Optimality for Periodic Control of Functional Differential Systems*

F. COLONIUS†

Division of Applied Mathematics, Brown University, Providence, Rhode Island 02912

1. INTRODUCTION

This paper is concerned with periodic solutions x of period $t_1 - t_0$ and periodic controls u of the same period for functional differential systems

$$\dot{x}(t) = f(x_t, u(t)), \qquad t \geqslant t_0 \tag{1.1}$$

such that the "average cost"

$$1/(t_1 - t_0) \int_{t_0}^{t_1} g(x(s), u(s))\, ds \tag{1.2}$$

is minimal; here $x_t(s) := x(t+s)$, $s \in [-r, 0]$, $r > 0$. The periodicity condition can be formalized by the requirement

$$x_{t_0} = x_{t_1}. \tag{1.3}$$

Then minimization may be performed on the interval $[t_0, t_1]$ only, where it is understood that x and u are periodically extended to $[t_0, \infty)$. These periodic extensions satisfy the system equation (1.1), since x_t is a complete state for this functional differential system. Hence it is sufficient to consider the optimal periodic control problem (1.1), (1.2) on $[t_0, t_1]$ with "mixed" boundary condition (1.3).

Optimal periodic control problems for systems governed by ordinary differential equations have been studied since the 1960s where the original motivation came from chemical engineering (in particular, control of continuous stirred tank reactors). Most of the applications lie in this area and

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† Present address: Institut für Dynamische Systeme, Universität Bremen, D-28 Bremen 33, West Germany.

in aircraft performance optimization (cf. [1, 21, 34, 40, 47]; for other applications see, e.g., [16, 28, 32, 48, 50]).

From the vast literature on optimal periodic control theory (cf. also [22, 24, 33]) we only cite the following small collection: A complete theory of first order necessary optimality conditions has been given in [20]. The most general treatment of second order necessary optimality conditions appears in [4] and sufficient optimality conditions are proven in [30, 54]. In [39], methods from nonlinear analysis are used to show the existence of periodic trajectories x and of optimal periodic solutions; see also [36, 45].

Except for [36] this work is concerned with systems governed by ordinary differential equations. Furthermore, there exists a well developed theory for discrete time systems (see, e.g., [8, 42, 49]). However, [46] is (as far as we know) the only paper, where optimality conditions for delay systems are treated. Its contribution will be discussed below.

It is the purpose of the present paper, to develop first and second order optimality conditions for optimal periodic control of functional differential systems. Our hope that this may contribute to applications of this theory is enhanced by the fact that functional differential equations frequently occur in mathematical models of chemical engineering [17, 44], which—as mentioned above—also is a major field of applications of periodic control. Other areas, as biological modelling [14, 50], appear promising, too.

Compared to the theory for ordinary differential equations, the proof of optimality conditions for periodic control of functional differential equations is complicated by two facts:

(a) The equality constraint corresponding to the periodicity requirement is infinite dimensional. This prohibits the use of Neustadt's theory of extremals for first order optimality conditions [38] and its generalization to higher order conditions in [5]. Observe that the proofs in [20] and [4] of optimality conditions for periodic control are based on [38, 5]. This point leads us to a use of mathematical programming theory in general Banach spaces where Frechet-differentiability is a necessary prerequisite. Hence we are restricted to weak variations of the control (in L_∞), instead of strong variations (in L_2) (alternatively, one might try to use methods from convex analysis: see [3]).[1]

(b) Duality theory of functional differential equations is rather involved, since the "adjoint equation" is not the functional analytic adjoint. In [13], this problem is dealt with using "structural operators" and their adjoints. Some results which are relevant here are cited in Section 2.

Note added in proof: Using Ekeland's Variational Principle, a global maximum principle (based on strong variations) for optimal periodic control of functional differential systems is proven in F. Colonius, Optimal Periodic Control, Habilitationsschrift, Universität Bremen, 1986.

Section 3 contains first order necessary optimality conditions. The result shows that for the considered class of systems the optimal periodic control problem is much more well-behaved than the fixed final state problem: Having struggled with the latter problem for some time, I was surprised to see that for optimal periodic control an approximate controllability property (or much less: see Theorem 3.2 and Proposition 3.1) is sufficient to get adjoint variables satisfying an adjoint equation, while for fixed final state problems approximate controllability without control constraint is useless and—even worse—characterizes the untreatable case (cf. [29, 10]).

The restriction to weak variations being a consequence of the considerations above is particularly important for a fundamental problem in optimal periodic control theory: When does time-dependent periodic control produce better performance than constant steady-state control? An optimum in the class of constant steady states is called proper if the average cost can be lowered (locally) by allowing periodic control. For systems governed by ordinary differential equations it is well known that first order optimality conditions based on weak variations are useless in ascertaining proper. The reason is that these conditions for periodic control coincide with the (static) optimality conditions for constant steady state solutions. In Section 4, we show that this generalizes to functional differential systems. This section also contains a discussion of the normality conditions which have to be assumed in the periodic and the static optimization problems. The papers [23, 7] pioneered a way to decide the question of properness on the basis of weak control variations: proper solutions do not satisfy second order necessary optimality conditions for optimal periodic solutions. Then a Fourier series expansions of the second order condition leads to the so-called Π-Criterion, which can more easily be applied. In Section 5, we prove second order optimality conditions for general optimal periodic solutions of functional differential systems.

In Section 6, a specialization to steady states and a Fourier series expansion lead to a generalized Π-Criterion for functional differential systems involving the characteristic matrix of the linearized system. This Π-Criterion coincides with the criterion given in [46] for the common range of applications (in [46] finitely many discrete delays are allowed which may depend on state and control). However, the arguments leading to the result in [46] are only heuristic. The critics of [6] on the arguments of [7] apply equally well to [46]: Since it is not clarified which variations are permissible, there may be collisions with abnormality. This is one of L. C. Young's "sad facts of life" (see [51, p. 218]; we remain silent about the others). Hence Section 6 contains the first rigorous proof of a Π-Criterion for delay systems.

Notation

The transpose of an element $x \in \mathbb{R}^n$ is denoted by x^{T}; similarly for matrices. For a map F between Banach spaces, $\mathscr{D}F$ denotes its Fréchet derivative, and $\mathscr{D}_i F$ denotes its partial Fréchet-derivative with respect to the ith argument. Frequently, we denote the value $x^*(x)$ of a continuous linear functional x^* in the dual X^* of a Banach space X at $x \in X$ by $\langle x^*, x \rangle$.

$\mathrm{NBV}(t_0, t_1; \mathbb{R}^n)$ is the space of normalized functions ψ of bounded variation on $[t_0, t_1]$, i.e., ψ is left continuous on (t_0, t_1) and $\psi(t_1) = 0$; furthermore $W_2^{(1)}(t_0, t_1; \mathbb{R}^n)$ is the Hilbert space of absolutely continuous function x on $[t_0, t_1]$ endowed with the norm $\|x\| := |(x(t_0), \|\dot{x}\|_{L^2})|$.

2. Preliminaries

We consider the following optimal periodic control problem:

(P1) Minimize $1/(t_1 - t_0) \int_{t_0}^{t_1} g(x(s), u(s), s)\, ds$ s.t.

$$\dot{x}(t) = f(x_t, u(t), t) \qquad \text{for a.a.} \quad t \in [t_0, t_1], \tag{2.1}$$

$$x_{t_0} = x_{t_1}, \tag{2.2}$$

$$u(t) \in \Omega(t) \subset \mathbb{R}^m, \qquad \text{a.a.} \quad t \in [t_0, t_1]; \tag{2.3}$$

here $x_t(s) := x(t+s) \in \mathbb{R}^n$, $s \in [-r, 0]$, $u(t) \in \mathbb{R}^m$, $f\colon C(-r, 0; \mathbb{R}^n) \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$, $g\colon \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ and $\Omega(t)$ is assumed to be closed and convex with $t \mapsto \Omega(t)$ measurable.

It is convenient to reformulate our problem as a minimization problem in Banach spaces. Hence we introduce the closed and convex set

$$Q := \{u \in L_\infty(t_0, t_1; \mathbb{R}^m)\colon u(t) \in \Omega(t) \text{ a.e.}\}$$

and the maps

$$G\colon \quad C(t_0, t_1; \mathbb{R}^n) \times L_\infty(t_0, t_1; \mathbb{R}^m) \to \mathbb{R},$$

$$G(x, u) := 1/(t_1 - t_0) \int_{t_0}^{t_1} g(x(s), u(s), s)\, ds,$$

$$R\colon C(t_0 - r, t_1; \mathbb{R}^n) \to C(-r, 0; \mathbb{R}^n),$$

$$Rx := x_{t_1}.$$

Throughout this paper the following assumptions are made:

(A1) For every $\phi \in C(-r, 0; \mathbb{R}^n)$ and $u \in L_\infty(t_0, t_1; \mathbb{R}^m)$ there exists a unique solution $x = S(\phi, u)$ of the initial value problem (2.1), (2.2) and the solution operator

$$S\colon \quad C(-r, 0; \mathbb{R}^n) \times L_\infty(t_0, t_1; \mathbb{R}^m) \to C(t_0 - r, t_1; \mathbb{R}^n)$$

is continuously Fréchet-differentiable with derivative

$$\mathscr{D}S(\phi^0, u^0).$$

(A2) G is continuously Fréchet-differentiable.

In view of (A1), we get the following equivalent problem formulation:

(P2) Minimize$_{u,\phi}$ $G(S(\phi, u), u)$ s.t.

$$RS(\phi, u) = \phi, \qquad u \in Q.$$

The following assumptions are introduced to get necessary optimality conditions in *normal* form for optimal periodic solutions (ϕ^0, u^0) (resp. optimal steady states $(x^1, u^1) \in \mathbb{R}^n \times \mathbb{R}^m$).

(A3) $\{\phi - R\mathscr{D}S(\phi^0, u^0)(\phi, u) \mid u = \alpha(v - u^0), \alpha \geqslant 0, v \in Q, \phi \in C(-r, 0; \mathbb{R}^n)\} = C(-r, 0; \mathbb{R}^n)$.

The following condition (A4) applies to the case, where f and Ω are independent of t, i.e., f is as in (1.1) and $\Omega(t) \equiv \Omega$.

