Linearizing Equations with State-Dependent Delays*

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Abstract. We investigate the differentiability of the composition operator and apply the result to equations with state-dependent delays.

1. Introduction

In this note we show how to linearize delay equations of the form

$$\dot{x}(t) = f(t, x(r(t, x(t)))), \tag{1}$$

$t \in [a, b] \subset \mathbf{R}$, where the velocity $\dot{x}(t)$ at time t depends on the "instantaneous state" $x(\tau)$ at time $\tau = r(t, x(t))$. More specifically, we consider the operator $T_{f,r}$, which maps a function x to the function defined by the right-hand side of (1), and we prove that $T_{f,r}$: $W^{1,\infty}[a, b] \to L^p[a, b]$, $1 \le p < \infty$, is continuously Fréchet-differentiable at functions x whose associated "delay" $r(t, x(t))$ is strictly increasing. Such a result is of interest, if we want to apply general optimization theory in Banach spaces in order to prove the existence of Lagrange multipliers for control systems of the form

$$\dot{x}(t) = f(u(t), x(r(t, x(t)))),$$

here Fréchet-differentiability of the function at the right-hand side with respect to the function x often is an indispensable prerequisite (see, e.g., [3], [7], [10], [2] and [6]). For examples of such control systems with state dependent delay see, e.g., [5]. Since the delay property $r(t, x(t)) \le t$ has no bearing on the question

* F. Colonius was supported by a Heisenberg grant from DFG at Universität Bremen.

of whether $T_{f,r}$ is differentiable or not, we do not assume this property here. Furthermore, we want to include the situation where $f(t, x)$ is discontinuous in t (since f may depend on the discontinuous control $u(t)$), so we cannot assume that the solution of (1) is more regular than $W^{1,\infty}$.

It is natural to consider $T_{f,r}$ as the composition of three operators F, K, R in the form

$$(T_{f,r}x)(t) = F(K(x, Rx))(t), \tag{2}$$

where

$$(Fx)(t) = f(t, x(t)),$$
$$K(x, y)(t) = x(y(t)),$$
$$(Rx)(t) = r(t, x(t)).$$

The operators F and R, which are called superposition or Nemitskij operators, have been studied by many authors (we refer to [1] for a recent extensive survey), and their differentiability properties relevant for this paper are known. We therefore concentrate on the differentiability of the composition operator

$$K(x, y) = x \circ y.$$

The arguments of this paper may presumably also be used for computing second derivatives. This would be useful in optimal periodic control in order to give a rigorous proof of a so-called Π-test for equations with state-dependent delays (see [9] and [2]).

Notations. Throughout this paper the norm of $x \in L^p$ for $1 \le p \le \infty$ is denoted by $\|x\|_p$.

2. Assumptions and Results

To simplify the exposition, we assume f and x to be scalar valued; generalizations to the vector case and to the case where finitely many r_i are involved in the right-hand side of (1), are immediate.

Assumption A. Let $f, r\colon [a, b] \times \mathbf{R} \to \mathbf{R}$. We assume that:

(i) $f(\cdot, x)$ is measurable for all x, $f(t, \cdot)$ is continuously differentiable for almost all t, and f and f_x are bounded for bounded arguments.
(ii) Function r is twice continuously differentiable.
(iii) $r(t, x) \in [a, b]$ for all $t \in [a, b]$, $x \in \mathbf{R}$.

We formulate the main result of this paper.

Theorem 2.1. *Let Assumption A hold, and let $x^0 \in W^{1,\infty}[a, b]$ with*

$$\frac{d}{dt} r(t, x^0(t)) \ge \varepsilon_0 > 0 \qquad \text{a.e. in } [a, b].$$

Then $T_{f,r}$: $W^{1,\infty}[a, b] \to L^p[a, b]$, $1 \le p < \infty$, *defined by*

$$(T_{f,r}x)(t) = f(t, x(r(t, x(t)))), \qquad t \in [a, b],$$

is continuously Fréchet-differentiable at x^0 *with derivative*

$$[DT_{f,r}(x^0)x](t) = f_x(t, x^0(r(t, x^0(t))))x(r(t, x^0(t))) + f_x(t, x^0(r(t, x^0(t))))\dot{x}^0(r(t, x^0(t)))r_x(t, x^0(t))x(t).$$

The proof is given at the end of this section.

