An Algebraic Characterization of Closed Small Attainability Subspaces of Delay Systems

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If small attainability subspaces of linear time delay systems are closed in a certain Sobolev space, the existence of Lagrange multipliers for optimal control to small solutions is guaranteed. This paper characterizes the required closedness property using an algebraic approach due to B. Jakubczyk. As a main result it turns out that closedness is—in an algebraic sense—generic in the variety of system matrices (A_0, A_1, B_0) with rank A_1 not greater than the dimension of the control space. This is in contrast to known results on closedness of attainability subspaces playing an analogous role for optimal control to fixed final states instead of small solutions.

1. INTRODUCTION

Consider the linear, autonomous time delay system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + B_0 u(t), \qquad t \geqslant 0; \tag{1.1}$$

here $T > h > 0$, $A_0, A_1 \in \mathbb{R}^{n \times n}$, $B_0 \in \mathbb{R}^{n \times m}$, the control functions u are elements of $L_p([0, T], \mathbb{R}^m)$, and $1 \leqslant p \leqslant \infty$. Define the attainability subspace $\mathscr{A}_T \subset W^{1,p}([-h, 0], \mathbb{R}^n)$ by

$$\mathscr{A}_T := \{x_T \colon x \text{ is a solution of (1.1) with initial state } x_0 = 0 \text{ for some control function } u \in L_p\};$$

here $x_t(s) := x(t+s)$, $s \in [-h, 0]$, $t \geqslant 0$. Then the *small attainability subspace* $\mathscr{A}_T^\alpha$ of order $\alpha \geqslant 0$ is defined by

$$\mathscr{A}_T^\alpha := \{x_{T+\alpha} \colon x \text{ is a solution of (1.1) on } [T, T+\alpha] \text{ with control function } u = 0 \text{ and initial state } x_T \in \mathscr{A}_T\}.$$

Observe that for $T' \geqslant T > nh$

$$\mathscr{A}_{T'}^\alpha = \mathscr{A}_T^\alpha \subset \mathscr{A}_T^0 = \mathscr{A}_T,$$

since $\mathscr{A}_T^\alpha \subset \mathscr{A}_{T+\alpha}$ and the attainability subspace does not increase after time nh (see [1, Corollary 5.1]). This paper gives an algebraic characterization for closedness of $\mathscr{A}_T^\alpha$, where $\alpha = kh$, $k \in \{0, 1, 2,...\}$ fixed, as a subspace of $W^{k+1,p}([-h, 0], \mathbb{R}^n)$. This property guarantees [2, 3] the existence of Lagrange multipliers for optimal control of the delay system (1.1) from a fixed initial state to small solutions, where the end condition has the form

$$x_{T+\alpha} = 0 \tag{1.2}$$

and $x(t)$, $t \geqslant T$, is the solution of (1.1) with zero control $u(t) = 0$, $t > T$, and initial state $x_T \in \mathscr{A}_T$. This means that the final state x_T at time T has to generate a small solution vanishing after time $t = T + \alpha - h$, i.e., the system "automatically" comes to equilibrium, without control action.

For $\alpha = 0$, this includes the fixed final state optimal control problem, where $x_T = 0$. For $\alpha = h$, it means that the *reduced state* $Fx_T = 0$ (see [5] and Remark 4.5 below). By a classical result due to Henry [6] all small solutions vanish after time $t = (n-1)h$. Hence (1.2) holds for $\alpha = nh$ iff it holds for any $\alpha > nh$.

Kurcyusz and Olbrot showed in [7] that $\mathscr{A}_T^0 = \mathscr{A}_T$ is closed in $W^{1,p}([-h, 0], \mathbb{R}^n)$ iff

$$A_1 A_0^i \mathscr{B} \subset \mathscr{B} \qquad \text{for all } i = 0, 1,..., n-1, \tag{1.3}$$

where $\mathscr{B} := \operatorname{Im} B_0$. This condition is *not generic* in the space of system matrices $(A_0, A_1, B_0) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$. In particular, if the pair (A_0, B_0) is controllable, (1.3) means that $\operatorname{Im} A_1 \subset \operatorname{Im} B_0$.

The results of this paper show that the small attainability subspace $\mathscr{A}_T^h$ is closed under much weaker conditions. Again, closedness is not generic if we allow arbitrary system matrices $(A_0, A_1, B_0) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$. However, it turns out that closedness is—in an algebraic sense—generic in the variety of all matrices (A_0, A_1, B_0) satisfying rank $A_1 \leqslant m$. This means that closedness of $\mathscr{A}_T^h$ is generic if the number of "independent delay terms" is not greater than the dimension of the control space. Then the existence of Lagrange multipliers for the corresponding optimal control problems with fixed reduced final state $Fx_T = 0$ is guaranteed. This "law of requisite variety in control" (cp. [11]) distinguishes an important class of delay systems, since in applications usually only certain state variables contain a delay. Furthermore, this result underlines the relevance of the structural operator F and the associated state concept.