(A4) $\{\mathscr{D}_1 f(\bar{x}^1, u^1)\, \bar{x} \mid x \in \mathbb{R}^n\} + \{\mathscr{D}_2 f(\bar{x}^1, u^1)\, u \mid = \alpha(v - u)^1), \alpha \geqslant 0, v \in \Omega\} = \mathbb{R}^n$,

$$\text{here} \quad \bar{x}^1, \bar{x} \in C(-r, 0; \mathbb{R}^n) \qquad \text{are defined by}$$

$$\bar{x}^1(s) = x^1, \quad \bar{x}(s) := x \qquad \text{for} \quad s \in [-r, 0].$$

Remark 2.1. For simplicity, the differentiability hypotheses (A1) and (A2) have been formulated on an abstract level in terms of S and G. One can easily give conditions in terms of f and g which ensure that (A1) and (A2) are satisfied (cf., e.g., [12, 25]).

Next, we collect some material on linear time-varying retarded functional differential equations in the state space $C(-r, 0; \mathbb{R}^n)$ which will be needed in the sequel. In particular, we cite some results on duality and structure theory from [13] (cf. also [25, 2, 26, 15]).

Consider the equation

$$\dot{x}(t) = L(t)\, x_t, \qquad t \geqslant t_0, \tag{2.4}$$

where $L(t)\colon C(-r, 0; \mathbb{R}^n) \to \mathbb{R}^n$ is bounded and linear with $t \mapsto L(t)$ measurable and essentially bounded. Then $L(t)$ admits a representation in the form

$$L(t)\, \phi = \int_{-r}^{0} d_\theta \eta(t, \theta)\, x(t + \theta), \tag{2.5}$$

where $\eta(t, \theta)$ is an $n \times n$ matrix function, measurable and essentially bounded with respect to t and of bounded variation with respect to s, normalized such that $s \mapsto \eta(t, s)$ is left continuous on $(-r, 0)$ with $\eta(t, 0) = 0$. Then η is measurable on the rectangle $[t_0, t_1] \times [-r, 0]$.

In presence of the initial condition

$$x_{t_0} = \phi \in C(-r, 0; \mathbb{R}^n), \tag{2.6}$$

eq. (2.4) is equivalent to

$$x(t) = \phi(0) + \int_{t_0}^{t} \int_{-r}^{0} d_\theta \eta(s, \theta)\, x(s+\theta)\, ds.$$

The evolution of the solution segment x_t in $C(-r, 0; \mathbb{R}^n)$ is described by the family of operators $\Phi(t, s)$, $t \geqslant s$, associating with $\phi \in C(-r, 0; \mathbb{R}^n)$ the solution segment x_t of (2.4) with initial condition $x_s = \phi$.

Define the "forcing term operator" $F(t_0)$,

$$F(t_0)\colon \quad C(-r, 0; \mathbb{R}^n) \to C(0, r; \mathbb{R}^n)$$

by

$$[F(t_0)\, \phi](\alpha) := \phi(0) + \int_{t_0}^{t_0+\alpha} \int_{-r}^{t_0 - s} d_\theta \eta(s, \theta)\, \phi(s + \theta - t_0)\, ds,$$
$$\alpha \in [0, r].$$

Then $F(t_0)\,\phi$ describes the "effect" of the initial condition (2.6) on the velocity and $F(t)\, x_{t_0}$ may be viewed as the "effective state" of (2.4) at time t_0.

Define $G(t_0)$: $C(0, r; \mathbb{R}^n) \to C(-r, 0; \mathbb{R}^n)$ as the operator associating with the "forcing term" $\psi \in C(0, r; \mathbb{R}^n)$ the solution segment $x_{t_0 + r}$ of

$$x(t) = \int_{t_0}^{t} \int_{-r}^{0} d_\theta \eta(s, \theta)\, x(s+\theta)\, ds + \psi(t - t_0), \quad t \in [t_0, t_0 + r],$$
$$x_{t_0} = 0.$$

By definition

$$\Phi(t_0 + r, t_0) = G(t_0)\, F(t_0).$$

The formal adjoint equation of (2.4) is the "backwards" equation

$$\frac{d}{ds}\left\{ y(s) + \int_{s}^{t_1 + r} \eta(\alpha, s - \alpha)^T\, y(\alpha)\, d\alpha \right\} = f(s). \tag{2.7}$$

Define for $s \leqslant t_1$,

$$y^s(\theta) = \begin{cases} y(s+\theta), & 0 \leqslant \theta < h \\ 0, & \theta = h. \end{cases} \tag{2.8}$$

A final condition for (2.7) is given by

$$y^{t_1} = \psi \in \mathrm{NBV}(0, r; \mathbb{R}^n).$$

The evolution of y^t in $\mathrm{NBV}(0, r; \mathbb{R}^n)$ is described by a family $\Phi^T(t, s)$, $s \leqslant t$, of operators on $\mathrm{NBV}(0, r; \mathbb{R}^n)$, and the solution y^s of (2.7) with final condition (2.8) is given by

$$y^s = \Phi^{\mathrm{T}}(t_1, s)\,\psi + \int_s^{t_1} \Phi^{\mathrm{T}}(\sigma, s)\, Y_0 f(\sigma)\, d\sigma, \tag{2.9}$$

where $Y_0 f(\sigma)$ is the following function in $\mathrm{NBV}(0, r; \mathbb{R}^n)$:

$$[Y_0 f(\sigma)](\alpha) := \begin{cases} -\displaystyle\int_\sigma^{t_1} f(\tau)\, d\tau, & \alpha = 0 \\ 0, & 0 < \alpha \leqslant r. \end{cases}$$

Consider the integrated version of the homogeneous equation (2.7),

$$y(s) - y(t_1) = -\int_s^{t_1+r} [\eta(\alpha, s-\alpha)^{\mathrm{T}} - \eta(\alpha, t_1-\alpha)^{\mathrm{T}}]\, y(\alpha)\, d\alpha, \qquad s \leqslant t_1, \tag{2.10}$$

and define the corresponding "forcing term" operator

$$F^{\mathrm{T}}(t) = \mathrm{NBV}(0, r; \mathbb{R}^n) \to \mathrm{NBV}(-r, 0; \mathbb{R}^n)$$

by

$$[F^{\mathrm{T}}(t)\,\psi](\theta) := \psi(0) - \int_0^r [\eta(t_1+\alpha, \theta-\alpha)^{\mathrm{T}} - \eta(t_1+\alpha, -\alpha)^{\mathrm{T}}]\, \psi(\alpha)\, d\alpha, \qquad \theta \in [-r, 0).$$

Then

$$F^{\mathrm{T}}(t_1) = F(t_1)^*. \tag{2.11}$$

Thus the adjoint of the forcing term operator of the original equation is the forcing term operator of the formal adjoint equation. The relation between the formal adjoint equation and the functional analytic adjoint of $\Phi(t, s)$ is elucidated by the following intertwining relation $(s \leqslant t)$:

$$\Phi^*(t, s) F^{\mathrm{T}}(t) = F^{\mathrm{T}}(s) \Phi^{\mathrm{T}}(t, s). \tag{2.12}$$

3. First Order Optimality Conditions for Periodic Control

In this section we begin the analysis of optimal periodic control. We prove first order necessary optimality conditions and analyze their concrete form using the structural theory of Section 2. We note the following:

Lemma 3.1. *Let x be defined by*

$$x := \mathscr{D}_1 S(\phi^0, u^0) \phi + \mathscr{D}_2 S(\phi^0, u^0) u. \tag{3.1}$$

Then x is the unique solution of the initial value problem

$$\begin{aligned} &x_{t_0} = \phi, \\ &\dot{x}(t) = \mathscr{D}_1 f(x_t^0, u^0(t), t) x_t + \mathscr{D}_2 f(x_t^0, u^0(t), t) u(t), \quad \text{a.e. } t \in [t_0, t_1]. \end{aligned} \tag{3.2}$$

Proof. Clear by the chain rule. ∎

Theorem 3.1. *Let (ϕ^0, u^0) be an optimal solution of* (P2), *or equivalently, of* (P1). *Then there exist nontrivial Lagrange multipliers $(l_0, l) \in \mathbb{R}_+ \times C(-r, 0; \mathbb{R}^n)^*$ such that with $x^0 := S(\phi^0, u^0)$*

$$\begin{aligned} &l_0 \mathscr{D}_1 G(x^0, u^0) \mathscr{D}_1 S(\phi^0, u^0) \phi + l_0 \mathscr{D}_1 G(x^0, u^0) \mathscr{D}_2 S(\phi^0, u^0) u \\ &\qquad + l_0 \mathscr{D}_2 G(x^0, u^0) u + \langle l, \phi - R\mathscr{D}_1 S(\phi^0, u^0) \phi - R\mathscr{D}_2 S(\phi^0, u^0) u \rangle \\ &\quad \geqslant 0 \end{aligned} \tag{3.3}$$

for all $\phi \in C(-r, 0; \mathbb{R}^n)$ and $u = \alpha(v - u^0)$, $\alpha \geqslant 0$, $v \in Q$. If, additionally, the normality assumption (A3) *is satisfied, we may choose $l_0 = 1$.*

Proof. In the normal case where (A3) holds, the theorem is an easy consequence of [52, Theorem 1] and the chain rule. Suppose (A3) is violated. Observe that by Lemma 3.1 (cf. Sect. 2)

$$R\mathscr{D}_1 S(\phi^0, u^0) = \Phi(t_1, t_0). \tag{3.4}$$

There exists $m \geqslant 1$ such that $(t_1 - t_0)^m \geqslant r$, and by periodicity of x^0, u^0,

$$\Phi(t_1 + (t_1 - t_0)^m, t_0) = \Phi(t_1, t_0)^{m+1}.$$

But $\Phi(t_1, t_0)^m$ is compact. Hence the range of

$$\mathrm{Id} - \Phi(t_1, t_0)^m = [\mathrm{Id} - \Phi(t_1, t_0)][\mathrm{Id} + \Phi(t_1, t_0) + \cdots + \Phi(t_1, t_0)^{m-1}]$$

has finite codimension. Then this is true also for

$$\mathrm{Id} - \Phi(t_1, t_0) = \mathrm{Id} - R \circ \mathscr{D}_1 S(\phi^0, u^0),$$

and clearly also for $\mathrm{Id} - R \circ \mathscr{D} S(\phi^0, u^0)$. Using this and the Hahn–Banach theorem one can show that there exists $0 \neq l \in C(-r, 0; \mathbb{R}^n)^*$ such that (3.1) holds. ▌

Let x be given by (3.2). Then (3.3) is equivalent to

$$l_0 \mathscr{D}_1 G(x^0, u^0)\, x + l_0 \mathscr{D}_2 G(x^0, u^0)\, u + \langle l, \phi - Rx \rangle \geqslant 0. \tag{3.5}$$

Remark 3.1. The optimality condition proved above holds for *local* optimal solutions. The same is true for all optimality conditions in this paper without further notion.