Remark 2.2. This result shows that the linearization of equation (1) at x^0 is a nonautonomous delay equation (with time—but not state—dependent delay) for x with $\dot{x}(t)$ given by the formula in Theorem 2.1.

We decompose $T_{f,r}$, as mentioned above, in the form

$$T_{f,r}x = F(K(x, Rx))$$

and consider F, R, and K separately.

Lemma 2.3. *Let, in addition to Assumption A, f and f_x be bounded on $[a, b] \times \mathbf{R}$. Then f defines a continuously Fréchet-differentiable superposition operator F: $L^p[a, b] \to L^q[a, b]$ for all $1 \le q < p \le \infty$ with derivative*

$$[DF(x^0)x](t) = f_x(t, x^0(t))x(t).$$

Proof. This is a special case of Theorem 20.2 in [4], if we recall the basic theorem on superposition operators in L^p spaces, namely that the growth condition

$$|f(t, x)| \le a_1(t) + a_2|x|^{p/q}, \qquad a_1 \in L^q, \quad a_2 \in \mathbf{R},$$

is necessary and sufficient for F being a continuous operator from L^p to L^q. □

Proposition 2.4. *Let Assumption A hold. Then r defines a continuously Fréchet-differentiable superposition operator R: $W^{1,\infty}[a, b] \to W^{1,\infty}[a, b]$ with derivative*

$$[DR(x^0)x](t) = r_x(t, x^0(t))x(t).$$

Proof. This is a tedious but straightforward extension of the standard proof in $L^\infty[a, b]$ (resp. $C[a, b]$) if we note that the chain rule yields

$$\frac{d}{dt}[r_x(t, x^0(t))x(t)] = [r_{tx}(t, x^0(t)) + r_{xx}(t, x^0(t))\dot{x}^0(t)]x(t) + r_x(t, x^0(t))\dot{x}(t).$$

□

We now state the result concerning the composition operator K.

Proposition 2.5. *Let I be a compact interval and let Y_0 be the open subset of $W^{1,\infty}[a, b]$ defined by*

$$Y_0 = \left\{ y \in W^{1,\infty}[a, b] \colon \operatorname*{ess\,inf}_{t \in [a,b]} \dot{y}(t) > 0, \text{ range } y \subset \operatorname{int}(I) \right\}.$$

Then the composition operator K, $K(x, y) = x \circ y$, maps bounded subsets of $W^{1,\infty}[I] \times Y_0$ into bounded subsets of $W^{1,p}[a, b]$, $1 \le p \le \infty$. As an operator

$$K\colon W^{1,\infty}[I] \times Y_0 \to L^p[a, b],$$

K is continuously Fréchet-differentiable for $1 \le p < \infty$ with derivative

$$[DK(x^0, y^0)(x, y)](t) = x(y^0(t)) + \dot{x}^0(y^0(t))y(t).$$

Proof. The proof is given in Section 3. □

Remark 2.6. For continuous differentiability we need ess inf $\dot{y} > 0$ for y near y^0. This entails the choice of $W^{1,\infty}$ as space for y^0, as observed by Manitius [6]. A first look at the formula for DK shows that some restriction on y^0 is needed in order that $\dot{x}^0 \circ y^0$ makes sense if $\dot{x}^0$ is not well defined on sets of measure zero.

Remark 2.7. The formula for $DK(x^0, y^0)$ involves the expression $\dot{x}^0(y^0(t))$, which does not depend continuously in L^∞ on y^0 in $Y_0 \subset W^{1,\infty}$. Hence, contrary to the proposal in [6], (continuous) Fréchet-differentiability of K (and hence of $T_{f,r}$) cannot be achieved if the range space of K is taken as L^∞ instead of L^p, $1 \le p < \infty$.