Our methods are algebraic and based on the paper by Jakubczyk [8]. Section 2 gives a brief summary of these tools. Section 3 characterizes closedness of $\mathscr{A}_T^{kh}$ using invariants of induced module homomorphisms over the ring $\mathbb{R}_p(s)$ of proper rational functions over $\mathbb{R}$. In Section 4 we get criteria for closedness in terms of the system matrices A_0, A_1, B_0. This generalizes the characterization (1.3) by Kurcyusz and Olbrot in [7] and yields the genericity statement.

The methods differ from those employed in [7] as well as from those used by Banks *et al.* in [1] and Colonius and Hinrichsen in [3] for the study of closedness of attainability subspaces.

Notation. For $k \in \mathbb{N} := \{1, 2, \ldots\}$, $W^{k,p}([t_0, t_1], \mathbb{R}^n)$ denotes the Banach space of continuous functions $\psi: [t_0, t_1] \to \mathbb{R}^n$ having an absolutely continuous $(k-1)$st derivative $\psi^{(k-1)}$ with derivative $\psi^{(k)} \in L_p([t_0, t_1], \mathbb{R}^n)$. The norm is given by

$$\|\psi\| := |(|\psi(t_0)|, |\psi^{(1)}(t_0)|, \ldots, |\psi^{(k-1)}(t_0)|, \|\psi^{(k)}\|_{L_p})|,$$

where $|\cdot|$ denotes the Euclidean norm in finite-dimensional space.

For square matrices A_i, $i \in \{1, 2, \ldots, k\}$,

$$\prod_{j=1}^{k} A_i := A_1 A_2 \cdots A_k,$$

and $\prod_{j=1}^{0} A_i$ is defined as the unit matrix.

2. Preliminaries

Based on the work of Jakubczyk [8] this section gives an algebraic framework for the study of closedness of small attainability subspaces.

We consider operators

$$K: L_p([0, t_1], \mathbb{R}^m) \to W^{k,p}([0, t_1], \mathbb{R}^n)$$

of the form

$$K = L + M. \tag{2.1}$$

Here M is a finite-dimensional operator (i.e., a bounded operator with finite-dimensional range) and L is a Volterra-type operator

$$(Lu)(t) := \int_0^t l(t-s)\, u(s)\, ds, \qquad t \in [0, t_1], \tag{2.2}$$

where l is a $n \times m$-matrix with entries l_{ij} of the form

$$l_{ij}(t) = C \exp(At) B, \qquad t \geqslant 0, \tag{2.3}$$

and A, B, C are matrices of dimensions $k \times k$, $k \times 1$, and $1 \times k$, $k \in \mathbb{N}$, respectively.

Observe that the decomposition (2.1) is unique.

Remark 2.1. The Laplace transform of l_{ij} is a strictly proper rational function over $\mathbb{R}$, and, conversely, by standard results in realization theory the inverse Laplace transform of each strictly proper rational function has the form (2.3).

Let $L(s)$ be the Laplace transform of L.

One can associate certain invariants q_i to operators of the form (2.2). Consider the formal power series expansion

$$L(s) = \sum_{j=0}^{\infty} L_j s^{-j},$$

where $L_j \in \mathbb{R}^{n \times m}$ (observe that $L_0 = 0$). Let

$$\mathscr{L}_i := \begin{pmatrix} L_0 & 0 & \cdots & 0 \\ L_1 & L_0 & & \vdots \\ \vdots & & & 0 \\ L_i & L_{i-1} & \cdots & L_0 \end{pmatrix}.$$

We define as a special case of [8, Remark 2]

$$\begin{aligned} q_0(L) &:= 0, \\ q_i(L) &:= \operatorname{rank} \mathscr{L}_i - \operatorname{rank} \mathscr{L}_{i-1}, \qquad i \in \mathbb{N}, \\ q_\infty(L) &:= \sup_i q_i(L). \end{aligned} \tag{2.4}$$

Remark 2.2. The invariants q_i are related to the Smith normal form of $L(s)$ over $\mathbb{R}_p(s)$ (see [8, Theorem 3 and Remark 1]).

Remark 2.3. The invariant q_i measures the rank of that part of the map L, which makes elements in L_p at most i times smoother, and q_∞ is the general rank.

