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OPTIMAL PERIODIC CONTROL: A SCENARIO FOR LOCAL PROPERNESS*

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Abstract. A fundamental problem in optimal periodic control is to decide whether proper periodic controls and trajectories yield better average performance than constant steady-state solutions. The present paper describes a situation where this holds true, because "nearby" the linearized system equation has a pair of eigenvalues on the imaginary axis. An example involving a retarded Liénard equation is discussed in detail.

Key words. optimal periodic control, local properness, functional differential equations, Hopf bifurcation

AMS(MOS) subject classifications. 49B34, 49D50

1. Introduction. In optimal periodic control theory, one looks for periodic controls and corresponding periodic trajectories of a control system described, for example, by a functional differential equation such that a certain average performance criterion is minimized. Suppose that a constant control u^0 and a corresponding steady state x^0 of the system are given, which are optimal among all such pairs (x, u). If it is possible to obtain better average performance in every neighborhood of (x^0, u^0) by allowing proper periodic controls and corresponding periodic trajectories x, then the pair (x^0, u^0) is called locally proper. It is the purpose of the present paper to explore a situation where one may expect local properness because of structural properties of the system equation. In particular these properties are related to those of a Hopf bifurcation. The guiding idea is that local properness will occur, if the considered system has "nearby" a "natural" periodic motion giving better performance.

A connection between Hopf bifurcation and optimal periodic control theory has already been observed by Russell [20]. He was interested in coupled nonlinear oscillators, where a Hopf bifurcation causes periodic motions which he wanted to dampen. Since this was not possible by linear regulator theory, he considered this problem as an optimal periodic control problem where the performance criterion is constructed in such a way as to minimize the amplitude of the oscillations.

Observe, however, that the spirit of the present paper is quite different: Instead of trying to dampen periodic motions we are willing to introduce them in order to get better performance. This is motivated by problems from chemical engineering (output maximization of chemical reactors [19], [24], [25]) and aircraft flight performance optimization (fuel optimal flight [22], [23]). Further references are given in [8], [17], [18].

In § 2 the optimal periodic control problem is formally defined for systems described by retarded functional differential equations. Furthermore, among other preliminaries, relevant information on necessary optimality conditions is cited from [8].

Section 3 exhibits a scenario for local properness. Theorem 3.6 contains the main result of this paper. In conclusion, § 4 discusses an example which, in fact, was the

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starting point of the present analysis. The importance of the result in § 3 is twofold: (i) It explains a mechanism by which local properness may occur and thus gives some insight into this phenomenon. (ii) It gives a hint, where to look for local properness, namely near equilibria, where the linearized system equation has a pair of eigenvalues on the imaginary axis.

It is worthwhile emphasizing that these results, which may be viewed as a contribution to the qualitative theory of optimal control, are also new for the special case of systems governed by ordinary differential equations.

Notation. The transpose of an element $x \in \mathbb{R}^n$ is denoted by x^T; similarly for matrices. For a map F between Banach spaces X and Y, $\mathscr{D}F(x^0)$ denotes its Fréchet-derivative at $x^0 \in X$. For maps between finite dimensional space we also use a subscript x in order to denote the partial derivative with respect to x. The second Fréchet-derivative at $x^0 \in X$ is denoted by $\mathscr{D}\mathscr{D}F(x^0)$. For an element $x \in \mathbb{R}^n$, $\bar{x}$ denotes the constant function $\bar{x}(s) \equiv x$ (in various function spaces).

2. Problem formulation and optimality conditions. In this section, a parameter dependent optimal periodic control problem $(\mathrm{OPC})^\alpha$ and the corresponding optimal steady-state problem $(\mathrm{OSS})^\alpha$ are formulated. Furthermore optimality conditions and results on smooth dependence of optimal solutions are cited, slightly modified for our purposes, from [8].

Consider the following optimal periodic control problem.

$(\mathrm{OPC})^\alpha$ Minimize $1/\tau \int_0^\tau g(x(s), u(s))\, ds$

over $(x, u) \in C(-r, \tau; \mathbb{R}^n) \times L^\infty(0, \tau; \mathbb{R}^m)$

subject to

$$\dot{x}(t) = f(x_t, u(t), \alpha) \quad \text{a.e. } t \in [0, \tau], \tag{2.1}$$

$$x_0 = x_\tau, \tag{2.2}$$

$$h(u(t)) \in \mathbb{R}^l_- \quad \text{a.e. } t \in [0, \tau], \tag{2.3}$$

$$\int_0^\tau k(x(t), u(t))\, dt = 0; \tag{2.4}$$

here $x_t(s) = x(t+s) \in \mathbb{R}^n$, $s \in [-r, 0]$, $r > 0$ is the length of the delay, $\alpha \in A$ is a parameter, $A \subset \mathbb{R}$ open, $f = (f^i): C(-r, 0; \mathbb{R}^n) \times \mathbb{R}^m \times A \to \mathbb{R}^n$, $g: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $h = (h^i): \mathbb{R}^m \to \mathbb{R}^l$, $k = (k^i): \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n_1}$.

The period length $\tau > 0$ is considered fixed here (we also allow $\tau < r$). The requirement (2.2) is imposed in order to allow periodic extensions of x and u to periodic solutions of (2.1) on $\mathbb{R}_+ := [0, \infty)$.

Abbreviate

$$\begin{aligned} &\Omega := \{u \in \mathbb{R}^m : h(u) \in \mathbb{R}^l_-\}, \\ &\mathscr{U}_{\mathrm{ad}} := \{u \in L^\infty(0, \tau; \mathbb{R}^m) : u(t) \in \Omega \text{ a.e. } t \in [0, \tau]\}. \end{aligned} \tag{2.5}$$

The corresponding steady-state problem has the following form:

$(\mathrm{OSS})^\alpha$ Minimize $g(x, u)$ over $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$

subject to

$$0 = f(\bar{x}, u, \alpha), \tag{2.6}$$

$$h(u) \in \mathbb{R}^l_-, \tag{2.7}$$

$$(2.8)\qquad 0=k(x,u);$$

here f, g, h and k are as in $(\mathrm{OPC})^\alpha$.