Remark 3.2. $\mathscr{D}_1 f$ admits the representation

$$\mathscr{D}_1 f(x_t^0, u^0(t), t)\, \phi = \int_{-r}^{0} d_s \eta(t, s)\, \phi(s) \tag{3.6}$$

with η as in (2.5).

Remark 3.3. In the following, the variational equation (3.2) is periodically extended to $[t_0, \infty)$ by setting for $k \in \mathbb{N}$, $t \in [0, t_1 - t_0]$, $x^0(t + k(t_1 - t_0) + t_0) := x^0(t + t_0)$ and similar extensions for the other terms.

The following lemma can be proven easily.

LEMMA 3.2. *For any $x \in C(t_0, t_1; \mathbb{R}^n)$ and $u \in L_\infty(t_0, t_1; \mathbb{R}^m)$*

$$\mathscr{D}_1 G(x^0, u^0)\, x = 1/(t_1 - t_0) \int_{t_0}^{t_1} \mathscr{D}_1 g(x^0(t), u^0(t), t)\, x(t)\, dt,$$

$$\mathscr{D}_2 G(x^0, u^0)\, u = 1/(t_1 - t_0) \int_{t_0}^{t_1} \mathscr{D}_2 g(x^0(t), u^0(t), t)\, u(t)\, dt.$$

Now Lemmas 3.1 and 3.2 show that (3.5) is equivalent to

$$l_0/(t_1 - t_0) \int_{t_0}^{t_1} \mathscr{D}_1 g(x^0(t), u^0(t), t)[\Phi(t, t_0)\, \phi](0)\, dt$$
$$+ \langle l, \phi - \Phi(t_1, t_0)\, \phi \rangle = 0 \qquad \text{for all } \phi \in C(-r, 0; \mathbb{R}^n) \tag{3.7}$$

and

$$l_0/(t_1-t_0)\int_{t_0}^{t_1}\left\{\mathscr{D}_1\, g(x^0(t), u^0(t), t)\right.$$
$$\times\left[\int_{t_0}^{t}\Phi(t,s)\, X_0\mathscr{D}_2 f(x_s^0, u^0(s), s)\, u(s)\, ds\right](0)$$
$$\left.+\mathscr{D}_2\, g(x^0(t), u^0(t), t)\, u(t)\right\} dt$$
$$-\left\langle l, \int_{t_0}^{t_1}\Phi(t_1, t)\, X_0\mathscr{D}_2 f(x_t^0, u^0(t), t)\, u(t)\, dt\right\rangle \geqslant 0 \tag{3.8}$$

for all $u=\alpha(v-u^0)$, $\alpha\geqslant 0$, $v\in Q$.

LEMMA 3.3. *Suppose that* $t_1-t_0\geqslant r$. *Then* $l\in \operatorname{Im} F^{\mathrm{T}}(t_1)$, *i.e., there exists* $\psi\in \mathrm{NBV}(0, r; \mathbb{R}^n)$ *with* $l=F^{\mathrm{T}}(t_1)\,\psi$.

Proof. By (3.7), we have for any $\phi\in C(-r, 0; \mathbb{R}^n)$,

$$\langle l, \phi\rangle = \langle \Phi(t_1, t_0)^*\, l, \phi\rangle$$
$$-\, l_0/(t_1-t_0)\int_{t_0}^{t_1}\mathscr{D}_1\, g(x^0(t), u^0(t), t)[\Phi(t, t_0)\,\phi](0)\, dt.$$

But

$$\Phi(t_1, t_0)^*\, l = [\Phi(t_1, t_0+r)\,\Phi(t_0+r, t_0)]^*\, l$$
$$= F(t_0)^*\, G(t_0)^*\, \Phi(t_1, t_0+r)^*\, l\in \operatorname{Im} F^{\mathrm{T}}(t_1)$$

since $F^{\mathrm{T}}(t_1)=F(t_1)^*=F(t_0)^*$, by (2.11) and periodicity. Furthermore observe that we can write

$$\int_{t_0}^{t_1}\mathscr{D}_1\, g(x^0(t), u^0(t), t)[\Phi(t, t_0)\,\phi](0)\, dt = \int_{t_0+r}^{t_1}\langle \tilde{g}(t), \Phi(t, t_0)\,\phi\rangle\, dt$$

for some bounded linear functional $\tilde{g}(t)$: $C(-r, 0; \mathbb{R}^n)\to\mathbb{R}^n$. Then we get for this expression

$$\int_{t_0+r}^{t_1}\langle \tilde{g}(t), \Phi(t, t_0+r)\,\Phi(t_0+r, t_0)\,\phi\rangle\, dt$$
$$=\int_{t_0+r}^{t_1}\langle F(t_0)^*\, G(t_0)^*\, \Phi(t, t_0+r)^*\, \tilde{g}(t), \phi\rangle\, dt$$
$$=\left\langle F(t_0)^*\int_{t_0}^{t_1} G(t_0)^*\, \Phi(t, t_0+r)^*\, \tilde{g}(t)\, dt, \phi\right\rangle.$$

This proves Lemma 3.3. ∎

The lemma above will enable us to interpret Eq. (3.7) as a periodicity condition for a solution of the formal adjoint equation.

We now formulate the main result of this section.

THEOREM 3.2. *Suppose that* $t_1 - t_0 \geqslant r$ *and let* (ϕ^0, u^0) *be an optimal solution of* (P1) (*or, equivalently of* (P2)) *with corresponding trajectory* x^0. *Then for some* $l_0 \in \mathbb{R}_+$, *the formal adjoint of the linearized system equation*

$$\frac{d}{ds}\left\{y(s) + \int_s^{t_1+r} \eta(\alpha, s-\alpha)^{\mathrm{T}}\, y(\alpha)\, d\alpha\right\}$$
$$= l_0/(t_1 - t_0)\, \mathscr{D}_1\, g(x^0(s), u^0(s), s),\ s \leqslant t_1 \tag{3.9}$$

has a solution y, *which is* $(t_1 - t_0)$*-periodic for* $s \leqslant t_1 + r$; *the periodic extensions of* x^0, u^0, f, g, $\Omega(\cdot)$ *and* y *to* $\mathbb{R}$ *satisfy the maximum condition*

$$y(t)^{\mathrm{T}}\, \mathscr{D}_2 f(x_t^0, u^0(t), t)\, u$$
$$+ l_0/(t_1 - t_0)\, \mathscr{D}_2\, g(x^0(t), u^0(t), t)\, u \geqslant 0 \tag{3.10}$$

for all $u = \alpha(v - u^0(t))$, $\alpha \geqslant 0$, $v \in \Omega(t)$, *and* a.a. $t \in \mathbb{R}$. *Furthermore, the nontriviality condition* $(l_0, F^{\mathrm{T}}(t_1)\, y^{t_1}) \neq (0, 0)$ *holds, where* $F^{\mathrm{T}}(t_1)$ *is the structural operator of* (3.9). *If, additionally,* (A3) *is satisfied, one may choose* $l_0 = 1$.

Proof. By Theorem 3.1, there exist nontrivial Lagrange multipliers (l_0, l). The function y defined by

$$y^s = \Phi^{\mathrm{T}}(t_1, s)\, \psi - \int_s^{t_1} \Phi^{\mathrm{T}}(t, s)\ Y_0 l_0/(t_1 - t_0)$$
$$\times \mathscr{D}_1\, g(x^0(t), u^0(t), t)\, dt, \qquad s \in [t_0, t_1] \tag{3.11}$$

satisfies the formal adjoint (3.9) (remember Lemma 3.3 for the definition of ψ). Using the intertwining relation (2.12), we find from (3.7) and (3.11),

$$0 = l_0/(t_1 - t_0) \int_{t_0}^{t_1} \langle Y_0 \mathscr{D}_1\, g(x^0(t), u^0(t), t),\, \Phi(t, t_0)\, \phi\rangle\, dt$$
$$+ \langle l, \phi - \Phi(t_1, t_0)\, \phi\rangle$$
$$= \int_{t_0}^{t_1} \langle \Phi(t, t_0)^*\, F(t)^*\ Y_0 l_0/(t_1 - t_0)\, \mathscr{D}_1\, g(x^0(t), u^0(t), t), \phi\rangle\, dt$$
$$+ \langle F(t_1)^{\mathrm{T}}\, \psi, \phi - \Phi(t_1, t_0)\, \phi\rangle$$

$$= \left\langle F(t_0)^* \int_{t_0}^{t_1} \Phi^{\mathrm{T}}(t, t_0)\, Y_0 l_0/(t_1 - t_0)\, \mathscr{D}_1 g(x^0(t), u^0(t), t)\, dt, \phi \right\rangle$$

$$+ \langle F(t_1)^{\mathrm{T}} \psi, \phi \rangle - \langle F(t_1)^{\mathrm{T}} \Phi^{\mathrm{T}}(t_1, t_0) \psi, \phi \rangle$$

$$= \langle F^{\mathrm{T}}(t_1)\, y^{t_1} - F^{\mathrm{T}}(t_0)\, y^{t_0}, \phi \rangle.$$

Thus

$$F^{\mathrm{T}}(t_1)\, y^{t_1} = F^{\mathrm{T}}(t_0)\, y^{t_0}.$$

Hence the effective state $F^{\mathrm{T}}(t)\, y^t$ of (3.9) is $(t_1 - t_0)$-periodic, and $y(t)$ is periodic for $t \leqslant t_1$.