We note the following lemma.

LEMMA 2.1. *Let X, Y be Banach spaces and $A, M: X \to Y$ be linear operators, where A is closed and M is a bounded finite-dimensional operator. Then* $\operatorname{Im} A$ *is closed iff* $\operatorname{Im}(A + M)$ *is closed.*

For a proof see, e.g., [8, Lemma 1].

The following two theorems are basic for this paper.

THEOREM 2.1. *Let K be an operator as in* (2.1). *Then K has a closed range in $W^{k,p}([t_0, t_1], \mathbb{R}^n)$ iff*

$$q_k(L) = q_\infty(L). \tag{2.5}$$

This is a special case of [8, Corollary 1].

DEFINITION 2.1. Let X_1 and X_2 be subspaces of a Banach space X_3. Then $X_1 \subset_f X_2$ iff there is a finite-dimensional subspace X of X_3 with $X_1 \subset X_2 + X$.

THEOREM 2.2. *Let $K_1 = L_1 + M_1$ and $K_2 = L_2 + M_2$ be operators as described in* (2.1) (*with possibly different dimensions m*). *Then the following conditions are equivalent*:

(i) $\operatorname{Im} K_1 \subset_f \operatorname{Im} K_2$;

(ii) $\operatorname{Im} L_1 \subset \operatorname{Im} L_2$;

(iii) $\operatorname{Im} L_1(s) \subset \operatorname{Im} L_2(s)$ *over* $\mathbb{R}_p(s)$;

(iv) $q_i([L_1, L_2]) = q_i(L_1)$ *for all* $1 \leqslant i \leqslant \infty$.

In (iv), $[L_1, L_2]$ *is the Volterra-type operator* $[L_1, L_2]\ (u) := (L_1 u, L_2 u)$.

The *proof* follows easily from [8, Theorem 5] taking into account the definition of q_i.

The relevance of these results for the theory of delay systems will become clearer by the following definitions.

Let S: $W^{1,p}([-h, 0], \mathbb{R}^n) \to W^{1,p}([-h, 0], \mathbb{R}^n)$ denote the operator which maps the initial state $\psi \in W^{1,p}([-h, 0], \mathbb{R}^n)$ onto the corresponding solution segment x_h of the uncontrolled system (1.1) (i.e., $u = 0$). Then by the variation of constants formula for ordinary differential systems we get

$$S\psi = A_f \psi + A_v \psi, \tag{2.6}$$

where

$$[A_f \psi](t) := \exp(A_0(t+h))\, \psi(0), \qquad t \in [-h, 0],$$

$$[A_v \psi](t) := \int_{-h}^{t} \exp(A_0(t-s))\, A_1 \psi(s)\, ds, \qquad t \in [-h, 0].$$

Hence S is an operator as described in (2.1).

Define for $u \in L_p([-h, 0], \mathbb{R}^m)$

$$(B_v u)(t) := \int_{-h}^{t} \exp(A_0(t-s))\, B_0 u(s)\, ds, \qquad t \in [-h, 0].$$

Then we get for the solution segment x_h of (1.1) corresponding to the initial state ψ

$$\begin{aligned} x_h &= S\psi + B_v u \\ &= A_f \psi + A_v \psi + B_v u. \end{aligned} \tag{2.7}$$

By Laplace transformation A_v and B_v induce module homomorphisms over $\mathbb{R}_p(s)$:

$$A(s)\colon \mathbb{R}_p^n(s) \to \mathbb{R}_p^n(s), \qquad A(s) := (sI - A_0)^{-1} A_1,$$

$$B(s)\colon \mathbb{R}_p^m(s) \to \mathbb{R}_p^n(s), \qquad B(s) := (sI - A_0)^{-1} B_0.$$

3. Closedness of Small Attainability Subspaces

In this section, we use the invariants q_i of induced module homomorphisms over $\mathbb{R}_p(s)$ to characterize closedness of small attainability subspaces.