We are interested in the behavior of $(\mathrm{OPC})^\alpha$ near an optimal solution $(x^0, u^0)\in \mathbb{R}^n\times\mathbb{R}^m$ of $(\mathrm{OSS})^{\alpha_0}$ (i.e., near the constant pair $(\bar{x}^0, \bar{u}^0)\in C(-r,\tau;\mathbb{R}^n)\times L^\infty(0,\tau;\mathbb{R}^m)$).

DEFINITION. A local solution (x^α, u^α) of problem $(\mathrm{OSS})^\alpha$ is called locally proper, if for all $\varepsilon>0$ there exist $(x,u)\in C(-r,\tau;\mathbb{R}^n)\times L^\infty(0,\tau;\mathbb{R}^m)$ with $\sup_{t\in[0,\tau]}|x^0-x(t)|<\varepsilon$ satisfying (2.1)-(2.4) and

$$1/\tau\int_0^\tau g(x(t),u(t))\,dt<g(x^\alpha,u^\alpha).$$

As is well known, first order necessary optimality conditions (based on weak variations) do not allow one to decide the question of local properness. Hence we will give below second order necessary optimality conditions for $(\mathrm{OPC})^\alpha$.

Let the Pontryagin function $H: C(-r,0;\mathbb{R}^n)\times\mathbb{R}^m\times\mathbb{R}^{n+n_1}\times A\to\mathbb{R}$ for $(\mathrm{OPC})^\alpha$ be

$$(2.9)\qquad H(\varphi,u,y,\alpha):=g(\varphi(0),u)+y^T\begin{pmatrix} f(\varphi,u,\alpha)\\ k(\varphi(0),u)\end{pmatrix}$$

and let the Lagrange function $\mathscr{L}:\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{n+n_1}\times\mathbb{R}^l\times A\to\mathbb{R}$ for $(\mathrm{OSS})^\alpha$ be

$$(2.10)\qquad \mathscr{L}(x,u,y,z,\alpha):=g(x,u)+y^T\begin{pmatrix} f(\bar{x},u,\alpha)\\ k(x,u)\end{pmatrix}+z^Th(u).$$

The following hypotheses will be used.

Hypothesis 2.1. The functions f, g, h and k are twice continuously Fréchet differentiable in a neighborhood of $(\bar{x}^0, u^0, \alpha_0)$ (respectively, (x^0,u^0), u^0, (x^0,u^0)); the function f and its first and second derivatives are bounded for bounded arguments; the set Ω is convex.

Hypothesis 2.2. There exist $(y^0,z^0)\in\mathbb{R}^{n+n_1}\times\mathbb{R}^l$ such that

$$(2.11)\qquad z^{0T}h(u^0)=0,$$

$$(2.12)\qquad \mathscr{D}_{1,2}\mathscr{L}(x^0,u^0,y^0,z^0,\alpha_0)=0,$$

$$(2.13)\qquad \mathscr{D}_{1,2}\mathscr{D}_{1,2}\mathscr{L}(x^0,u^0,y^0,z^0,\alpha_0)((x,u),(x,u))>0$$

for all $(x,u)\in\mathbb{R}^n\times\mathbb{R}^m$ with $\mathscr{D}_{1,2}f(\bar{x}^0,u^0,\alpha_0)(\bar{x},u)=0$, $k_{x,u}(x^0,u^0)(x,u)=0$, $h^i_u(u^0)u<0$ if $h^i(u^0)=0$, $i\in\{1,\cdots,l\}$.

Hypothesis 2.3. The gradients in $\mathbb{R}^{n+m}$

$$(2.14)\qquad \begin{aligned} &\mathscr{D}_{1,2}f^i(\bar{x}^0,u^0,\alpha_0), && i=1,\cdots,n,\\ &(0,h^i_u(u^0))\quad\text{with } h^i(u^0)=0, && i\in\{1,\cdots,l\},\\ &k^i_{x,u}(x^0,u^0), && i=1,\cdots,n_1\end{aligned}$$

are linearly independent and the multiplier $z^0=(z^{0,i})$ from Hypothesis 2.2 satisfies $z^{0,i}>0$ if $h^i(u^0)=0$, $i\in\{1,\cdots,l\}$.

Hypothesis 2.4. For all $\alpha\neq\alpha^0$ in a neighborhood of α_0, the linearized equation

$$(2.15)\qquad \dot{x}(t)=\mathscr{D}_1f(\bar{x}^\alpha,u^\alpha,\alpha)x_t,\qquad t\geqq 0$$

has only the trivial τ-periodic solution; here (x^α,u^α) are elements in $\mathbb{R}^n\times\mathbb{R}^m$ to be determined in Theorem 2.7, below.

Next we comment on these hypotheses.

Remark 2.5. Hypothesis 2.4 is equivalent to

$$(2.16)\qquad \operatorname{rank}\Delta(j\omega,\alpha)=n\quad\text{for } \omega=2k\pi/\tau,\quad k\in\mathbb{Z},$$

where $\Delta(z, \alpha)$ is the characteristic function of (2.15),

$$\Delta(z, \alpha) = zI - \mathcal{D}_1 f(\bar{x}^\alpha, u^\alpha, \alpha)(e^{z\cdot}I), \qquad z \in \mathbb{C}.$$

This hypothesis will guarantee that all Lagrange multipliers for $(\mathrm{OPC})^\alpha$ can be obtained from Lagrange multipliers for $(\mathrm{OSS})^\alpha$.

Remark 2.6. Hypotheses 2.3 and 2.2 are a constraint qualification and a second order sufficient optimality condition, respectively, for the steady-state problem $(\mathrm{OSS})^{\alpha_0}$. Note that in (2.14) $f(\cdot, u^0, \alpha_0)$ is considered as a map $\mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto f(\bar{x}, u^0, \alpha_0)$.

First we analyze the steady-state problem $(\mathrm{OSS})^\alpha$.

THEOREM 2.7. *Suppose that* $(x^0, u^0) \in \mathbb{R}^n \times \mathbb{R}^m$ *satisfy Hypotheses* 2.1–2.3. *Then*

(i) *The pair* (x^0, u^0) *is an isolated local minimum of Problem* $(\mathrm{OSS})^{\alpha_0}$, *and the Lagrange multipliers* $(y^0, z^0) \in \mathbb{R}^{n+n_1} \times \mathbb{R}^l$ *are uniquely determined by* (2.11) *and* (2.12).