We still have to prove the maximum condition (3.10). From (3.8) we find

$$0 \leqslant \int_{t_0}^{t_1} \left\langle Y_0 l_0/(t_1 - t_0)\, \mathscr{D}_1 g(x^0(t), u^0(t), t), \right.$$

$$\left. F(t) \int_{t_0}^{t} \Phi(t, s)\, X_0 \mathscr{D}_2 f(x_s^0, u^0(s), s)\, u(s)\, ds \right\rangle dt$$

$$+ l_0/(t_1 - t_0) \int_{t_0}^{t_1} \mathscr{D}_2 g(x^0(t), u^0(t), t)\, u(t)\, dt$$

$$- \int_{t_0}^{t_1} \langle \Phi(t_1, t)^* F(t_1)^* \psi, X_0 \mathscr{D}_2 f(x_t^0, u^0(t), t)\, u(t) \rangle\, dt$$

$$= \int_{t_0}^{t_1} \int_{s}^{t_1} \langle \Phi(t, s)^* F(t)^* Y_0 l_0/(t_1 - t_0)\, \mathscr{D}_1 g(x^0(t), u^0(t), t),$$

$$X_0 \mathscr{D}_2 f(x_s^0, u^0(s), s)\, u(s) \rangle\, dt\, ds$$

$$+ l_0/(t_1 - t_0) \int_{t_0}^{t_1} \mathscr{D}_2 g(x^0(t), u^0(t), t)\, u(t)\, dt$$

$$- \int_{t_0}^{t_1} \langle F^{\mathrm{T}}(t_1)\, \Phi^{\mathrm{T}}(t_1, s) \psi, X_0 \mathscr{D}_2 f(x_s^0, u^0(s), s)\, u(s) \rangle\, ds$$

$$= \int_{t_0}^{t_1} \left\langle \int_{s}^{t_1} \Phi^{\mathrm{T}}(t, s)\, Y_0 l_0/(t_1 - t_0)\, \mathscr{D}_1 g(x^0(t), u^0(t), t)\, dt, \right.$$

$$\left. X_0 \mathscr{D}_2 f(x_s^0, u^0(s), s)\, u(s) \right\rangle ds$$

$$+ l_0/(t_1 - t_0) \int_{t_0}^{t_1} \mathscr{D}_2 g(x^0(t), u^0(t), t)\, u(t)\, dt$$

$$- \int_{t_0}^{t_1} \langle \Phi^{\mathrm{T}}(t_1, s) \psi, X_0 \mathscr{D}_2 f(x_s^0, u^0(s), s)\, u(s) \rangle\, ds$$

$$= \int_{t_0}^{t_1} \langle y^s, X_0 \mathscr{D}_2 f(x_s^0, u^0(s), s)\, u(s) \rangle \, ds$$

$$+ l_0/(t_1 - t_0) \int_{t_0}^{t_1} \mathscr{D}_2 \, g(x^0(t), u^0(t), t)\, u(t)\, dt,$$

by definition of y. Hence, by standard arguments, (3.10) follows for $t \in [t_0, t_1]$.

If necessary, y is redefined in $[t_1, t_1 + r]$ such that $y^{t_1} = y^{t_0}$. This does not change $y(s)$ for $s \leqslant t_1$, because of the periodicity condition for $F(s)\, y^s$. Then the periodic extensions satisfy the maximum condition (3.10) for a.a. $t \in \mathbb{R}$. The normality assertion is a direct consequence of Theorem 3.1. ∎

Assumption (A3) is not only important as a normality condition for first order optimality conditions in the theorem above, but it will be crucial for the proof of second order optimality conditions in Section 5. The following proposition gives sufficient conditions for (A3) (cf. also the discussion in Sect. 4).

PROPOSITION 3.1. *Each of the following conditions is sufficient for* (A3):

(i) *For every $m \in \mathbb{N}$ the equation*

$$\dot{x}(t) = \mathscr{D}_1 f(x_t^0, u^0(t), t)\, x_t, \qquad t \geqslant 0$$

has only the trivial $m(t_1 - t_0)$-periodic solution.

(ii) *The linearized system*

$$\dot{x}(t) = \mathscr{D}_1 f(x_t^0, u^0(t), t)\, x_t + \mathscr{D}_2 f(x_t^0, u^0(t), t)\, u(t), \; t \in [t_0, t_1],$$

$$x_{t_0} = 0,$$

where $u \in L_\infty(t_0, t_1; \mathbb{R}^m)$ satisfies the positivity constraint

$$u(t) = \alpha(v(t) - u^0(t)) \qquad \text{a.e., where } \alpha \geqslant 0 \text{ and}$$

$$v(t) \in \Omega(t) \text{ a.e.,}$$

is approximately controllable to $C(-r, 0; \mathbb{R}^n)$, i.e., the set of all x_{t_1}, x a trajectory of the system above, contains a dense subspace *of this space.*

Proof. Sufficiency of (i) follows from the fact that $[R \circ \mathscr{D}_1 S(\phi^0, u^0)]^m$ is a compact operator arguing similarly as in the proof of Theorem 3.1. Concerning condition (ii), observe that

$$\phi - R \circ \mathscr{D} S(\phi^0, u^0)(\phi, u) = [\mathrm{Id} - R \circ \mathscr{D}_1 S(\phi^0, u^0)]\, \phi - R \circ \mathscr{D}_2 S(\phi^0, u^0)\, u.$$

The image of $\mathrm{Id} - R \circ \mathscr{D}_1 S(\phi^0, u^0)$ has a finite dimensional complement; the controllability assumption (ii) guarantees that the image set under $R \circ \mathscr{D}_2 S$ (ϕ^0, u^0) contains a dense subspace, since $\mathscr{D}_2 S(\phi^0, u^0)\, u$ is the solution of

$$\begin{aligned} \dot{x}(t) &= \mathscr{D}_1 f(x_t^0, u^0(t), t)\, x_t + \mathscr{D}_2 f(x_t^0, u^0(t), t)\, u(t), \\ x_{t_0} &= 0. \end{aligned}$$

The sum of a space with finite dimensional complement and a dense subspace coincides with the whole space. Hence (A3) holds. ∎

Remark 3.4. Conditions which ensure approximate controllability of delay systems to $C(-r, 0; \mathbb{R}^n)$ have been given, e.g., in [41].

Remark 3.5. Theorem 3.2 and Proposition 3.1 show that for functional differential systems the optimal periodic control problem is much more well-behaved than the fixed final state problem: For the latter problem, a very strong regularity condition is needed, even to get optimality conditions in Fritz–John form, where l_0 is allowed to be zero (cf. [10]). For optimal periodic control, as we have seen, no additional regularity condition is necessary to get the Fritz–John-type condition, and the normality condition (A3) is, e.g., satisfied, if an *approximate* controllability condition is satisfied. Contrarily, for the fixed *final* state problem, an *exact* controllability condition has to be satisfied [29, 10].

The following proposition shows that the requirement $t_1 - t_0 \geqslant r$ in Theorem 3.2 is not very restrictive, since "nothing is lost by considering large periods." More precisely, the proposition says that the infimal value of Problem (P1) does not increase for multiples of $t_1 - t_0$.

PROPOSITION 3.2. *Define for* $\tau = t_1 - t_0$

$$J(\tau) := \inf \frac{1}{\tau} \int_{t_0}^{t_1} g(x(t), u(t))\, dt,$$

where the infimum is taken over all pairs (x, u) *satisfying* (2.1)–(2.3), *where* $g, f,$ *and* Ω *are independent of* t. *Then*

$$J(\tau) \geqslant J(k\tau) \qquad \textit{for all} \quad k \in \mathbb{N}.$$

Proof. The proof of this proposition is an easy consequence of the fact that τ-periodic solutions are also $k\tau$-periodic.

We have reformulated the original optimal control problem (P1) as the optimization problem (P2) in Banach spaces. One of the numerous alternatives to this reformulation will turn out to be particularly convenient in

Section 5 for the derivation of 2*nd order* optimality conditions. Hence we will briefly sketch this alternative.

Define $F: C(t_0, t_1; \mathbb{R}^n) \times L_\infty(t_0, t_1; \mathbb{R}^m) \times C(-r, 0; \mathbb{R}^n) \to C(t_0, t_1; \mathbb{R}^n)$ by

$$[F(x, u, \phi)](t) := \phi(0) + \int_{t_0}^{t} f(x_s, u(s), s)\, ds, \ t \in [t_0, t_1],$$

where it is understood that $x_{t_0} := \phi$ and the right-hand side is well defined. Then (P1) is equivalent to

(P3) $\text{Minimize}_{(x,u,\phi)}\, G(x, u)$ s.t.

$$x = F(x, u, \phi),$$

$$Rx = \phi, \qquad u \in Q.$$

It is an easy exercise in the chain rule to see that Lagrange multipliers $(l_0, l_1, l_2) \in \mathbb{R}_+ \times C(t_0, t_1, \mathbb{R}^n)^* \times C(-r, 0; \mathbb{R}^n)^*$ for an optimal solution (x^0, u^0, ϕ^0) of (P3) must satisfy

$$l_1 = \mathscr{D}_1 F(x^0, u^0, \phi^0)^* \, l_1 + l_0 \mathscr{D}_1 G(x^0, u^0) - R^* l, \tag{3.12}$$

$$l_0 \mathscr{D}_2 G(x^0, u^0) - \langle l_1, \mathscr{D}_2 F(x^0, u^0, \phi^0)\, u \rangle \geqslant 0 \tag{3.13}$$

for all $u = \alpha(v - u^0)$, $\alpha \geqslant 0$, $v \in Q$.

$$l_2 = -\mathscr{D}_3 F(x^0, u^0, \phi^0)^* \, l_1. \tag{3.14}$$

Lagrange multipliers for (P2) and (P3) are nicely related, as it should be. This is the content of:

PROPOSITION 3.3. *Let (u^0, ϕ^0) with trajectory x^0 be an optimal solution of* (P1).

(i) *Suppose that (l_0, l) are Lagrange multipliers for* (P2), *i.e., satisfy* (3.3). *Define l_1 as the unique solution of*

$$l_1 = \mathscr{D}_1 F(x^0, u^0, \phi^0)^* \, l_1 + l_0 \mathscr{D}_1 G(x^0, u^0)^* - R^* l. \tag{3.15}$$

Then (l_0, l_1, l) are Lagrange multipliers for (P3), *i.e., satisfy* (3.12)–(3.14).

(ii) *Conversely, suppose that (l_0, l_1, l_2) are Lagrange multipliers for* (P3), *i.e., satisfy* (3.12)–(3.14). *Then (l_0, l_2) are Lagrange multipliers for* (P2), *i.e., satisfy* (3.3).

(iii) *Suppose that y is a solution of* (3.9). *Then y satisfies Eq.* (3.15) *with $l = F^{\mathrm{T}}(t_1)\, y^{t_1}$, where $F(t_1)$ is the structural operator of the linearized system equation. If y is $(t_1 - t_0)$-periodic, also* (3.14) *holds with (l_1, l_2) as in* (i).