By the variation of constants formula (2.7) and induction, one finds for $x_{Nh} \in \mathscr{A}_{Nh}$, $N \in \mathbb{N}$,

$$X_{Nh} = \sum_{i=0}^{N-1} S^i B_v u_{(N-i)h}, \tag{3.1}$$

where $u_t(s) := u(t+s)$, $s \in [-h, 0]$ for $t \geqslant h$. If we identify $L_p([0, Nh], \mathbb{R}^m)$ appropriately with $L_p([-h, 0], \mathbb{R}^{mN})$, we get

$$\mathscr{A}_{Nh} = \operatorname{Im}[B_v, SB_v, \ldots, S^{N-1}Bv]. \tag{3.2}$$

Define for $k = 0, 1, 2, \ldots$ and $N \in \mathbb{N}$

$$K_N^{(k)} := S^k [B_v, SB_v, \ldots, S^{N-1}B_v],$$

$$L_N^{(k)} := A_v^k [B_v, A_v B_v, \ldots, A_v^{N-1}B_v].$$

Then

$$\operatorname{Im} K_N^{(k)} = \mathscr{A}_{Nh}^{kh} \tag{3.3}$$

and

$$M_N^{(k)} := K_N^{(k)} - L_N^{(k)}$$

is a finite-dimensional operator.

The following theorem gives a first characterization of the closedness property.

Theorem 3.1. *The small attainability subspace $\mathscr{A}_{Nh}^{kh}$ is closed in*

$$W^{k+1,p}([-h, 0], \mathbb{R}^n)$$

iff

$$q_{k+1}(L_N^{(k)}) = q_\infty(L_N^{(k)}). \tag{3.4}$$

Proof. Observe that by Remark 2.1 $K_N^{(k)}$, $L_N^{(k)}$, and $M_N^{(k)}$ are operators as described in (2.1). By Theorem 2.1, (3.4) holds iff $\mathscr{A}_{Nh}^{kh} = \operatorname{Im} K_N^{(k)}$ is closed in $W^{k+1,p}([-h, 0], \mathbb{R}^n)$. Now we will analyze property (3.4).

LEMMA 3.1. *Equation* (3.4) *holds for all* $N \in \mathbb{N}$ *iff the following two conditions are satisfied*:

(i) $q_{k+1}(A_v^k B_v) = q_\infty(A_v^k B_v)$;

(ii) $q_j(L_{N+1}^{(k)}) = q_j(A_v^k B_v)$ *for all* $1 \leqslant j \leqslant \infty$ *and all* $N \in \mathbb{N}$.

Proof. Observe that

$$A_v^k B_v = L_1^{(k)}.$$

Hence (i) means that $q_{k+1}(L_1^{(k)}) = q_\infty(L_1^{(k)})$. The use of conditions (ii), (i), and (ii) again shows that for $N > 1$

$$\begin{aligned} q_\infty(L_N^{(k)}) &= q_\infty(A_v^k B_v) \\ &= q_{k+1}(A_v^k B_v) \\ &= q_{k+1}(L_N^{(k)}). \end{aligned}$$

Conversely, we only have to prove (ii) for $N > 1$. Observe that

$$(sI - A_0)^{-1} = \sum_{i=1}^{\infty} A_0^{i-1} s^{-i}. \tag{3.5}$$

Hence, for $j \geqslant 0$, the first $k + j + 1$ coefficients in the power series expansion of

$$A(s)^{k+j} B(s) = [(sI - A_0)^{-1} A_1]^{k+j} (sI - A_0)^{-1} B_0 \tag{3.6}$$

vanish. Furthermore

$$[A_v^k B_v, L_N^{(k+1)}] = L_{N+1}^{(k)}. \tag{3.7}$$

Then by definition of q_j,

$$q_0(A_v^k B_v) = \cdots = q_k(A_v^k B_v) = 0 \tag{3.8}$$

and

$$q_0([A_v^k B_v, L_N^{(k+1)}]) = \cdots = q_k([A_v^k B_v, L_N^{(k+1)}]) = 0, \tag{3.9}$$

since in the formal power series expansion $L_N^{(k+1)}$ gives no contribution to the coefficient of $s^{-j}, j \leqslant k+1$. Thus we get, from (3.4),

$$\begin{aligned} q_\infty(L_{N+1}^{(k)}) &= q_{k+1}(L_{N+1}^{(k)}) \\ &= q_{k+1}([A_v^k B_v, L_N^{(k+1)}]) \\ &= q_{k+1}(A_v^k B_v) \\ &= q_\infty(A_v^k B_v), \end{aligned} \tag{3.10}$$

where the last equality follows from (i). The equalities (3.8), (3.9), and (3.10) prove (ii), taking into account the definition of q_∞.

The following theorem gives an interpretation of Lemma 3.1 in terms of structural properties of the system.

