if $(x_n, y_n) \to (x^0, y^0)$ in $W^{1,p}(I) \times Y_0$, then the bracketed term converges to zero again by Lemma 3.1. Hence $D_y K$ is continuous.

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