The High-Frequency Π-Criterion for Retarded Systems

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Abstract—**A test for local properness in optimal periodic control of functional differential (including delay differential) equations is proven. If it is satisfied, the performance near an optimal steady state can be improved by controls and trajectories of sufficiently high frequencies.**

I. Introduction

The fundamental problem of optimal periodic control may be formulated as follows. Suppose one has an optimal steady state x^o correspond-

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ing to a constant u^o. Can performance be improved by allowing for trajectories x and controls u which are periodic with some common period $\tau > 0$? If this is the case, the problem is called *proper.*

For systems governed by ordinary differential equations, the so-called Π-criterion is a test for (local) properness. It had been proposed in [1] and proven in [3]; the most general version can be found in [2]. The paper [13] gave a version of the Π-criterion (called "singular control test") which tests properness for sufficiently high frequencies and requires significantly less computational effort. The Π-criterion has found applications in chemical engineering and aircraft performance optimization [9], [12]. See also the survey papers [7] and [8]. It is the purpose of the present paper to extend the "high-frequency criterion" of [12] to retarded systems. The proof is based on the recently obtained Π-criterion in [4] (see also [10]).

First we formulate the optimal periodic control problem and the optimal steady-state problem. Then we cite the Π-criterion and derive from it the high-frequency version.

Notation

The Frechet-derivative of a map F between Banach spaces is denoted by $\mathfrak{D}F$. The index i in $\mathfrak{D}_iF$ means that the derivative is taken with respect to the ith variable. $C(a, b; \mathbb{R}^n)$ denotes the Banach space of continuous functions on $[a, b]$ with values in $\mathbb{R}^n$ endowed with the supremum norm $\|\cdot\|_\infty$.

$L_\infty(a, b; \mathbb{R}^n)$ denotes the Banach space of (equivalence classes of) essentially bounded functions u on $[a, b]$ with values in $\mathbb{R}^n$ endowed with the norm $\|u\|_\infty := \text{ess sup}\{|u(t)|: t \in [a, b]\}$. Finally, we let $j := (-1)^{1/2}$ and denote the transpose of a matrix A by A'.

II. Problem Formulation and Π-Criterion

Consider the following optimal periodic control problem:

$$\text{(OPC)} \qquad \text{Minimize } \frac{1}{\tau}\int_0^\tau g(x(t), u(t))\, dt \quad \text{s.t.}$$

$$\dot{x}(t) = f(x_t, u(t)) \text{ a.a } t \in [0, \tau] \tag{2.1}$$

$$x_0 = x_\tau \tag{2.2}$$

where $\tau > 0$, $x_t(s) := x(t+s) \in \mathbb{R}^n$, $s \in [-h, 0]$, $u(t) \in \mathbb{R}^m$, and $h > 0$ is the length of the retardation. The maps $f: C(-h, 0; \mathbb{R}^n) \times \mathbb{R}^m \to \mathbb{R}^n$ and $g: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ are assumed to be twice continuously Fréchet-differentiable. The controls are taken in $L_\infty(0, \tau; \mathbb{R}^m)$. Equation (2.1) comprises, e.g., the delay equation

$$\dot{x}(t) = f(x(t), x(t-h_1), \cdots, x(t-h_k), u(t))$$

where $0 < h_1 < h_2 < \cdots < h_k = h$.

Since we do not want to go into technical details concerning unique solvability of (2.1) we impose (without further mentioning) throughout this note the following assumption (cf. [5, pp. 37]).

Assumption: For every initial function $x_o = \psi \in C(-h, 0; \mathbb{R}^n)$ and every control $u \in L_\infty(0, \tau; \mathbb{R}^m)$ (2.1) has a unique absolutely continuous solution x.

The optimal steady-state problem corresponding to (OPC) has the following form:

$$\text{(OSS)} \qquad \text{Minimize } g(x, u) \text{ over } x \in \mathbb{R}^n,\ u \in \mathbb{R}^m \quad \text{s.t.}$$

$$0 = f(\bar{x}, u) \tag{2.3}$$

where $\bar{x} \in C(-h, 0; \mathbb{R}^n)$ is the constant function $\bar{x}(s) := x$.