Proof. Using the chain rule, we compute

$$\mathscr{D}_1 S(\phi^0, u^0)\,\phi = [\mathrm{Id} - \mathscr{D}_1 F(x^0, u^0, \phi^0)]^{-1}\, \mathscr{D}_3 F(x^0, u^0, \phi^0)\,\phi, \tag{3.16}$$

$$\mathscr{D}_2 S(\phi^0, u^0)\,u = [\mathrm{Id} - \mathscr{D}_1 F(x^0, u^0, \phi^0)]^{-1}\, \mathscr{D}_2 F(x^0, u^0, \phi^0)\,u. \tag{3.17}$$

For (i) observe that (3.12) is satisfied by definition. Insertion of (3.16) and (3.17) into (3.3) yields

$$\begin{aligned} & l_0 \mathscr{D}_1 G(x^0, u^0)[\mathrm{Id} - \mathscr{D}_1 F(x^0, u^0, \phi^0)]^{-1}\, \{\mathscr{D}_3 F(x^0, u^0, \phi^0)\,\phi \\ &\quad + \mathscr{D}_2 F(x^0, u^0, \phi^0)\,u\} + l_0 \mathscr{D}_2 G(x^0, u^0)\,u \\ &\quad + \langle l, \phi - R[\mathrm{Id} - \mathscr{D}_1 F(x^0, u^0, \phi^0)]^{-1} \\ &\quad \times \{\mathscr{D}_3 F(x^0, u^0, \phi^0)\,\phi + \mathscr{D}_2 F(x^0, u^0, \phi^0)\,u\}\rangle \\ &\geqslant 0 \qquad \text{for all } \phi,\ u = \alpha(v - u^0), \text{ where } \alpha \geqslant 0, v \in Q. \end{aligned}$$

Collecting terms with ϕ (resp. u) we get

$$\begin{aligned} & \langle \mathscr{D}_3 F(x^0, u^0, \phi^0)^*\, [\mathrm{Id} - \mathscr{D}_1 F(x^0, u^0, \phi^0)^*]^{-1} \\ &\quad \times \{l_0 \mathscr{D}_1 G(x^0, u^0) - R^* l\} + l, \phi\rangle = 0 \end{aligned}$$

and

$$\begin{aligned} & \langle [\mathrm{Id} - \mathscr{D}_1 F(x^0, u^0, \phi^0)^*]^{-1}\, \{l_0 \mathscr{D}_1 G(x^0, u^0) - R^* l\}, \\ &\quad \times \mathscr{D}_2 F(x^0, u^0, \phi^0)\,u\rangle + l_0 \mathscr{D}_2 G(x^0, u^0)\,u \geqslant 0. \end{aligned}$$

Then, with l_1 defined by (3.15), conditions (3.13) and (3.14) are satisfied.

Assertion (ii) follows in the same way. Assertion (iii) is proven by computing the adjoint operators appearing in Eq. (3.15). In particular one has for $s \in [t_0 - r, t_1]$,

$$\begin{aligned} [R^* l](s) &= [R^* F(t_1)^{\mathrm{T}}\, y^{t_1}](s) \\ &= \begin{cases} [F(t_1)^{\mathrm{T}}\, y^{t_1}](-r), & s \in [t_0 - r, t_1 - r] \\ [F(t_1)^{\mathrm{T}}\, y^{t_1}](s - t_1), & s \in [t_1 - r, t_1]. \end{cases} \end{aligned}$$ ∎

Remark 3.6. There is no essential difficulty in allowing for additional isoperimetric constraints of the form

$$\int_{t_0}^{t_1} h_i(s(t), u(t), t)\, dt \leqslant 0, \qquad i = 1, \ldots, r.$$

Using standard devices in optimal control theory one obtains terms with additional Lagrange multipliers $\lambda_i \geqslant 0$, $i=1,\dots,r$, in the adjoint equation and the maximum condition. Such additional constraints are often imposed in optimal periodic control problems.

4. First Order Optimality Conditions for Steady States

In the rest of this paper, we restrict our analysis to autonomous problems, where the system equation has the form (1.1) and g as well as Ω does not depend explicitly on time t.

This section is concerned with steady states which are optimal among steady states. More precisely, we consider the following problem:

(P4) $\text{Minimize}_{(x,u)\in\mathbb{R}^n\times\mathbb{R}^m}\, g(x,u)$ s.t.

$$0=f(\bar{x},u), \tag{4.1}$$

$$u\in\Omega\subset\mathbb{R}^m, \tag{4.2}$$

where $\bar{x}(s):=x$, $s\in[-r,0]$, and f, g are as in Section 1.

THEOREM 4.1. *Let* (x^1,u^1) *be an optimal solution of* (P4). *Then there exist nontrivial Lagrange multipliers* $(\lambda_0,\lambda)\in\mathbb{R}_+\times\mathbb{R}^n$ *such that*

$$\lambda_0\mathscr{D}_1 g(x^1,u^1)+\mathscr{D}_1 f(\bar{x}^1,u^1)^{\mathrm{T}}\lambda=0, \tag{4.3}$$

$$[\lambda_0\mathscr{D}_2 g(x^1,u^1)+\lambda^{\mathrm{T}}\mathscr{D}_2 f(\bar{x}_1,u^1)]\,u\geqslant 0 \tag{4.4}$$

for all $u=\alpha(v-u^1)$, $\alpha\geqslant 0$, $v\in\Omega$.

If, additionally, (A4) *holds, one may choose*

$$\lambda_0=1.$$

Proof. Clear by Lagrange multiplier theorem. ∎

Along with (P4), consider as dynamic version of this problem the following optimal periodic control problem (P5), which is a special case of (P1):

(P5) Minimize $1/\tau\int_0^\tau g(x(t),u(t))\,dt$ s.t. (4.5)

$$\dot{x}(t)=f(x_t,u(t)),\qquad t\in[0,\tau], \tag{4.6}$$

$$x_0=x_\tau, \tag{4.7}$$

$$u(t)\in\Omega. \tag{4.8}$$

Suppose that the steady state solution $(x^1,u^1)\in\mathbb{R}^n\times\Omega$ of (4.6) is a minimizer of (4.5) among all *periodic* solutions (x, u), i.e., pairs (x, u) satisfying (4.6)–(4.8). Then the content of Theorem 3.2 is that there exist $l_0 \in \mathbb{R}_+$ and a τ-periodic solution of the formal adjoint

$$\dot{y}(s) = -\int_{-r}^{0} d\eta^{\mathrm{T}}(\alpha)\, y(s-\alpha) + l_0/\tau \mathscr{D}_1\, g(x^1, u^1) \tag{4.9}$$

satisfying the maximum condition

$$y(t)^{\mathrm{T}}\, \mathscr{D}_2 f(\bar{x}^1, u^1)\, u + l_0/\tau \mathscr{D}_2\, g(x^1, u^1)\, u \geqslant 0 \tag{4.10}$$

for all $u = \alpha(v - u^1)$, $\alpha \geqslant 0$, $v \in \Omega$.

Now observe that by Theorem 4.1 already optimality in the class of steady states implies the existence of multipliers (λ_0, λ) satisfying (4.3) and (4.4). Then the pair $(l_0, l) = (\lambda_0, \lambda\tau)$ satisfies (4.9) and (4.10). Such a Lagrange multiplier l will always lie in the range of $F^{\mathrm{T}}(t_1)$. Hence the assumption $t_1 - t_0 \geqslant r$ of Theorem 3.2 can be omitted in this case.

These remarks show that first order optimality conditions do not allow to discern optimal periodic solutions, which happen to be steady state, from optimal steady states, which are not optimal among periodic solutions. This phenomenon already appeared for systems governed by ordinary differential equations (cf. [7, 6]). The remedy in our context is the same: In the next section, we turn to second order optimality conditions.

The rest of this section is concerned with an analysis of the normality conditions (A4) and (A3) at a steady state (x^1, u^1). The linearized system equation has the form

$$\dot{x}(t) = Lx_t + B_0 u(t), \qquad t \in [0, \tau], \tag{4.11}$$

where $L := \mathscr{D}_1 f(\bar{x}^1, u^1)$ and $B_0 := \mathscr{D}_2 f(\bar{x}^1, u^1)$.

We want to make use of some notions from the theory of autonomous linear retarded systems in the state space $M^2 = \mathbb{R}^n \times L_2(-r, 0; \mathbb{R}^n)$ (see, e.g., [31]). For Proposition 4.1 we might use as well the state space $C(-r, 0; \mathbb{R}^n)$; however, not for the discussion following that proposition.

It is well known that Eq. (4.11) induces a strongly continuous semigroup $S(t)$, $t \geqslant 0$, of operators on M^2. For $\phi \in M^2$, let ϕ^0, ϕ^1 denote its $\mathbb{R}^n$ and $L_2(-r, 0; \mathbb{R}^n)$ components, respectively. Let $x(t)$ be a solution of (4.11) corresponding to some initial conditions $x(0) = \phi^0$, $x(\theta) = \phi^1(\theta)$, $\theta \in [-r, 0)$, where $\phi \in M^2$, and to some control $u \in L_2(0, \tau; \mathbb{R}^m)$. Then $z(t) = (x(t), x_t) \in M^2$ is the mild solution of the abstract differential equation

$$z(0) = \phi, \qquad \dot{z}(t) = Az(t) + Bu(t), \qquad t \geqslant 0,$$

where $A: \mathscr{D}(A) \subset M^2 \to M^2$ is the infinitesimal generator of $S(t)$, $t \geqslant 0$, and $B: \mathbb{R}^m \to M^2$ is the bounded linear operator $Bu := (B_0 u, 0)$; here the domain $\mathscr{D}(A)$ of A is the image of $W_2^{(1)}(-r, 0; \mathbb{R}^n)$ under the natural embedding and $A\phi = (L\phi, \dot{\phi})$. Let $\Delta(\lambda)$ be the characteristic matrix

$$\Delta(\lambda) = \lambda I - L(e^{\lambda \cdot}).$$

We recall that the spectrum of A is $\sigma(A) := \{\lambda \in \mathbb{C} \mid \det \Delta(\lambda) = 0\}$ and $\rho(A) := \mathbb{C} \backslash \sigma(A)$ is the resolvent set of A. For $\lambda \in \sigma(A)$ let $\mathscr{M}_\lambda$ denote the generalized eigenspace of A corresponding to λ, that is $\mathscr{M}_\lambda = \bigcup_{k \in \mathbb{N}} \ker(\lambda I - A)^k$.