(ii) *There exists a continuously differentiable function* $\alpha \to (x^\alpha, u^\alpha, y^\alpha, z^\alpha) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{n+n_1} \times \mathbb{R}^l$ *defined on a neighborhood of* α_0 *such that* (x^α, u^α) *is an isolated minimum of* $(\mathrm{OSS})^\alpha$, $(x^{\alpha_0}, u^{\alpha_0}, y^{\alpha_0}, z^{\alpha_0}) = (x^0, u^0, y^0, z^0)$ *and* $(x^\alpha, u^\alpha, y^\alpha, z^\alpha)$ *satisfy conditions* (2.11)–(2.13) *with* α_0 *replaced by* α.

Proof. This follows from a result in Fiacco [9, § 3.2]. □

Next we state second order necessary optimality conditions for $(\mathrm{OPC})^\alpha$.

THEOREM 2.8. *Let* $(x^0, u^0) \in \mathbb{R}^n \times \mathbb{R}^m$ *satisfy Hypotheses* 2.1–2.3 *and suppose that* (x^α, u^α) *determined by Theorem* 2.7 *satisfy Hypothesis* 2.4. *There exists a neighborhood* A_0 *of* α_0 *with the following property. Let* $\alpha \in A_0$, $\alpha \neq \alpha_0$ *and assume that the constant functions* $(\bar{x}^\alpha, \bar{u}^\alpha) \in C(-r, \tau; \mathbb{R}^n) \times L^\infty(0, \tau; \mathbb{R}^m)$ *are a local minimum of* $(\mathrm{OPC})^\alpha$.

Then for all $(x, u) \in C(-r, \tau; \mathbb{R}^n) \times L^\infty(0, \tau; \mathbb{R}^m)$ *with*

$$\int_0^\tau [g_x(x^\alpha, u^\alpha)x(t) + g_u(x^\alpha, u^\alpha)u(t)]\, dt \leqq 0 \tag{2.17}$$

and

$$x_0 = x_\tau, \quad \dot{x}(t) = \mathcal{D}_1 f(\bar{x}^\alpha, u^\alpha, \alpha)x_t + f_u(\bar{x}^\alpha, u^\alpha, \alpha)u(t) \quad \text{a.e. } t \in [0, \tau], \qquad \underline{u}^0 + u \in \operatorname{int} \mathcal{U}_{\mathrm{ad}} \tag{2.18}$$

it follows that

$$\int_0^\tau [\mathcal{D}_1\mathcal{D}_1 H(\bar{x}^\alpha, u^\alpha, y^\alpha, \alpha)(x_t, x_t) + 2\mathcal{D}_1\mathcal{D}_2 H(\bar{x}^\alpha, u^\alpha, y^\alpha, \alpha)(x_t, u(t)) + \mathcal{D}_2\mathcal{D}_2 H(\bar{x}^\alpha, u^\alpha, y^\alpha, \alpha)(u(t), u(t))]\, dt \geqq 0. \tag{2.19}$$

Sketch of proof. By continuity Hypotheses 2.1–2.3 hold for α in a neighborhood of α_0. Problem $(\mathrm{OPC})^\alpha$ can be reformulated as an optimization problem over $(\varphi, u) \in C(-r, 0; \mathbb{R}^n) \times L^\infty(0, \tau; \mathbb{R}^m)$ (with $\varphi := x_0$) using the implicit function theorem near $(\bar{x}^\alpha, \bar{u}^\alpha) \in C(-r, \tau; \mathbb{R}^n) \times L^\infty(0, \tau; \mathbb{R}^m)$ (cf. [8, Chap. 5]). Application of optimization theory in Banach spaces (cf. [8, Chap. 2] or [16]) yields second order necessary optimality conditions for $(\varphi^\alpha, \bar{u}^\alpha)$ with $\varphi^\alpha := \bar{x}^\alpha$, involving Lagrange multipliers $(l^\alpha, y^\alpha, z^\alpha) \in C(-r, 0; \mathbb{R}^n)^* \times \mathbb{R}^{n+n_1} \times \mathbb{R}^l$. Since by assumption $\bar{u}^0 + u \in \operatorname{int} \mathcal{U}_{\mathrm{ad}}$, the term with z vanishes. Hypothesis 2.4 yields that the Lagrange multipliers for $(\mathrm{OPC})^\alpha$ can be obtained from Lagrange multipliers for $(\mathrm{OSS})^\alpha$ (cf. [8, Prop. VII.2.7]); these, however, are unique by Theorem 2.7.

For more details see Theorem VII.3.1 of [8].

Theorem 2.8 furnishes a test for local properness: If there are (x, u) satisfying (2.17) and (2.18) but violating (2.19), then $(\bar{x}^\alpha, \bar{u}^\alpha)$ cannot be a local optimal solution of $(\mathrm{OPC})^\alpha$. Using Hypothesis 2.4 it is advantageous to consider special (sinusoidal) test functions (x, u). First we introduce the following abbreviations ($\omega \in \mathbb{R}_+$, $\alpha \in A_0$):

$$
\begin{aligned}
P(\omega, \alpha) &:= \mathscr{D}_1\mathscr{D}_1 H(\bar{x}^\alpha, u^\alpha, y^\alpha, \alpha)(e^{j\omega\cdot}I, e^{-j\omega\cdot}I),\\
Q(\omega, \alpha) &:= \mathscr{D}_2\mathscr{D}_1 H(\bar{x}^\alpha, u^\alpha, y^\alpha, \alpha)(e^{j\omega\cdot}I),\\
R(\alpha) &:= \mathscr{D}_2\mathscr{D}_2 H(\bar{x}^\alpha, u^\alpha, y^\alpha, \alpha),\\
B(\alpha) &:= \mathscr{D}_2 f(\bar{x}^\alpha, u^\alpha, \alpha)
\end{aligned}
\tag{2.20}
$$

and for later purposes

$$L(\alpha) := \mathscr{D}_1 f(\bar{x}^\alpha, u^\alpha, \alpha).$$

Identify $P(\omega, \alpha)$, $Q(\omega, \alpha)$ and $R(\alpha)$ with elements in $\mathbb{C}^{n\times n}$, $\mathbb{C}^{n\times m}$ and $\mathbb{R}^{m\times m}$, respectively. Define

$$
\begin{aligned}
\Pi(\omega, \alpha) = {} & B(\alpha)^T\Delta(-j\omega, \alpha)^T P(\omega, \alpha)\Delta(j\omega, \alpha)B(\alpha)\\
&+ Q(-\omega, \alpha)^T\Delta(j\omega, \alpha)B(\alpha)\\
&+ B(\alpha)^T\Delta(-j\omega, \alpha)^T Q(\omega, \alpha) + R(\alpha).
\end{aligned}
\tag{2.21}
$$