Let $(x^o, u^o) \in \mathbb{R}^n \times \mathbb{R}^m$ be a steady-state, i.e., satisfy (2.3). Then we can linearize the system equation (2.1) around $(\bar{x}^o, \bar{u}^o)$ (here $\bar{x}^o$ and $\bar{u}^o$ denote constant functions on $[0, \tau]$) and find

$$\dot{x}(t) = Lx_t + Bu(t), \qquad \text{a.e. } t \in [0, \tau] \tag{2.4}$$

where

$$L := \mathfrak{D}_1 f(\bar{x}^o, u^o): C(-h, o; \mathbb{R}^n) \to \mathbb{R}^n,\ B := \mathfrak{D}_2 f(\bar{x}^o, u^o) \in \mathbb{R}^{n\times m}. \tag{2.5}$$

Then L has a representation $(L\phi)^T = \int_{-h}^0 \phi(\theta)^T d\eta(\theta)^T$, where η is an $n\times n$ matrix whose elements are functions of bounded variation on $[-h, 0]$.

The characteristic matrix $\Delta(z)$ for (2.4) is given by

$$\Delta(z) := zI - L(e^{z\cdot}), \qquad z \in \mathbb{C} \tag{2.6}$$

where $e^{z\cdot}$ denotes the function $\exp(z\theta)$, $\theta \in [-h, 0]$, and I is the $n \times n$ identity matrix.

Introduce the function $H: C(-h, 0; \mathbb{R}^n) \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$

$$H(\psi, u, \lambda) := g(\psi(0), u) + \lambda' f(\psi, u). \tag{2.7}$$

Then the following expressions exist:

$$P(\omega) := \mathfrak{D}_1\mathfrak{D}_1 H(\bar{x}^o, u^o, \lambda)(e^{j\omega\cdot}, e^{-j\omega\cdot}),$$

$$Q(\omega) := \mathfrak{D}_2\mathfrak{D}_1 H(\bar{x}^o, u^o, \lambda)(e^{j\omega\cdot})\ R := \mathfrak{D}_2\mathfrak{D}_2 H(\bar{x}^o, u^o, \lambda). \tag{2.8}$$

We identify $P(\omega)$, $Q(\omega)$, and R with elements in $\mathbb{C}^{n\times n}$, $\mathbb{C}^{n\times m}$, and $\mathbb{R}^{m\times m}$, respectively. Observe that R is symmetric, $P(\omega)$ is Hermitian, and $Q(\omega)' = \mathfrak{D}_1\mathfrak{D}_2 H(\bar{x}^o, u^o, \lambda)(e^{j\omega\cdot})$. Furthermore, for all $\omega \in \mathbb{R}$

$$\|P(\omega)\| \le \|\mathfrak{D}_1\mathfrak{D}_1 H(\bar{x}^o, u^o, \lambda)\| < \infty,$$

$$\|Q(\omega)\| \le \|\mathfrak{D}_2\mathfrak{D}_1 H(\bar{x}^o, u^o, \lambda)\| < \infty. \tag{2.9}$$

Define for $\omega \in \mathbb{R}$ the complex $m \times m$-matrix $\Pi(\omega)$ by

$$\begin{aligned} \Pi(\omega) := {} & B'\Delta^{-1}(-j\omega)' P(\omega)\Delta^{-1}(j\omega)B \\ & + Q(-\omega)'\Delta^{-1}(j\omega)B + B'\Delta^{-1}(-j\omega)'Q(\omega) + R. \end{aligned} \tag{2.10}$$

The matrix $\Pi(\omega)$ is Hermitian.

Then the following Π-criterion is valid (see [4, Theorem 6.1]).

Theorem 2.1: Suppose that $(x^o, u^o) \in \mathbb{R}^n \times \mathbb{R}^m$ is an optimal solution of (OSS) and that $jk\omega$, $k \in \mathbb{Z}$, is not a zero of det $\Delta(z)$, for $\omega = 2\pi/\tau$.