Proof of Theorem 2.1. Theorem 2.1 is a direct consequence of Propositions 2.4 and 2.5 and the chain rule if we note that, by Propositions 2.4 and 2.5,

$$\|K(x, Rx)\|_\infty \le C$$

in a neighborhood of x^0, and we can therefore assume f and f_x are globally bounded. □

3. The Composition Operator

The aim of this section is to prove Proposition 2.5. For $y \in Y_0$, we have

$$\begin{aligned} \left\| \frac{d}{dt} K(x, y) \right\|_p^p &= \int_a^b |\dot{x}(y(t))\dot{y}(t)|^p \, dt \le \|\dot{y}\|_\infty^{p-1} \int_a^b |\dot{x}(y(t))|^p \dot{y}(t) \, dt \\ &\le \|\dot{y}\|_\infty^{p-1} \int_I |\dot{x}(s)|^p \, ds = \|\dot{y}\|_\infty^{p-1} \|\dot{x}\|_p^p, \end{aligned}$$

so K maps bounded subsets of $W^{1,p}(I) \times Y_0$ into bounded subsets of $W^{1,p}[a, b]$. It is now sufficient to show that the partial derivatives

$$D_x K(x^0, y^0)\colon W^{1,p}(I) \to L^p[a, b],$$

$$D_y K(x^0, y^0)\colon Y_0 \to L^p[a, b]$$

exist, are continuous with respect to (x^0, y^0), and have the form

$$[D_x K(x^0, y^0)x](t) = x(y^0(t)),$$

$$[D_y K(x^0, y^0)y](t) = \dot{x}^0(y^0(t))y(t).$$

We need the following lemma.

Lemma 3.1. *Let $g \in L^p(I)$, $y \in Y_0$, and set*

$$\varepsilon(y) = \operatorname*{ess\,inf}_{t \in [a,b]} \dot{y}(t).$$

Then

$$\int_a^b |g(y(t))|^p\, dt \le \frac{1}{\varepsilon(y)} \|g\|_p^p.$$

Moreover, if $y_n \to y$ in Y_0 in the norm of $W^{1,\infty}$, then

$$\lim_{n \to \infty} \int_a^b |g(y_n(t)) - g(y(t))|^p\, dt = 0.$$

Proof. We have

$$\int_a^b |g(y(t))|^p\, dt = \int_a^b |g(y(t))|^p \dot{y}(t) \frac{1}{\dot{y}(t)}\, dt \le \frac{1}{\varepsilon(y)} \|g\|_p^p.$$

We prove the second assertion at first if g is the characteristic function of an interval $E = [\alpha, \beta]$, i.e., $g(x) = \chi_E(x) = 1$ if $x \in E$ and 0 otherwise. Obviously, $\chi_E(y_n(t)) \ne \chi_E(y(t))$ implies that either $|y(t) - \alpha| \le \|y_n - y\|_\infty$ or $|y(t) - \beta| \le \|y_n - y\|_\infty$, therefore we have

$$\operatorname{meas}\{t\colon \chi_E(y_n(t)) \ne \chi_E(y(t))\} \le \frac{4}{\varepsilon(y)} \|y_n - y\|_\infty.$$

This proves the assertion for the case $g = \chi_E$. The triangle inequality yields the result if g is a step function. Now if $g \in L^p(I)$, for any step function $s \in L^p(I)$ we know from the first part of the lemma that

$$\int_a^b |g(y(t)) - s(y(t))|^p\, dt \le \frac{1}{\varepsilon(y)} \|g - s\|_p^p,$$

$$\int_a^b |g(y_n(t)) - s(y_n(t))|^p\, dt \le \frac{1}{\varepsilon(y_n)} \|g - s\|_p^p.$$

This proves the lemma, since the step functions are dense in $L^p(I)$ and since $\varepsilon(y_n)\to\varepsilon(y)$. □

We now consider the partial derivatives separately.