COROLLARY 2.9 (Π-Test). *Let the assumptions of Theorem* 2.8 *be satisfied. Then* (x^α, u^α) *is locally proper, if there exist* $\nu_0, \nu_1 \in \mathbb{C}^m$ *with* ($\omega = 2\pi/\tau$)

$$
\begin{aligned}
&[g_x(x^\alpha, u^\alpha)\Delta(0, \alpha)B(\alpha) + g_u(x^\alpha, u^\alpha)]\nu_0\\
&\quad + [g_x(x^\alpha, u^\alpha)\Delta(j\omega, \alpha)B(\alpha) + g_u(x^\alpha, u^\alpha)]\nu_1 \leqq 0,\\
&h(u^\alpha + \nu_0 + \operatorname{Re}(\nu_1 e^{j\omega t})) \in \operatorname{int}\mathbb{R}^l_- \quad \text{for all } t \in [0, \tau],
\end{aligned}
\tag{2.22}
$$

$$\nu_0^T\Pi(0, \alpha^0)\nu_0 + 2\nu_1^T\Pi(\omega, \alpha)\nu_1 < 0. \tag{2.23}$$

Sketch of proof. Choose $u(t) := \nu_0 + \operatorname{Re}(\nu_1 e^{j\omega t})$, $t \in [0, \tau]$. Then (2.22) ensures that (2.17) and (2.18) are satisfied. Computation of the expression in (2.19) yields the one in (2.23) (cf. [8, Thm. VII.3.3]).

3. A scenario for local properness. Now we will relate local properness to structural changes in the system equation. The analysis is motivated by the following consideration. Suppose a Hopf bifurcation occurs at $\alpha = \alpha_0$. See Hale [12] or Hassard, Kazarinoff and Wang [13] for an exposition of Hopf bifurcation theory of functional differential equations. If the generated periodic solution is "better" than the steady-state solution, one will expect local properness at $\alpha = \alpha_0$. It turns out that under a controllability condition this is true for all α close to α_0. The controllability condition guarantees that the free periodic motion can be approximated by forced periodic motions for $\alpha \neq \alpha_0$. In fact it is not necessary that a Hopf bifurcation actually occur; instead some weaker properties stated below are sufficient.

Throughout this section we assume that Hypotheses 2.1–2.4 hold and hence the assertions of Theorems 2.7, 2.8, and Corollary 2.9 hold. Recall that the characteristic function of the linearized equation (with $L(\alpha) := \mathscr{D}_1 f(\bar{x}^\alpha, u^\alpha, \alpha)$)

$$\dot{x}(t) = L(\alpha)x_t, \qquad t \geqq 0 \tag{3.1}$$

is given by

$$\Delta(z, \alpha) = zI - L(\alpha)(e^{z\cdot}I), \qquad z \in \mathbb{C}. \tag{3.2}$$

LEMMA 3.1. *Suppose that for a pair* $(\omega_0, \alpha_0) \in (0, \infty) \times A$

$$
\begin{aligned}
&\operatorname{rank}\Delta(j\omega_0, \alpha_0) = n - 1,\\
&\operatorname{rank}\Delta(j\omega, \alpha) = n \quad \textit{for all } (\omega, \alpha) \neq (\omega_0, \alpha_0) \textit{ close to } (\omega_0, \alpha_0).
\end{aligned}
\tag{3.3}
$$

Then for all α *in a neighborhood of* α_0, (3.1) *has a simple eigenvalue* $z(\alpha)$ *and* $z(\alpha)$ *has a continuous derivative* $z'(\alpha_0)$ *at* $\alpha=\alpha_0$.

Proof. By Theorem 2.7, the map $\alpha \to L(\alpha)$ is continuously Fréchet differentiable, and Hale [12, Lemma 2.2, p. 171] implies the assertion. □

Remark 3.2. Condition (3.3) does not require that an eigenvalue actually cross the imaginary axis at $\alpha=\alpha_0$.

LEMMA 3.3. *Condition* (3.3) *implies that there exists a nontrivial* τ*-periodic solution of* (3.1) *with* $\alpha=\alpha_0$, $\tau:=2\pi/\omega_0$; *furthermore, there exists* $p_1 \in \mathbb{C}^n$ *such that for every such* τ*-periodic solution* p

$$p(t)=2\gamma \operatorname{Re}(e^{j\omega t}p_1), \qquad t \geqq 0, \tag{3.4}$$

for some $\gamma \in \mathbb{R}$.

Proof. By assumption the eigenspace corresponding to $z=j\omega_0$ is one-dimensional and the assertion follows (cf. Hale [12]). □

LEMMA 3.4. *Suppose that condition* (3.3) *is satisfied. Then the following two conditions are equivalent*:

$$\text{There exists } \nu_1 \in \mathbb{C}^m \quad \text{with } p_1=[\operatorname{Adj}\Delta(j\omega_0,\alpha_0)]B(\alpha_0)\nu_1 \tag{3.5}$$

where p_1 *is given by Lemma* 3.3 *and* Adj *denotes the adjunct*;

$$[\operatorname{Adj}\Delta(j\omega_0,\alpha_0)]B(\alpha_0)\neq 0. \tag{3.6}$$

Proof. Recall that

$$\Delta(j\omega_0,\alpha_0)[\operatorname{Adj}\Delta(j\omega_0,\alpha_0)]=\det\Delta(j\omega_0,\alpha_0)\cdot I$$

(see, e.g., Kowalsky [15, Kap. 4]). Thus the range of

$$[\operatorname{Adj}\Delta(j\omega_0,\alpha_0)]B(\alpha_0)$$

is contained in the kernel of $\Delta(j\omega_0,\alpha_0)$ which is spanned by p_1. □

Condition (3.5) may be viewed as a "controllability condition" for the periodic solution (3.4).