i) Then there exists $\lambda \in \mathbb{R}^n$ such that

$$0 = \mathfrak{D}_1 H(\bar{x}^o, u^o, \lambda\tau),\ 0 = \mathfrak{D}_2 H(\bar{x}^o, u^o, \lambda\tau). \tag{2.11}$$

ii) Let $\lambda \in \mathbb{R}^n$ satisfy (2.11), and suppose that there is $\eta \in \mathbb{R}^m$ with

$$\eta'\Pi(\omega)\eta < 0. \tag{2.12}$$

Then (x^o, u^o) is locally proper, i.e., for all $\epsilon > 0$, there exists x and u satisfying the constraints of (OPC) with $\|x - \bar{x}^o\|_\infty < \epsilon$, $\|u - \bar{u}^o\|_\infty < \epsilon$, and

$$\frac{1}{\tau}\int_0^\tau g(x(t), u(t))\, dt < g(x^o, u^o).$$

III. The High-Frequency Π-Criterion

In this section we derive from Theorem 2.1 a version of the Π-criterion which tests properness for sufficiently high frequencies. The proof is based on a series expansion of $\Delta^{-1}(z)$ which yields a series expansion of $\Pi(\omega)$. We impose the following condition.

$$\det \Delta(z) \text{ has no zeros in the closed right half plane } \{z \in \mathbb{C} : \operatorname{Re} z \ge 0\}. \tag{3.1}$$

Then it follows from [6, Prop. 5.7] that for Re $z > -\delta$, $\delta > 0$, $\Delta^{-1}(z)$ is the Laplace transform of the fundamental solution $Y(t)$ of (2.4).

For $0 \ne \omega \in \mathbb{R}$ we may write (cp. [6, p. 10]) with η as in (2.5)

$$\Delta(j\omega) = j\omega\left(I - 1/(j\omega)\int_{-h}^0 \exp(j\omega\theta)\, d\eta(\theta)\right)$$

and

$$\left\|\frac{1}{j\omega}\int_{-h}^0 \exp(j\omega\theta)\, d\eta(\theta)\right\| \le \|L\|/|\omega|.$$

Thus, for $|\omega| > \|L\|$, $\Delta^{-1}(j\omega)$ is given by

$$\Delta^{-1}(j\omega)=1/(j\omega)\sum_{k=0}^{\infty}\left(1/(j\omega)\int_{-h}^{0}\exp(j\omega\theta)\,d\eta(\theta)\right)^k$$

the series converging absolutely. Moreover, for any $\epsilon > 0$ the series is uniformly converging for $|\omega| \geq \|L\| + \epsilon$. Introduce for $0 \neq \omega \in \mathbb{R}$ the $n \times n$-matrix $A(\omega)$ over $\mathbb{C}$ by

$$A(\omega) := \int_{-h}^{0}\exp(j\omega\theta)\,d\eta(\theta). \tag{3.2}$$

Then $\overline{A(\omega)} = A(-\omega)$.

The facts mentioned above imply that for $|\omega| > \|L\|$

$$\|1/(j\omega)A(\omega)\| \leq \|L\|/|\omega| < 1 \tag{3.3}$$

and

$$\Delta^{-1}(j\omega)=1/(j\omega)\sum_{k=0}^{\infty}(j\omega)^{-k}A(\omega)^k \tag{3.4}$$

the series converging absolutely and uniformly for $|\omega| > \|L\| + \epsilon$.

Inserting (3.4) into (2.10), one obtains for $|\omega| > \|L\|+\epsilon$

$$\begin{aligned}\Pi(\omega)&=B^t\sum_{i=0}^{\infty}(-j\omega)^{-i-1}A^t(-\omega)^iP(\omega)\sum_{l=0}^{\infty}(j\omega)^{-l-1}A(\omega)^lB\\&\quad+Q^t(-\omega)\sum_{k=0}^{\infty}(j\omega)^{-k-1}A(\omega)^kB\\&\quad+B^t\sum_{k=0}^{\infty}(-j\omega)^{-k-1}A^t(-\omega)^kQ(\omega)+R\\&=\sum_{k=0}^{\infty}(j\omega)^{-k-1}\Bigg\{\sum_{\substack{i+l=k\\ i,l\geq 0}}B^t(-A^t(-\omega))^iP(\omega)/(j\omega)A(\omega)^lB\\&\quad+Q^t(-\omega)A(\omega)^kB+B^t(-A^t(-\omega))^kQ(\omega)\Bigg\}+R.\end{aligned}$$

Define

$$R_0 := R$$

$$R_k := [Q^t(-\omega)B^t]\begin{bmatrix}A(\omega) & 0\\ -P(\omega)/(j\omega) & -A^T(-\omega)\end{bmatrix}^k\begin{bmatrix}B\\ -Q(\omega)\end{bmatrix}.$$