DEFINITION 4.1. The generalized eigenspace $\mathscr{M}_\lambda$ is called controllable if the canonical projection of (4.11) on $\mathscr{M}_\lambda$ is completely controllable.

We obtain the following interpretation of (A4) if no control constraint is present.

PROPOSITION 4.1. *Let $\Omega = \mathbb{R}^m$. Then condition* (A4) *holds* iff $\lambda = 0$ *is in the resolvent set $\rho(A)$ or the generalized eigenspace $\mathscr{M}_0$ is controllable.*

Proof. Remember that $\mathscr{M}_\lambda$ is controllable iff

$$\begin{aligned} n &= \operatorname{rank}[\Delta(\lambda), B_0] \\ &= \operatorname{rank}[\lambda I - L(e^{\lambda \cdot}), B_0]. \end{aligned}$$

For $\lambda = 0$, this means

$$\{L(\bar{x}) \mid \bar{x} \in \mathbb{R}^n\} + \operatorname{Im} B_0 = \mathbb{R}^n, \qquad \text{i.e., (A4).} \quad \blacksquare$$

The following remarks are also restricted to the case $\Omega = \mathbb{R}^m$. Then for systems governed by ordinary differential equations, the normality conditions which are obtained by a specialization of (A3) and (A4), respectively, are equivalent. However, the proof of this result, given in [6, Theorem 4.3], breaks down for functional differential systems, as can be seen from the following discussion.

One can easily show (cf. [31, p. 531]) that the condition rank $[\Delta(\lambda), B_0] = n$ is equivalent to

$$\operatorname{Im}(\lambda I - A) + \operatorname{Im} B = M^2.$$

Hence (A4) is equivalent to

$$\operatorname{Im} A + \operatorname{Im} B = M^2. \tag{4.12}$$

Furthermore (cf., e.g., [43, p. 51),

$$[S(\tau)-\mathrm{Id}_M]\,z = A\int_0^\tau S(\sigma)\,z\,d\sigma,\ z\in M^2. \tag{4.13}$$

Hence $\mathrm{Im}(S(\tau)-\mathrm{Id}_M)\subset \mathrm{Im}\,A$. Let the attainable subspace $\mathscr{A}$ be defined by

$$\begin{aligned}\mathscr{A} := \{(x(t_1), x_{t_1})\colon & \text{ there exists } u\in L_\infty(0,\tau;\mathbb{R}^m)\\ & \text{such that } x_{t_0}=0,\ \dot x(t)=L(x_t)+Bu(t),\\ & t\in[0,\tau]\}.\end{aligned}$$

Then (A3) means (cf. the proof of Proposition (3.1)),

$$\mathrm{Im}[S(\tau)|_C-\mathrm{Id}_C]+\mathscr{A}=C(-r,0;\mathbb{R}^n)\subset M^2. \tag{4.14}$$

Hence (A3) implies by (4.13)

$$\mathrm{Im}\,A+\mathscr{A}\supset C(-r,0;\mathbb{R}^n). \tag{4.15}$$

But $\mathscr{A}\subset W_2^{(1)}(-r,0;\mathbb{R}^n)\subset M^2$. Hence $\mathscr{A}\cap \mathrm{Im}\,B=0$. Thus there is not way to conclude from (4.15) that (4.12) holds. Contrarily, for ordinary differential systems, $\mathscr{A}=\mathrm{Im}[B, AB,..., A^{n-1}B]$. Hence (4.15) is equivalent to $\mathrm{Im}\,A+\mathrm{Im}\,B=\mathbb{R}^n$, i.e., (A4).

Conversely observe that by definition of the integral

$$\int_0^\tau S(\sigma)\,z\,d\sigma \in \text{closure} \bigcup_{\sigma\in(0,\tau)} \mathrm{Im}\,S(\sigma).$$

For general delay systems, this is a proper subset of $\mathscr{D}(A)$. Hence (4.13) does not imply $\mathrm{Im}[S(\tau)-\mathrm{Id}_M]=\mathrm{Im}\,A$. Contrarily, for ordinary differential equations, (4.13) implies that $\mathrm{Im}[S(\bar\tau)-\mathrm{Id}]=\mathrm{Im}[e^{A\bar\tau}-I]=\mathrm{Im}\,A$ for all but a finite number of values of $\bar\tau$ in $[0, T]$. Then we conclude from (4.12) that

$$\begin{aligned}\mathbb{R}^n &= \mathrm{Im}\,A+\mathrm{Im}\,B\\ &= \mathrm{Im}\,A+\mathrm{Im}[B, AB,..., A^{n-1}B]\\ &= \mathrm{Im}[e^{A\bar\tau}-I]+\mathscr{A},\end{aligned}$$

i.e., (A3) follows for all but a finite set of values of $\bar\tau$ in $[0, T]$.

5. Second Order Optimality Conditions for Periodic Control

In this section, second order necessary optimality conditions for periodic control will be derived. As a basic tool, we cite the following result [35, Theorems 3.3 and 5.6]:

Let X and Y be Banach spaces, $f: X \to \mathbb{R}$ and $g: X \to Y$, and consider the problem

$$\text{Minimize } f(x) \text{ s.t. } \quad g(x) = 0.$$

Theorem 5.1. *Suppose that f and g are twice continuously Fréchet-differentiable at $\bar{x}$ with $\mathscr{D}g(\bar{x})$ surjective.*

(i) *If $\bar{x}$ is optimal, then for any Lagrange multiplier l, i.e., any $l \in Y^*$ with $\mathscr{D}f(\bar{x}) + l \circ \mathscr{D}g(\bar{x}) = 0$ it follows that*

$$\mathscr{D}\mathscr{D}g(\bar{x})(h, h) + \langle l, \mathscr{D}\mathscr{D}g(\bar{x})(h, h) \rangle \geqslant 0$$

for any $h \in X$ with $\mathscr{D}g(\bar{x})\, h = 0$.

(ii) *Conversely, if a Lagrange multiplier $l \in Y^*$ satisfies*

$$\mathscr{D}\mathscr{D}f(\bar{x})(h, h) + \langle l, \mathscr{D}\mathscr{D}g(\bar{x})(h, h) \rangle \geqslant \delta \|h\|^2$$

for some $\delta > 0$ and all $h \in X$ with $\mathscr{D}g(\bar{x})\, h = 0$ then $\bar{x}$ is a (local) minimum.

We will apply this result to problem (P3) and impose the following assumption.

(A5) F and G are twice continuously Fréchet-differentiable.

Remark 5.1. Assumption (A5) is satisfied under standard differentiability and boundedness conditions for f and g.

Let $l_1 \in C(t_0, t_1; \mathbb{R}^n)^*$ and $l_2 \in C(-r, 0; \mathbb{R}^n)^*$ and consider the map

$$(x, u, \phi) \mapsto G(x, u) + \langle l_1, x - F(x, u, \phi) \rangle + \langle l_2, \phi - Rx \rangle. \tag{5.1}$$

Lemma 5.1. *Under the assumption* (A5), *the second derivative of the map in* (5.1) *exists, is continuous, and has the following form*:

$$\begin{aligned}
&\mathscr{D}\mathscr{D}G(x^0, u^0)((x, u)(x, u)) \\
&\quad - \langle l_1, \mathscr{D}_1\mathscr{D}_1 F(x^0, u^0, \phi^0)(x, x) + \mathscr{D}_2\mathscr{D}_2 F(x^0, u^0, \phi^0)(u, u) \\
&\quad + \mathscr{D}_3\mathscr{D}_3 F(x^0, u^0, \phi^0)(\phi, \phi) + 2\mathscr{D}_2\mathscr{D}_2 F(x^0, u^0, \phi^0)(x, u) \\
&\quad + \mathscr{D}_1\mathscr{D}_3 F(x^0, u^0, \phi^0)(x, \phi) + \mathscr{D}_3\mathscr{D}_1 F(x^0, u^0, \phi^0)(\phi, x) \\
&\quad + 2\mathscr{D}_2\mathscr{D}_3 F(x^0, u^0, \phi^0)(u, \phi) \rangle
\end{aligned} \tag{5.2}$$

Proof. Observe that the first derivative of the map in (5.1) has the following form (remember (3.12)–(3.14)):

$$\begin{aligned}\mathscr{D}G(x^0, u^0)(x, u) + \langle l_1, x - \mathscr{D}_1 F(x^0, u^0, \phi^0)\, x \\ - \mathscr{D}_2 F(x^0, u^0, \phi^0)\, u - \mathscr{D}_3 F(x^0, u^0, \phi^0)\, \phi\rangle \\ + \langle l_2, \phi - Rx\rangle.\end{aligned}$$

Then (5.2) follows similarly, taking into account continuous Fréchet-differentiability. ∎

In the following series of lemmas, the concrete form of the derivatives appearing in (5.2) will be analyzed. The proofs follow by direct computation using the assumptions and are—expect for Lemma 5.3—omitted.

LEMMA 5.2. *The second derivative of G is given by*

$$\begin{aligned}\mathscr{D}\mathscr{D}G(x^0, u^0)(x, u) = \frac{1}{t_1 - t_0}\int_{t_0}^{t_1} [x(t)^{\mathrm{T}}\, g_{xx}(x^0(t), u^0(t))\, x(t) \\ + 2x(t)^{\mathrm{T}}\, g_{xu}(x^0(t), u^0(t))\, u(t) \\ + u(t)^{\mathrm{T}}\, g_{uu}(x^0(t), u^0(t))\, u(t)]\, dt.\end{aligned}$$

LEMMA 5.3. *Identify $l_1 \in C(t_0, t_1; \mathbb{R}^n)^*$ with a normalized function y of bounded variation. Then*

$$\begin{aligned}\langle l_1, \mathscr{D}_1\mathscr{D}_1 F(x^0, u^0, \phi^0)(x, x) + \mathscr{D}_1\mathscr{D}_3 F(x^0, u^0, \phi^0)(x, \phi) \\ + \mathscr{D}_3\mathscr{D}_1 F(x^0, u^0, \phi^0)(\phi, x) + \mathscr{D}_3\mathscr{D}_3 F(x^0, u^0, \phi^0)(\phi, \phi)\rangle \\ = -\int_{t_0}^{t_1} x(\phi)_t^{\mathrm{T}}\, \mathscr{D}_1\mathscr{D}_1[y(t)^{\mathrm{T}} f(x_t^0, u^0(t))]\, x(\phi)_t\, dt;\end{aligned}$$

here $x(\phi)$ is defined by

$$x(\phi)_{t_0} := \phi, \qquad x(\phi)(t) := x(t), \qquad t \in [t_0, t_1]. \tag{5.4}$$

Remark 5.2. Observe that $y(t)^{\mathrm{T}} f(x_t^0, u^0(t))$ is scalar. Hence the second derivative with respect to x_t^0, denoted by $\mathscr{D}_1\mathscr{D}_1[y(t)^{\mathrm{T}} f(x_t^0, u^0(t))]$ is a bilinear form on $C(-r, 0; \mathbb{R}^n) \times C(-r, 0; \mathbb{R}^n)$.