LEMMA 3.5. *Let condition* (3.3) *be satisfied. Then*

$$\bar{p}_1^T P(\omega_0,\alpha_0)p_1=\int_0^\tau \mathcal{D}_{1,2}\mathcal{D}_{1,2}H(\bar{x}^0,u^0,y^0,\alpha^0)((p_t,0),(p_t,0))\,dt$$

where p_1, $p(\cdot)$ *are as in Lemma* 3.3.

Proof. Obvious from the definitions and Lemma 3.3. □

The next theorem establishes the connection to local properness.

THEOREM 3.6. *Let* $(x^0,u^0)\in\mathbb{R}^n\times\mathbb{R}^m$ *satisfy the constraints of Problem* $(\mathrm{OSS})^{\alpha_0}$ *and suppose Hypotheses* 2.1–2.4 *hold. Furthermore, assume that conditions* (3.3) *and* (3.5) *are satisfied, and that there exists* $\nu_0\in\mathbb{C}^m$ *such that* ν_0 *and* ν_1 *satisfy* (2.22) *with* $\alpha=\alpha_0$. *Let* $p_0:=[\operatorname{Adj}\Delta(0,\alpha_0)]B(\alpha_0)\nu_0$ *and assume for* P *given by* (2.20)

$$\bar{p}_0^T P(0,\alpha_0)p_0+\bar{p}_1^T P(\omega_0,\alpha_0)p_1<0. \tag{3.7}$$

Then there exists a neighborhood $\mathcal{N}$ *of* (ω_0,α_0) *such that the steady states* (x^α,u^α) *being isolated local minima of* $(\mathrm{OSS})^\alpha$ *are locally proper and* (2.22), (2.23) *hold for all* $(\omega,\alpha)\in\mathcal{N}$, $(\omega,\alpha)\neq(\omega_0,\alpha_0)$.

Proof. In view of Theorem 2.7 and Corollary 2.9 it only remains to establish (2.22) and (2.23). By continuity, (2.22) is satisfied for (ω,α) near (ω_0,α_0) and ν_0, ν_1 replaced by some elements $\nu_0^\alpha, \nu_1^\alpha$, which depend continuously on α. Furthermore $B(\alpha)$, $\operatorname{Adj} \Delta(j\omega, \alpha)$ and $P(\omega, \alpha)$ are continuous with respect to (ω, α) and

$$[\det \Delta(j\omega, \alpha)]^2 > 0$$

for $(\omega, \alpha) \neq (\omega_0, \alpha_0)$ in a neighborhood of (ω_0, α_0). We have

$$\begin{aligned}
&\bar{\nu}_0^{\alpha T} B(\alpha)^T \Delta^{-1}(0, \alpha)^T P(0, \alpha) \Delta^{-1}(0, \alpha) B(\alpha) \nu_0^\alpha \\
&\qquad + \bar{\nu}_1^{\alpha T} B(\alpha)^T \Delta^{-1}(-j\omega, \alpha)^T P(\omega, \alpha) \Delta^{-1}(j\omega, \alpha) B(\alpha) \nu_1^\alpha \\
&\quad = [\det \Delta(0, \alpha)]^{-2} \{\bar{\nu}_0^\alpha B(\alpha)^T [\operatorname{Adj} \Delta(0, \alpha)^T] P(0, \alpha) [\operatorname{Adj} \Delta(0, \alpha)] B(\alpha) \nu^\alpha\} \\
&\qquad + [\det \Delta(j\omega, \alpha)]^{-2} \{\bar{\nu}_1^\alpha B(\alpha)^T [\operatorname{Adj} \Delta(-j\omega, \alpha)^T] P(\omega, \alpha) [\operatorname{Adj} \Delta(j\omega, \alpha)] B(\alpha) \nu_1^\alpha\}.
\end{aligned}$$

For $(\omega, \alpha) \to (\omega_0, \alpha_0)$ we have that $\det \Delta(j\omega, \alpha)$ tends to zero, while the second factor $\{\cdots\}$ in the second summand converges to $\bar{p}_1^T P(\omega_0, \alpha_0) p_1 < 0$.

Now consider the definition (2.21) of $\Pi(\omega, \alpha)$: For $(\omega, \alpha) \to (\omega_0, \alpha_0)$ the first summand tends to minus infinity with $[\det \Delta(j\omega, \alpha)]^{-2}$, the others tend to infinity with at most $|\det \Delta(j\omega, \alpha)|^{-1}$.

Thus the first summand becomes dominant and hence

$$\bar{\nu}^{0T} \Pi(0, \alpha) \nu^0 + \bar{\nu}^{1T} \Pi(j\omega, \alpha) \nu^1 < 0 \tag{3.8}$$

for all $(\omega, \alpha) \neq (\omega_0, \alpha_0)$ in a neighborhood of (ω_0, α_0). □

COROLLARY 3.7. *Let* $(x^0, u^0) \in \mathbb{R}^n \times \mathbb{R}^m$ *satisfy the constraints of problems* $(\mathrm{OPC})^{\alpha_0}$ *and* $(\mathrm{OSS})^{\alpha_0}$ *without control constraints* (*i.e.*, $h \equiv 0$) *and assume that Hypotheses* 2.1–2.4 *hold and conditions* (3.3) *and* (3.5) *are satisfied. If*

$$\bar{p}_1^T P(\omega_0, \alpha_0) p_1 < 0 \tag{3.9}$$

where p_1 *is given by Lemma* 3.3, *then there exists a neighborhood* $\mathcal{N}$ *of* (ω_0, α_0) *such that the steady states* (x^α, u^α) *being isolated local minima of* $(\mathrm{OSS})^\alpha$ *are locally proper and*

$$\bar{\nu}_1^T \Pi(\omega, \alpha) \nu_1 < 0 \quad \text{for all } (\omega, \alpha) \in \mathcal{N}, \quad (\omega, \alpha) \neq (\omega_0, \alpha_0), \tag{3.10}$$

where ν_1 *is given by Lemma* 3.4.

Proof. Follows from Lemma 3.4 and Theorem 3.6.

Remark 3.8. In Corollary 3.7, Condition (3.5) may be replaced by (3.6).