By induction one finds

$$\begin{aligned}R_k(\omega)&=[Q^t(-\omega)B^t]\\&\quad\cdot\begin{bmatrix}A(\omega)^k & 0\\ \sum_{\substack{i+l=k-1\\ i,l\geq 0}}(-A^t(-\omega))^iP(\omega)/(j\omega)A(\omega)^l & (-A^t(-\omega))^k\end{bmatrix}\\&\quad\cdot\begin{bmatrix}B\\ -Q(\omega)\end{bmatrix}\\&=Q^t(-\omega)A(\omega)^kB\\&\quad-B^t\sum_{\substack{i+l=k-1\\ i,l\geq 0}}(-A^t(-\omega))^iP(\omega)/(j\omega)A(\omega)^lB\\&\quad-B^t(-A^t(-\omega))^kQ(\omega).\end{aligned} \tag{3.6}$$

This comparison yields the following expansion for $\Pi(\omega)$:

$$\Pi(\omega)=\sum_{k=0}^{\infty}(j\omega)^{-k}R_k(\omega). \tag{3.7}$$

Lemma 3.1: For each $k \geq 0$ and each $\omega \in \mathbb{R}$ one has

$$R_{2k}(\omega)^t=\overline{R_{2k}(\omega)}\ \text{ and }\ R_{2k+1}(\omega)^t=\overline{-R_{2k+1}(\omega)},$$

i.e., R_{2k} is Hermitian and R_{2k+1} is skew-Hermitian.

Proof: Clearly, the real matrix $R_o = R$ is symmetric. For $k \geq 1$, the proof follows by inspection of formula (3.6) and the properties of $A(\omega)$, $P(\omega)$, and $Q(\omega)$ mentioned above.

Lemma 3.2: Suppose that for some $l \in \{0, 1, 2, \cdots\}$ and all ω large enough, the following assumption holds:

$$R_k(\omega)=0 \qquad \text{for all } 0\leq k<l \text{ and } R_l(\omega)\neq 0.$$

Then there exists $\omega_0 > 0$ such that the following conditions are equivalent.

i) There exists $\delta > 0$ such that for all $\omega \geq \omega_0$ there is $\eta \in \mathbb{R}^m$ with $|\eta| = 1$ and $\eta^t\Pi(\omega)\eta < -\delta/\omega^l$;

ii) There exists $\delta > 0$ such that for all $\omega \geq \omega_0$ there is $\eta \in \mathbb{R}^m$ with $|\eta| = 1$ and $\eta^tj^{-l}R_l(\omega)\eta < -\delta/\omega^l$.

Proof: Suppose that i) holds. Then for all $\omega \geq \omega_0$, there is $\eta \in \mathbb{R}^m$ with

$$\begin{aligned}-\delta&>\eta^t\Pi(\omega)\eta\\&=\eta^t\sum_{k=l}^{\infty}(j\omega)^{-k}R_k(\omega)\eta\end{aligned}$$

by (3.7) and assumption. But from (3.3), (3.6), and (2.9) it follows for all ω with $|\omega| > \|L\|$ that

$$\begin{aligned}&\sum_{k=l+1}^{\infty}\|(j\omega)^{-k}R_k(\omega)\|\\&\leq|\omega|^{-1}\sum_{k=l+1}^{\infty}\{2\|Q(\omega)\|(\|L\|/|\omega|)^{k-1}\|B\|\\&\quad+\|B\|(\|L\|/|\omega|)^{k-1}\|P(\omega)\|\}\\&\leq c|\omega|^{-l-1}(1-\|L\|/|\omega|)^{-1}\end{aligned}$$

for some constant c which is independent of ω. Hence, ii) follows. The converse can be seen in the same way.

Now one easily obtains the following high-frequency Π-criterion.

Theorem 3.1: Let $l \in \{0, 1, 2, \cdots\}$. Then either of the following conditions implies that the optimal steady-state (x^o, u^o) is locally proper.

i) There exist $\omega_0 > 0$ and $\delta > 0$ such that for all $\omega \geq \omega_0$ and all $k = 0, 1, \cdots, 2l-1$ one has $R_k(\omega) = 0$ and there exists $\eta \in \mathbb{R}^m$ with $|\eta| = 1$ such that $(-1)^l\eta^tR_{2l}(\omega)\eta < -\delta/\omega^{2l}$.

ii) There exist $\omega_0 > 0$ and $\delta > 0$ such that for all $\omega \geq \omega_0$ and all $k = 0, 1, \cdots, 2l$ one has $R_k(\omega) = 0$ and there exists $\eta \in \mathbb{R}^m$ with $|\eta| = 1$ such that $(-1)^{l+1}j\eta^tR_{2l+1}(\omega)\eta < -\delta/\omega^{2l+1}$.