By an extension of the Riesz representation theorem, such bilinear forms can be represented as repeated Riemann–Stieltjes integrals [19]. This yields

$$\mathscr{D}_1\mathscr{D}_1[y(t)^{\mathrm{T}} f(x_t^0, u^0(t))](\phi, \psi) = \int_{-r}^{0} \phi(s)^{\mathrm{T}}\, d_s \int_{-r}^{0} d_\tau K(s, \tau)\, \psi(\tau),$$

where $K(s, \tau)$ is a $n \times n$-matrix function; each component of K has a finite F-variation on $[-r, 0] \times [-r, 0]$ (cf. also [37]).

Proof of Lemma 5.3. We obtain for the considered expression

$$\int_{t_0}^{t_1} dy(t)^{\mathrm{T}} \left\{ \int_{t_0}^{t} [\mathscr{D}_1 \mathscr{D}_1 f(x_s^0, u^0(s))(x_s, x_s) + \mathscr{D}_1 \mathscr{D}_1 f(x_s^0, u^0(s))(x_s, z_s) + \mathscr{D}_1 \mathscr{D}_1 f(x_s^0, u(s))(z_s, x_s) + \mathscr{D}_1 \mathscr{D}_1 f(x_s^0, u^0(s))(z_s, z_s)]\, ds \right\},$$

where $z_{t_0} := \phi$, $z(t) = 0$, $t \in (t_0, t_1]$. By partial integration, one computes

$$-\int_{t_0}^{t_1} y(t)^{\mathrm{T}} [\mathscr{D}_1 \mathscr{D}_1 f(x_t^0, u^0(t))(x_t, x_t) + \mathscr{D}_1 \mathscr{D}_1 f(x_t^0, u^0(t))(x_t, z_t) + \mathscr{D}_1 \mathscr{D}_1 f(x_t^0, u^0(t))(z_t, x_t) + \mathscr{D}_1 \mathscr{D}_1 f(x_t^0, u^0(t))(z_t, z_t)]\, dt.$$

Now the assertion follows by definition of $x(\phi)$. ∎

LEMMA 5.4. *Under the same identification as above*

$$\langle l_1, \mathscr{D}_2 \mathscr{D}_2 F(x^0, u^0, \phi^0)(u, u) \rangle = -\int_{t_0}^{t_1} u(t)^{\mathrm{T}} \mathscr{D}_2 \mathscr{D}_2 [y(t)^{\mathrm{T}} f(x_t^0, u^0(t))]\, u(t)\, dt.$$

LEMMA 5.5. *Under the same identification as above*

$$\langle l_1, \mathscr{D}_1 \mathscr{D}_2 F(x^0, u^0, \phi^0)(x, u) + \mathscr{D}_2 \mathscr{D}_3 F(x^0, u^0, \phi^0)(x, \phi) \rangle = -\int_{t_0}^{t_1} u(t)^{\mathrm{T}} \mathscr{D}_1 \mathscr{D}_2 [y(t)^{\mathrm{T}} f(x_t^0, u^0(t))]\, x(\phi)_t\, dt,$$

where $x(\phi)$ is defined in (5.4).

Define a function

$$H: \quad C(-r, 0; \mathbb{R}^n) \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$$

by

$$H(\phi, u, \lambda) := \frac{1}{\tau} g(\phi(0), u) - \lambda^{\mathrm{T}} f(\phi, u).$$

Then we get the following fundamental result on second order optimality conditions for periodic control.

THEOREM 5.2. *Suppose that* (x^0, u^0, ϕ^0) *is an optimal* τ*-periodic solution (i.e., a solution of* (P5)), *with* $\Omega = \mathbb{R}^m$ *and* $\tau \geqslant r$. *Let the assumptions* (A3) *and* (A5) *be satisfied and suppose that* y *is a* τ*-periodic solution of the formal adjoint*

$$\frac{d}{ds}\left\{y(s) + \int_s^\tau \eta(\alpha, s-\alpha)^{\mathrm{T}}\, y(\alpha)\, d\alpha\right\} = 1/\tau\, \mathscr{D}_1\, g(x^0(s), u^0(s)), \qquad s \leqslant \tau \tag{5.5}$$

satisfying the maximum condition

$$\mathscr{D}_2 H(x_s^0, u^0(s), y(s)) = 0, \qquad s \in \mathbb{R}. \tag{5.6}$$

Then the following second order optimality condition holds:

$$\begin{aligned}\int_0^\tau &\{\mathscr{D}_1 \mathscr{D}_1 H(x_t^0, u^0(t), y(t))(x_t, x_t)\\ &+ 2\mathscr{D}_1 \mathscr{D}_2 H(x_t^0, u^0(t), y(t))(x_t, u(t))\\ &+ \mathscr{D}_2 \mathscr{D}_2 H(x_t^0, u^0(t), y(t))(u(t), u(t))\}\, dt \geqslant 0\end{aligned} \tag{5.7}$$

for all τ*-periodic solutions* (x, u) *of the variational equation*

$$\dot{x}(t) = \mathscr{D}_1\, f(x_t^0, u^0(t))\, x_t + \mathscr{D}_2\, f(x_t^0, u^0(t))\, u(t), \qquad t \in \mathbb{R}. \tag{5.8}$$

Proof. By Theorem 5.1 the values of the map (5.2) are positive for any Lagrange multipliers (l_1, l_2) and any triple (x, u, ϕ) with

$$x = \mathscr{D}F(x^0, u^0, \phi^0)(x, u, \phi) \qquad \text{and} \qquad \phi = Rx. \tag{5.9}$$

By definition, (5.9) means that x is a τ-periodic solution of the variational equation (5.8). The second derivative of (5.1) has been computed in Lemmas 5.1–5.5. We get

$$\begin{aligned}\frac{1}{\tau}\int_0^\tau &[x(t)^{\mathrm{T}}\, g_{xx}(x^0(t), u^0(t))\, x(t)\\ &+ 2x(t)^{\mathrm{T}}\, g_{xu}(x^0(t), u^0(t))\, u(t)\\ &+ u(t)^{\mathrm{T}}\, g_{uu}(x^0(t), u^0(t))\, u(t)]\, dt\\ &- \int_0^\tau x(\phi)_t^{\mathrm{T}}\, \mathscr{D}_1 \mathscr{D}_1 [y(t)^{\mathrm{T}} f(x_t^0, u^0(t))]\, x(\phi)_t\, dt\\ &- 2\int_0^\tau u(t)^{\mathrm{T}}\, \mathscr{D}_1 \mathscr{D}_2 [y(t)^{\mathrm{T}} f(x_t^0, u^0(t)]\, x(\phi)_t\, dt\\ &- \int_0^\tau u(t)^{\mathrm{T}}\, \mathscr{D}_2 \mathscr{D}_2 [y(t)^{\mathrm{T}} f(x_t^0, u^0(t))]\, u(t)\, dt \geqslant 0.\end{aligned}$$

Taking into account the definition of H and the fact that a τ-periodic solution y of the formal adjoint equation satisfying the maximum condition (5.6) defines Lagrange multipliers (l_1, l_2) the assertion follows. ∎

Remark 5.3. One can easily prove that for an optimal steady state (x^1, u^1), the following necessary optimality condition holds:

Let (A4) be satisfied. Then for any Lagrange multiplier $\lambda \in \mathbb{R}^n$, satisfying (4.3) and (4.4) (with $\lambda_0 = 1$) one has

$$\mathscr{D}_1 \mathscr{D}_1 H(\bar{x}^1, u^1, \lambda)(x, x) + 2\mathscr{D}_1 \mathscr{D}_2 H(\bar{x}^1, u^1, \lambda)(x, u) \\ + \mathscr{D}_2 \mathscr{D}_2 H(\bar{x}^1, u^1, \lambda)(u, u) \geqslant 0$$

for any pair $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$ with

$$0 = \mathscr{D}_1 f(\bar{x}^1, u^1)\, x + \mathscr{D}_2 f(\bar{x}^1, u^1, \lambda)\, u.$$

Hence condition (5.7) need not be satisfied for all τ-periodic solutions of the variational equation (5.8), but only for steady state solutions of (5.8). This, clearly, is much weaker. In particular, the test function u employed in the proof of the Π-criterion (Theorem 6.1) is not allowed for optimal steady states.

6. The Π-Criterion

In this section, we generalize the so-called Π-Criterion which was proved for systems governed by ordinary differential equations in [6] to functional differential systems. The Π-Criterion proved below coincides with the criterion formulated in [45], for the class of systems to which both criteria apply.

Throughout this section, we consider the problem without control constraint, i.e., $\Omega = \mathbb{R}^m$. Based on a Fourier series expansion of the optimality conditions in Theorem 5.2, the Π-Criterion gives a handy means to discern optimal periodic and optimal steady state solutions. We impose the following assumption for an optimal steady state (x^1, u^1):

(A6) The characteristic function $\Delta(\lambda)$ of $\dot{x}(t) = \mathscr{D}_1 f(\bar{x}^1, u^1)\, x_t$ which is given by

$$\Delta(\lambda) := \lambda I - \mathscr{D}_1 f(\bar{x}^1, u^1)(e^{\lambda \cdot} I)$$

is invertible for all $\lambda = jk\omega$, $k \in \mathbb{Z}$, where $\omega := 2\pi/\tau$.

Condition (A6) implies [25, p. 209], that for any τ-periodic control u, the variational equation

$$\dot{x}(t) = \mathscr{D}_1 f(\bar{x}^1, u^1)\, x_t + \mathscr{D}_2 f(\bar{x}^1, u^1)\, u(t) \tag{6.1}$$

has a unique τ-periodic solution x. Furthermore (A6) implies (A4) and, by arguments similar to those employed during the proof of Theorem 5.1, even (A3).

We obtain the following Fourier series expansions (cf. for the following, e.g., [9]).