Remark 3.9. The second order sufficient optimality condition for the steady-state problem (i.e., Hypothesis 2.2) and the "complementary slackness" condition in Hypothesis 2.3 are needed in order to guarantee smooth dependence of $(x^\alpha, u^\alpha, y^\alpha)$ on α. If this can be guaranteed by other arguments (e.g., if the steady-state problem is independent of α as in the example of § 4, below) we can replace Hypothesis 2.2 by the assumption that (x^α, u^α) are a local minimum of $(\mathrm{OSS})^\alpha$.

The following result is a partial converse of Corollary 3.7.

THEOREM 3.10. *Let the assumptions of Corollary* 3.7 *be satisfied. If there exists a sequence* $(\omega_n, \alpha_n) \to (\omega_0, \alpha_0)$, $(\omega_n, \alpha_n) \neq (\omega_0, \alpha_0)$ *with*

$$\bar{\nu}^T \Pi(\omega_n, \alpha_n) \nu > 0 \tag{3.11}$$

where $\nu = \nu_1$ *is given by Lemma* 3.4, *then*

$$\bar{p}_1^T P(\omega_0, \alpha_0) p_1 \geqq 0 \tag{3.12}$$

where p_1 *is given by Lemma* 3.3.

Proof. Condition (3.12) and (2.21) imply

$$\begin{aligned} 0 &< \bar{\nu}^T \Pi(\omega_n, \alpha_n)\nu \\ &= \bar{\nu}^T B(\alpha_n)^T \Delta^{-1}(-j\omega_n, \alpha_n)^T P(\omega_n, \alpha_n)\Delta^{-1}(j\omega_n, \alpha_n)B(\alpha_n)\nu \\ &\quad + \bar{\nu}^T \{B(\alpha_n)^T \Delta^{-1}(-j\omega_n, \alpha_n)^T Q(\omega_n, \alpha_n) \\ &\qquad + Q(-\omega_n, \alpha_n)^T \Delta^{-1}(j\omega_n, \alpha_n)B(\alpha_n) + R(\alpha_n)\}\nu. \end{aligned}$$

The first summand equals

$$[\det \Delta^{-1}(-j\omega_n, \alpha_n)]^{-2}\{\bar{\nu}^T B(\alpha_n)\ [\mathrm{Adj}\,\Delta(-j\omega_n, \alpha_n)]^T P(\omega_n, \alpha_n)\ \mathrm{Adj}\,\Delta(j\omega_n, \alpha_n)B(\alpha_n)\nu\}.$$

Again $[\det \Delta(-j\omega_n, \alpha_n)]^2 > 0$, and the second factor converges to

$$\begin{aligned} &\bar{\nu}^T B(\alpha_0)[\mathrm{Adj}\,\Delta(-j\omega_0, \alpha_0)]^T P(\omega_0, \alpha_0)\ \mathrm{Adj}\,\Delta(j\omega_0, \alpha_0)B(\alpha_0)\nu \\ &\quad = \bar{p}_1^T P(\omega_0, \alpha_0)p_1. \end{aligned}$$

Arguing as in the proof of Theorem 3.6, we obtain (3.12). □

Remark 3.11. Suppose that a Hopf bifurcation occurs at $\alpha = \alpha_0$ (cf. Hale [12, Thm. 1.1, p. 246]). Then Theorem 3.6 may be interpreted as follows: At $\alpha = \alpha_0$, a "natural" periodic solution of $\dot{x}(t) = f(x_t, u^\alpha, \alpha)$ bifurcates from the steady state x^α, $\alpha = \alpha_0$. By (3.7), this periodic motion shows better average performance than the steady state. Condition (3.3) is satisfied and the controllability condition (3.5) guarantees (by continuity) that for all α near α_0 the periodic trajectory can be approximated by trajectories corresponding to a sinusoidal control. Hence, for α near α_0, the points (x^α, u^α) are locally proper. Suppose nontrivial periodic trajectories exist for $\alpha > \alpha_0$. Then, also for $\alpha < \alpha_0$, where no free periodic trajectory exists, we can generate periodic trajectories by appropriate sinusoidal controls. Thus it is not surprising that the assumption can be weakened by requiring only the assumptions of Theorem 3.6: it is not necessary that the nonlinear equation actually has a free periodic trajectory. In view of this discussion, it seems feasible to me to use the expression "Controlled Hopf Bifurcation" if conditions (3.3) and (3.5) are satisfied.

Remark 3.12. The stability properties of the periodically forced equations near $\alpha = \alpha_0$ may be very complicated; cf. Gambaudo [10] for a classification in the case of two-dimensional ordinary differential equations.

4. An example. In this section we consider an optimal periodic control problem for a retarded Lienard equation where Corollary 3.7 applies. First results for this problem were obtained in [6]. The problem is the following:

$$\text{Minimize} \quad -\frac{1}{\tau}\int_0^\tau x(s)\,ds + \frac{1}{2\tau}\int_0^\tau u(s)^2\,ds$$

subject to

$$\ddot{x}(t) + f(x(t))\dot{x}(t) + g(x(t-r)) = u(t) \quad \text{a.e. } t \in [0, \tau], \tag{4.1}$$

$$x_0 = x_\tau, \qquad (\dot{x})_0 = (\dot{x})_\tau, \tag{4.2}$$

$$\int_0^\tau u(t)\,dt = 0; \tag{4.3}$$

here f and $g: \mathbb{R} \to \mathbb{R}$; $x(t), u(t) \in \mathbb{R}$, and $r, \tau > 0$. We require that

(4.4) f and g are C^2-functions in a neighborhood of zero with $f(0) = g(0) = 0$, $g'(0) = 1$, $g''(0) = -1$, and $f(x) \neq 0$ for $x > 0$.

Writing (4.1) as a system of first order equations and applying the time transformation $t := tr$, we get

$$\dot{x}_1(t) = \frac{1}{r} x_2(t),$$

(4.5)

$$\dot{x}_2(t) = \frac{1}{r}[-f(x_1(t))x_2(t) - g(x(t-1)) + u(t)].$$

Consider $\alpha = 1/r > 0$ as bifurcation parameter.

The corresponding steady-state problem is

(OSS) Minimize $-x_1 + \frac{1}{2}u^2$ subject to

$$0 = x_2,$$

(4.6) $$0 = -f(x_1)x_2 - g(x_1) + u,$$

$$0 = u.$$

The assumptions in (4.4) guarantee that $(x^0, u^0) = (0, 0)$ is the unique optimal solution of (OSS). Observe that (OSS) is independent of α; hence Remark 3.9 applies and we can omit Hypothesis 2.2.