3.1. *The Partial Derivative D_xK*

Since the mapping $x\to x\circ y^0$ obviously defines a linear continuous operator from $C(I)$ to $C[a,b]$, only the continuity of D_xK remains to be shown. Let $(x_n, y_n)\to(x^0, y^0)$ in $W^{1,p}(I)\times Y_0$, then we have, for all $x\in W^{1,p}(I)$,

$$
\begin{aligned}
&\|D_xK(x_n, y_n)x - D_xK(x^0, y^0)x\|_p^p\\
&\quad=\int_a^b |x(y_n(t))-x(y^0(t))|^p\,dt\\
&\quad=\int_a^b\left|\int_0^1 \dot{x}(y^0(t)+u(y_n(t)-y^0(t)))\cdot(y_n(t)-y^0(t))\,du\right|^p dt\\
&\quad\le\|y_n-y^0\|_\infty^p\int_0^1\int_a^b|\dot{x}(y^0(t)+u(y_n(t)-y^0(t)))|^p\,dt\,du\\
&\quad\le\|y_n-y^0\|_\infty^p\cdot\frac{1}{\varepsilon_0}\|\dot{x}\|_p^p
\end{aligned}
$$

by Lemma 3.1, if $\varepsilon_0=\min\{\varepsilon(y^0), \min_{n\in\mathbb{N}}\varepsilon(y_n)\}$, which we may assume to be positive since $y_n\to y^0$ in the norm of $W^{1,\infty}$. This implies

$$\|D_xK(x_n, y_n)-D_xK(x^0, y^0)\|\le\frac{1}{\varepsilon_0}\|y_n-y^0\|_\infty,$$

therefore D_xK is continuous.

3.2. *The Partial Derivative D_yK*

We define A: $W^{1,\infty}[a,b]\to L^p[a,b]$ by

$$(Ah)(t)=\dot{x}^0(y^0(t))h(t).$$

By Lemma 3.1, we have $\dot{x}^0\circ y^0\in L^p[a,b]$, so A is well defined, linear, and continuous. We estimate the remainder as follows:

$$
\begin{aligned}
&\|K(x^0, y^0+h)-K(x^0, y^0)-Ah\|_p^p\\
&\quad=\int_a^b|x^0(y^0(t)+h(t))-x^0(y^0(t))-\dot{x}^0(y^0(t))h(t)|^p\,dt\\
&\quad=\int_a^b|h(t)|^p\cdot\left|\int_0^1[\dot{x}^0(y^0(t)+uh(t))-\dot{x}^0(y^0(t))]\,du\right|^p dt\\
&\quad\le\|h\|_\infty^p\int_0^1\int_a^b|\dot{x}^0(y^0(t)+uh(t))-\dot{x}^0(y^0(t))|^p\,dt\,du.
\end{aligned}
$$

By Lemma 3.1, the inner integral

$$I_h(u)=\int_a^b |\dot{x}^0(y^0(t)+uh(t))-\dot{x}^0(y^0(t))|^p\,dt$$

converges to 0 pointwise in u, if $h\to 0$ in $W^{1,\infty}[a,b]$, and

$$|I_h(u)|^{1/p}\le\left(\frac{1}{\varepsilon(y^0+uh)}+\frac{1}{\varepsilon(y^0)}\right)\|\dot{x}^0\|_p,$$

so $I_h\to 0$ in $L^1[0,1]$, if $h\to 0$ in $W^{1,\infty}[a,b]$. This proves that $A=D_yK(x^0,y^0)$. For the continuity of D_yK, we observe that

$$\begin{aligned}
&\|D_yK(x_n,y_n)h-D_yK(x^0,y^0)h\|_p^p\\
&=\int_a^b |\dot{x}_n(y_n(t))-\dot{x}^0(y^0(t))|^p|h(t)|^p\,dt\\
&\le\|h\|_\infty^p\left[\int_a^b |\dot{x}_n(y_n(t))-\dot{x}^0(y_n(t))|^p\,dt+\int_a^b |\dot{x}^0(y_n(t))-\dot{x}^0(y^0(t))|^p\,dt\right]\\
&\le\|h\|_\infty^p\left[\frac{1}{\varepsilon(y_n)}\|\dot{x}_n-\dot{x}_0\|_p^p+\int_a^b |\dot{x}^0(y_n(t))-\dot{x}^0(y^0(t))|^p\,dt\right]
\end{aligned}$$

by Lemma 3.1, and if $(x_n,y_n)\to(x^0,y^0)$ in $W^{1,p}(I)\times Y_0$, then the bracketed term converges to zero again by Lemma 3.1. Hence D_yK is continuous.

Acknowledgments

We thank an anonymous referee for pointing out an error in an earlier version of this paper. Furthermore, the first author wants to thank Jürgen Appell for helpful discussions.

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