Let u be an element in L_τ^2, i.e., an equivalence class of τ-periodic functions with

$$\|u\|^2 := \frac{1}{\tau} \int_0^\tau |u(t)|^2\, dt < \infty.$$

Then u has the expansion

$$u(t) = \sum_{k=-\infty}^{\infty} u\hat{\ }(k)\, e^{ik\omega t}, \tag{6.2}$$

where

$$\begin{aligned} u\hat{\ }(k) &:= 1/\tau \int_0^\tau u(t)\, e^{-ik\omega t}\, dt \\ &= 1/2\pi \int_{-\Pi}^{\Pi} u\left(\frac{t}{\omega}\right) e^{-ikt}\, dt \end{aligned} \tag{6.3}$$

and x has the expansion

$$x(t) = \sum_{k=-\infty}^{\infty} x\hat{\ }(k)\, e^{ik\omega t} \tag{6.4}$$

with

$$x\hat{\ }(k) = \Delta^{-1}(ik\omega)\, \mathscr{D}_2 f(\bar{x}^1, u^1)\, u\hat{\ }(k). \tag{6.5}$$

Since the trajectory x is absolutely continuous, hence of bounded variation and continuous, the convergence in (6.4) is uniform (cf. [18, Remark 2, p. 151]).

DEFINITION 6.1. Let $(x^1, u^1) \in \mathbb{R}^n \times \mathbb{R}^m$ be a steady state solution of

$$\dot{x}(t) = f(x_t, u(t)),$$

i.e., let $0 = f(\bar{x}^1, u^1)$. The pair (x^1, u^1) is called *proper*, if it is optimal among all steady states (i.e., a solution of (P4)), while the triple (x^0, u^0, ϕ^0) defined by

$$\begin{aligned} x^0(t) &:= x^1 \qquad \text{and} \qquad u^0(t) = u^1 \ \text{ for } \ t \in [0, \tau] \\ \phi^0(t) &:= x^1, \qquad t \in [-r, 0], \end{aligned} \tag{6.6}$$

is not optimal among periodic solutions (i.e., not a solution of (P5)).

For a steady state (x^1, u^1), conditions (5.7) and (5.8) in Theorem 5.2 have the following form (recall the discussion at the end of Sect. 4):

Let λ be a Lagrange multiplier satisfying (4.3) and (4.4). For all τ-periodic solutions (x, u) of

$$\dot{x}(t) = \mathscr{D}_1 f(\bar{x}^1, u^1)\, x_t + \mathscr{D}_2 f(\bar{x}^1, u^1)\, u(t), \qquad t \in \mathbb{R}, \tag{6.7}$$

it follows that

$$\begin{aligned} \int_0^\tau \{ &\mathscr{D}_1 \mathscr{D}_1 H(\bar{x}^1, u^1, \lambda/\tau)(x_t, x_t) \\ &+ 2\mathscr{D}_1 \mathscr{D}_2 H(\bar{x}^1, u^1, \lambda/\tau)(x_t, u(t)) \\ &+ \mathscr{D}_2 \mathscr{D}_2 H(\bar{x}^1, u^1, \lambda/\tau)(u(t), u(t))\}\, dt \geqslant 0. \end{aligned} \tag{6.8}$$

Define for $\omega \in \mathbb{R}_+$ a $m \times m$-matrix $\Pi(\omega)$ by

$$\begin{aligned} \Pi(\omega) := {} & \mathscr{D}_2 f(\bar{x}^1, u^1)^{\mathrm{T}} \Delta^{-1}(-i\omega)^{\mathrm{T}} \\ & \times \mathscr{D}_1 \mathscr{D}_1 H(\bar{x}^1, u^1, \lambda/\tau)(e^{i\omega\cdot}, e^{-i\omega\cdot})\, \Delta^{-1}(i\omega)\, \mathscr{D}_2 f(\bar{x}^1, u^1) \\ & + \mathscr{D}_1 \mathscr{D}_2 H(\bar{x}^1, u^1, \lambda/\tau)(e^{i\omega\cdot})\, \Delta^{-1}(i\omega)\, \mathscr{D}_2 f(\bar{x}^1, u^1) \\ & + \mathscr{D}_2 f(\bar{x}^1, u^1)^{\mathrm{T}} \Delta^{-1}(-i\omega)^{\mathrm{T}}\, \mathscr{D}_2 \mathscr{D}_1 H(\bar{x}^1, u^1, \lambda/\tau)(e^{i\omega\cdot}) \\ & + \mathscr{D}_2 \mathscr{D}_2 H(\bar{x}^1, u^1, \lambda/\tau); \end{aligned} \tag{6.9}$$

here we have used the identification

$$\begin{aligned} &\mathscr{D}_1 \mathscr{D}_2 H(\bar{x}^1, u^1, \lambda/\tau)(e^{i\omega\cdot}, u) \\ &\quad = u^{\mathrm{T}}\, \mathscr{D}_1 \mathscr{D}_2 H(\bar{x}^1, u^1, \lambda/\tau)(e^{i\omega\cdot}) \qquad \text{for} \quad u \in \mathbb{R}^m; \end{aligned}$$

similarly for $\mathscr{D}_2 \mathscr{D}_1 H(\bar{x}^1, u^1, \lambda/\tau)$.

THEOREM 6.1 (Π-Criterion). *Let the assumptions* (A5) *and* (A6) *be satisfied for an optimal steady state* (x^1, u^1) (*i.e., a solution of* (P4)). *Then* (x^1, u^1) *is proper* (*i.e., no solution of* (P5)), *if there exists* $\eta \in \mathbb{R}^m$ *such that for* $\omega = 2\pi/\tau$,

$$\eta^{\mathrm{T}}\, \Pi(\omega)\, \eta < 0.$$

Proof. If (x^1, u^1) is an optimal solution of (P5), condition (6.8) must hold. Take

$$u(t) = 2\eta \cos \omega t, \qquad \eta \in \mathbb{R}^m$$

as a test function. Then u has Fourier coefficients

$$u\hat{}(1) = u\hat{}(-1) = \eta, \qquad u\hat{}(k) = 0,\ k \neq 1, -1.$$

and

$$\begin{aligned} x\hat{}(1) &= \Delta^{-1}(i\omega)\, \mathscr{D}_2 f(\bar{x}^1, u^1)\, \eta, \\ x\hat{}(-1) &= \Delta^{-1}(-i\omega)\, \mathscr{D}_2 f(\bar{x}^1, u^1)\, \eta, \\ x\hat{}(k) &= 0,\ k \neq 1, -1. \end{aligned}$$

We compute, using uniform convergence and orthonomality of $\{e^{ik\cdot}, k \in Z\}$

$$\begin{aligned} &\frac{1}{2\tau}\int_0^\tau \mathscr{D}_1\mathscr{D}_1 H(\bar{x}^1, u^1, \lambda/\tau)(x_t, x_t)\, dt \\ &= \frac{1}{2\tau}\int_0^\tau \mathscr{D}_1\mathscr{D}_1 H(\bar{x}^1, u^1, \lambda/\tau) \\ &\quad \times \left(\sum_{k=-\infty}^{\infty} x\hat{}(k)\, \hat{e}^{ik\omega(t+\cdot)}, \sum_{k=-\infty}^{\infty} x\hat{}(k)\, e^{ik\omega(t+\cdot)} \right) \\ &= \frac{1}{2\tau}\int_0^\tau \sum_{k,k'=-\infty}^{\infty} x\hat{}(k)^{\mathrm{T}}\, e^{ik\omega t}\, \mathscr{D}_1\mathscr{D}_1 H(\bar{x}^1, u^1, \lambda/\tau) \\ &\quad \times (e^{ik\omega\cdot}, e^{ik'\omega\cdot})\, x\hat{}(k')\, e^{ik'\omega t}\, dt \\ &= \eta^{\mathrm{T}} \mathscr{D}_2 f(\bar{x}^1, u^1)^{\mathrm{T}}\, \Delta^{-1}(-i\omega)^{\mathrm{T}}\, \mathscr{D}_1\mathscr{D}_1 H(\bar{x}^1, u^1, \lambda/\tau)(e^{i\omega\cdot}, e^{-i\omega\cdot}) \\ &\quad \times \Delta^{-1}(i\omega)\, \mathscr{D}_2 f(\bar{x}^1, u^1)\, \eta. \end{aligned}$$

Similar expressions are obtained for the other terms in (6.8). Inserting the definition (6.9) of Π, the theorem is proven. ▌

COROLLARY (Legendre condition). *Under the same assumptions as in Theorem* 6.2, *let* (x^1, u^1) *be a steady state, which is optimal among periodic solutions of any period* $\tau > 0$. *Then*

$$\mathscr{D}_2\mathscr{D}_2 H(\bar{x}^1, u^1, \lambda/\tau) \geqslant 0$$

(*i.e., this matrix is positive semidefinite*).

Proof. Optimality implies that inequality (6.8) holds. Taking the same test functions as in the proof above one finds that the first two summands in (6.8) contain $\Delta^{-1}(\pm i\omega)$ as a factor, the other factors being independent of ω. Now $\Delta^{-1}(\lambda)$ is the Laplace transform of an integrable ($n \times n$-valued) function [27, Theorem 5.7 and p. 10].

Hence the Lemma of Riemann–Lebesgue implies $\Delta^{-1}(\lambda) \to 0$ for $|\lambda| \to \infty$. Letting $\tau \to \infty$, we obtain

$$\mathscr{D}_2 \mathscr{D}_2 H(\bar{x}^1, u^1, \lambda)(\eta, \eta) \geqslant 0$$

for all $\eta \in \mathbb{R}^m$. This proves the corollary. ∎

Remark 6.1. Application of the Π-Criterion requires computation of the Lagrange multiplier λ for the static *finite dimensional* optimization problem (P4). Hence the dynamic *infinite dimensional* optimization problem (P5) has not to be solved to recognize properness of an optimal steady state.

Remark 6.2. In [11], the Π-Criterion above is applied to optimal periodic control of retarded Liénard equations and in [46] Bailey and Sincic analyzed a problem with a linear delay equation.

Remark 6.3. The continuous bilinear form $\mathscr{D}_1 \mathscr{D}_1 H(\bar{x}^1, u^1, \lambda(\tau))$ on $C(-r, 0; \mathbb{R}^n) \times C(-r, 0; \mathbb{R}^n)$ appearing in the definition (6.9) of Π can be represented similarly as in Remark 5.2.

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