Furthermore Hypothesis 2.3 is satisfied and the corresponding Lagrange multipliers are

$$y_1 = -g'(0)^{-1} f(0) = 0,$$

(4.7) $$y_2 = -g'(0)^{-1} = -1,$$

$$y_3 = g'(0) = 1.$$

The linearized system equation is

(4.8) $$\dot{x}(t) = \alpha \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \alpha \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} x(t-1) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$

or

(4.9) $$\ddot{x}(t) + \alpha^2 x(t-1) = u(t).$$

Thus the characteristic equation is

(4.10) $$\det \Delta(z, \alpha) = z^2 + \alpha^2 e^{-z} = 0.$$

LEMMA 4.1. (i) *There exists an eigenvalue z of* (4.8) *on the imaginary axis if and only if $\alpha = \alpha_n = 1/(2n\pi)$, $n \in \mathbb{N}$.*

(ii) *If $\alpha = \alpha_n$ for some $n \in \mathbb{N}$, then the eigenvalues z on the imaginary axes are $z = \pm j$.*

(iii) *For $\alpha \to 0$ all eigenvalues in the right half-plane tend to the origin.*

(iv) *For α close to α_n, there exists a C^1-function $\alpha \to z(\alpha)$ such that $z(\alpha)$ is a simple eigenvalue of* (4.8) *and $z(\alpha_n) = j$, $z'(\alpha_n) > 0$ and* Re $z(\alpha) < 0$ *for $\alpha < \alpha_n$.*

Proof. The proof follows by an elementary analysis of (4.10).

The lemma shows that for $\alpha = \alpha_n$, $n \in \mathbb{N}$, a Hopf bifurcation occurs with frequency $\omega_0 = 1$.

A nontrivial periodic solution of

$$\dot{x}(t) = \alpha_n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \alpha_n \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} x(t-1)$$

with period $\tau = 2\pi$ is given by

$$p(t) = 2\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}.$$

The Fourier coefficients of p are

$$p_1 = p\hat{\ }(1) = \begin{pmatrix} 1 \\ j \end{pmatrix}, \qquad \bar{p}_1 = p\hat{\ }(-1) = \begin{pmatrix} 1 \\ -j \end{pmatrix},$$

$$p\hat{\ }(k) = 0 \quad \text{for } k \neq \pm 1.$$

The function $H: C(-1, 0; \mathbb{R}^2) \times \mathbb{R} \times \mathbb{R}^3 \times (0, \infty) \to \mathbb{R}$ is given by

$$H(\phi, u, y, \alpha) := -\phi_1(0) + \frac{1}{2}u^2 + \alpha y^T \begin{pmatrix} \phi_2(0) \\ -f(\phi_1(0))\phi_2(0) - g(\phi_1(-1)) + u \\ u \end{pmatrix}.$$

We compute

$$\bar{p}_1^T P(\omega_0, \alpha_0) p_1 = (1 \quad -j) \begin{pmatrix} -1 & f'(0) \\ f'(0) & 0 \end{pmatrix} \begin{pmatrix} 1 \\ j \end{pmatrix}$$
$$= -1 < 0;$$

thus (3.9) holds.

It only remains to show the controllability condition (3.6) (cf. Remark 3.8). We easily compute

$$\operatorname{Adj}[\Delta(j\omega, \alpha)] B_0(\alpha) = \begin{pmatrix} j\omega & 1 \\ -\exp(-j\omega, \alpha) & j\omega \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ j\omega \end{pmatrix}.$$

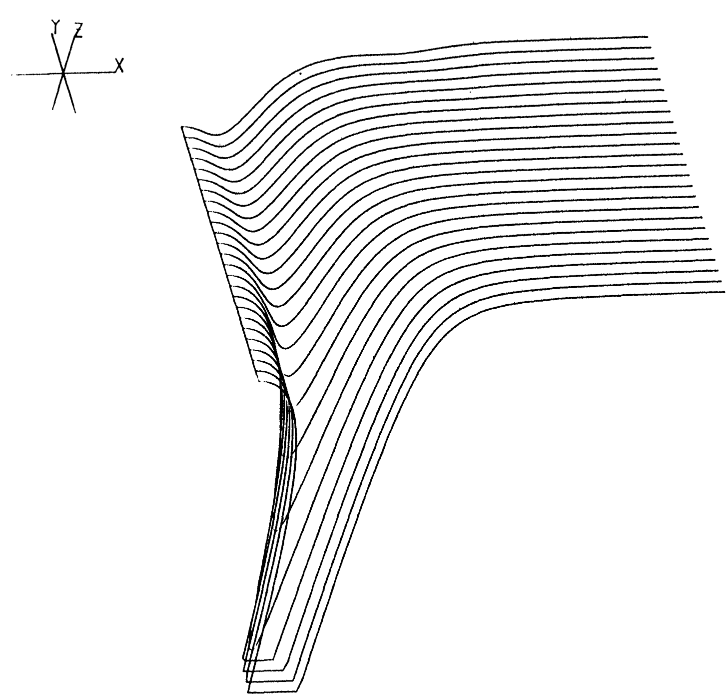

FIG. 1. *Shows* $\Pi(\omega, r)$, $0 \leqq \omega \leqq 4$, *for different values of* r *between* $r = 0$ *and* $r = 3$ ($X = \omega$, $Y = r$, $Z = \Pi(\omega, r)$). *The function values are cut off for* $z < -3$.