Proof: By Lemma 3.2, $\Pi(\omega)$ is partially negative iff $j^{-l}R_l(\omega)$ is. This together with Theorem 2.1 implies that i) is sufficient for local properness.

The assertion for odd coefficients follows in the same way noting that

$$j^{-(2l+1)}R_{2l+1}(\omega)=(-1)^{l+1}jR_{2l+1}(\omega).$$

It is helpful to illustrate the formula above for the special equation

$$\dot{x}(t)=f(x(t),\ x(t-h),\ u(t))$$

where $f(x, y, u) \in \mathbb{R}^n$ for $x, y \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and f is twice continuously differentiable in its arguments. Let $(x^o, u^o) \in \mathbb{R}^n \times \mathbb{R}^m$ be a steady state.

Abbreviate

$$A_0 := f_x(x^o, x^o, u^o),\ A_1 := f_y(x^o, x^o, u^o),\ B := f_u(x^o, x^o, u^o).$$

In the following, all derivatives are evaluated at (x^o, x^o, u^o). The function H of (2.7) is

$$H(x, y, u, \lambda) := g(x, u) + \lambda' f(x, y, u).$$

By (2.8)

$$P(\omega) = H_{xx} + 2H_{xy} \exp(-j\omega h) + H_{yy}$$

$$Q(\omega) = H_{ux} + H_{uy} \exp(-j\omega h),\ R = H_{uu}.$$

Thus,

$$\begin{aligned}\Pi(\omega) = {} & B'\{-j\omega I - A_o - A_1 \exp(j\omega h)\}^{-1} P(\omega) \\ & \cdot \{j\omega I - A_o - A_1 \exp(-j\omega h)\}^{-1} B \\ & + Q(-\omega)'\{(j\omega I - A_o - A_1 \exp(-j\omega h)\}^{-1} B \\ & + B'\{-j\omega I - A_o - A_1 \exp(j\omega h)\}^{-1\prime} Q(\omega) + R\end{aligned}$$

and

$$R_o = R$$

$$R_k(\omega) = [Q'(-\omega) B'] \begin{bmatrix} A_o + A_1 \exp(-j\omega h) & 0 \\ -P(\omega)/(j\omega) & -A_o - A_1 \exp(j\omega h) \end{bmatrix}^k \begin{bmatrix} B \\ -Q(\omega) \end{bmatrix}.$$

Thus computation of $R_k(\omega)$ is significantly simpler than that of $\Pi(\omega)$, since it is not necessary to compute the transfer matrix of (2.4), that is, to invert the matrix $j\omega I - A_o - A_1 \exp(-j\omega h)$.

Naturally, one still has to compute the Lagrange multiplier λ *for the* optimal steady-state problem (which, in general, is needed anyway in order to compute the optimal steady-state). The following simple example illustrates the usefulness of the criterion.

Example:

$$\text{minimize } 1/\tau \int_0^\tau [x_1(t)^2 - 2x_2(t)^2 + u(t)]\, dt \text{ s.t.}$$

$$\dot{x}_1(t) = x_2(t),\ \dot{x}_2(t) = -a x_2(t) - x_1(t-1) + u(t).$$

By a standard result [5, Theorem A6], the stability condition (3.1) is satisfied provided that $a > \sin \xi/\xi$, where ξ is the unique root of the equation $\xi^2 = \cos \xi$, $0 < \xi < \Pi/2$. The corresponding steady-state problem

$$\text{minimize } x_1^2 - 2x_2^2 + u$$

$$\text{s.t. } 0 = x_2,\ 0 = -x_1 + u$$

has $(x_1, x_2, u) = (-1/2, 0, -1/2)$ as optimal solution. The high-frequency Π-criterion Theorem 3.1 ii) applies since $R_0 = H_{uu} = 0$ and $-jR_1(\omega) = -4/\omega$ (observe that for scalar controls, the choice of η poses no problem; furthermore, the steady-state Lagrange multipliers are not needed here, since the system equation is linear). The problem is thus locally proper for all sufficiently high frequencies.

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