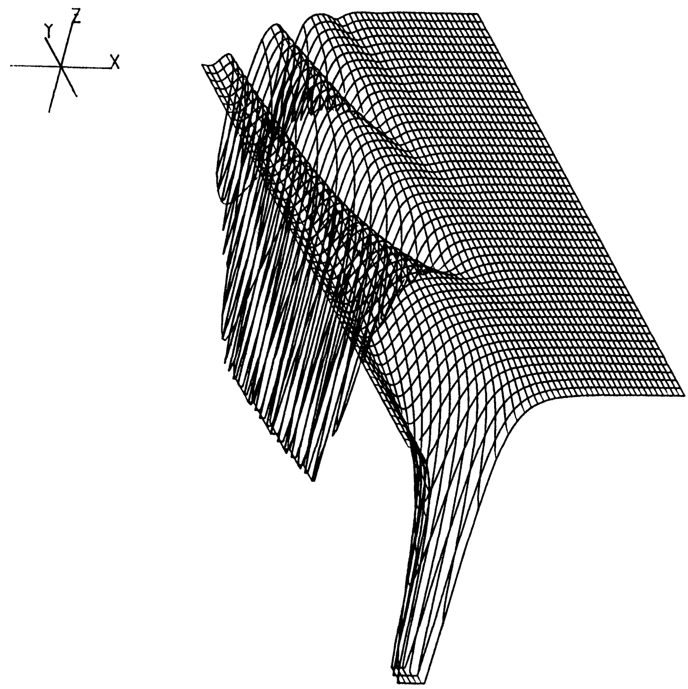

FIG. 2. *Shows* $\Pi(\omega, r)$, $0 \leqq \omega \leqq 4$, *for different values of r between* $r = 0$ *and* $r = 10$.

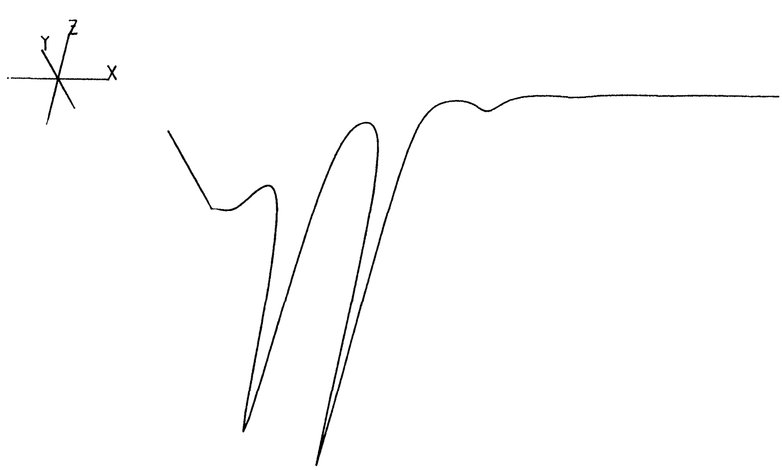

FIG. 3. *Shows* $\Pi(\omega, r)$, $0 \leqq \omega \leqq 4$, *for* $r = 10$.

Clearly

$$p_1 = \begin{pmatrix} 1 \\ j\omega_0 \end{pmatrix} \in \mathcal{R}[\mathrm{Adj}\,[\Delta(\alpha_0, j\omega_0)]B_0(\alpha_0)].$$

Thus all the assumptions of Corollary 3.7 are verified. It is advantageous to write Π as a function of the delay r and the frequency ω. Then a simple computation yields

$$\Pi(\omega, r) = 1 - 1/[\omega^4 - 2\omega^2 \cos(\omega r) + 1] \tag{4.11}$$

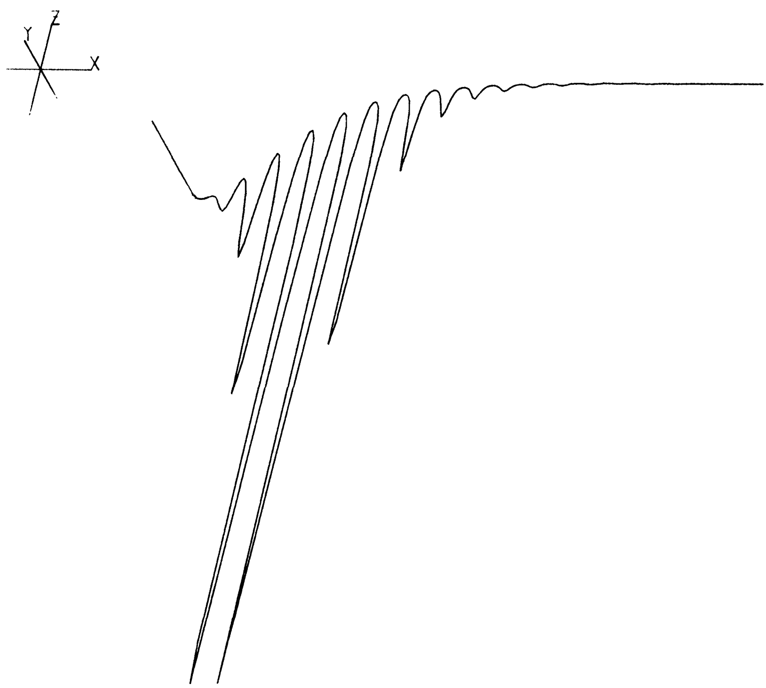

FIG. 4. *Shows* $\Pi(\omega, r)$, $0 \leqq \omega \leqq 4$, *for* $r = 30$.

for $(\omega, r) \neq (1, r_n)$, where $r_0 = 1$, $r_n := 1/\alpha_n$, $n \in \mathbb{N}$. Zones of local properness, indicated by $\Pi(\omega, r) < 0$ occur for (ω, r) close to $(1, r_n)$.

An analysis of the function Π given by (4.11) yields that for all $\omega, r \in \mathbb{R}_+$

$$1 - 1/[\omega^2 - 1]^2 \leqq \Pi(\omega, r) \leqq 1 - 1/[\omega^2 + 1]^2,$$
$$\Pi(0, r) = 0, \qquad \lim_{\omega \to \infty} \Pi(\omega, r) = 1.$$

Figures 1–4 show $\Pi(\omega, r)$ for different values of r (here $X = \omega$, $Y = r$, $Z = \Pi(\omega, r)$). A significant feature of this example is that the zones of properness (i.e., the ω-intervals where $\Pi(\omega, r) < 0$) which occur at a Hopf bifurcarion at $r = r_n$ (indicated by a negative pole of $\Pi(\omega, r)$ at $\omega = 1$) do not vanish for increasing r. Thus for large r, $\Pi(\omega, r)$ becomes very oscillatory (see Fig. 4).

Remark 4.2. It is easy to check that the function $\Pi: \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ has no local minima besides $(\omega, r) = (1, r_n)$. This may be interpreted in the following way: Local properness in this problem occurs *only* via the mechanism described by Corollary 3.7. Naturally, this may not be true for other problems (e.g., local properness may be due to nonlinearities in the performance criterion).

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