THEOREM 3.2. *The small attainability subspace* $\mathscr{A}_{Nh}^{kh}$ *is closed in* $W^{k+1,p}([-h,0],\mathbb{R}^n)$ *for all* $N \in \mathbb{N}$ *iff the following two conditions are satisfied*:

(i) *The small attainability subspace* $\mathscr{A}_h^{kh}$ *is closed in* $W^{k+1,p}([-h,0],\mathbb{R}^n)$;

(ii) *for all* $N \in \mathbb{N}$

$$\mathscr{A}_{Nh}^{(k+1)h} \subset_f \mathscr{A}_h^{kh}.$$

Furthermore, condition (ii) *holds iff*

$$\operatorname{Im}(A^{k+1}(s)\,B(s)) \subset \operatorname{Im}(A^k(s)\,B(s)) \qquad \text{over } \mathbb{R}_p(s).$$

Proof. Remember

$$\mathscr{A}_h^{kh} = \operatorname{Im} K_1^{(k)}, \qquad \mathscr{A}_{Nh}^{(k+1)h} = \operatorname{Im} K_N^{(k+1)}.$$

Hence Lemma 2.1, Theorem 2.1, and Theorem 2.2 show the equivalence of (i) and (ii) above with conditions (i) and (ii) in Lemma 3.1, respectively. Furthermore, by Theorem 2.2, condition (ii) is equivalent to

$$\operatorname{Im}[A_v^k B_v,\ldots, A_v^{k+N-1} B_v] \subset \operatorname{Im}(A_v^k B_v) \qquad \text{for all } N \in \mathbb{N},$$

i.e.,

$$\operatorname{Im}(A_v^{k+1} B_v) \subset \operatorname{Im}(A_v^k B_v).$$

Then Theorem 2.2 again shows the last assertion.

Remark 3.1. For general linear retarded systems, conditions (i) and (ii) above are still known to be sufficient for closedness of $\mathscr{A}_{Nh}^{kh}$ [3, Theorem 3.2].

The following theorem shows that condition (ii) in Theorem 3.2 holds for $k = n$ iff it holds for $k > n$. I could not prove a similar statement for condition (i).

THEOREM 3.3. *The condition*

$$\operatorname{Im}(A^{k+1}(s)\,B(s)) \subset \operatorname{Im}(A^k(s)\,B(s)) \tag{3.11}$$

holds for $k = n$ iff it holds for any $k > n$.

Proof. It is a trivial consequence of the definitions that (3.11) holds for k'' with $k'' > k' \geqslant 0$ if it holds for k'. For the converse, it suffices to prove that (3.11) implies

$$\operatorname{Im}(A^k(s)\,B(s)) \subset \operatorname{Im}(A^{k-1}(s)\,B(s))$$

if $k \geqslant n + 1$.

The Cayley–Hamilton Theorem applied over $\mathbb{R}_p(s)$ shows that

$$A^n(s) = \sum_{j<n} q_j(s)\,A^j(s),$$

where $q_j(s) \in \mathbb{R}_p(s)$. In the non-trivial case

$$l := \min\{j: q_j(s) \neq 0\} < n$$

exists. Hence for any $w(s) \in \mathbb{R}_p^m(s)$

$$\begin{aligned} A^k(s)\,B(s)\,w(s) &= 1/q_l(s)\,A^{k-l}(s)\,q_l(s)\,A^l(s)\,B(s)\,w(s) \\ &= 1/q_l(s)\,A^{k-l}(s)\left|A^n(s) - \sum_{j>l} q_j(s)\,A^j(s)\right|\,B(s)\,w(s) \\ &= 1/q_l(s)\left|A^{n-l-1}(s) - \sum_{j>l} q_j(s)\,A^{-l+j-1}(s)\right| \\ &\quad \times A^{k+1}(s)\,B(s)\,w(s). \end{aligned}$$

(Observe that $k + n - l \geqslant k + 1$, $k - l + j \geqslant k + 1$). By assumption there is $w'(s) \in \mathbb{R}_p^m(s)$ such that

$$\begin{aligned} A^k(s)\,B(s)\,w(s) &= 1/q_l(s)\left[A^{n-l-1}(s) - \sum q_j(s)\,A^{-l+j-1}(s)\right] A^k(s)\,B(s)\,w'(s) \\ &= 1/q_l(s)\,A^{k-l-1}(s)\left[A^n(s) - \sum q_j(s)\,A^j(s)\right] B(s)\,w'(s) \\ &= A^{k-1}(s)\,B(s)\,w'(s), \end{aligned}$$

i.e., $\operatorname{Im}(A^k(s)\,B(s)) \subset \operatorname{Im}(A^{k-1}(s)\,B(s))$.

4. Closedness Criteria in Terms of the System Matrices

In this section we establish criteria for the properties (i) and (ii) in Theorem 3.2, and prove a genericity statement.

THEOREM 4.1. *The following two conditions are equivalent*:

(i) *For all* $N \in \mathbb{N}$, $\mathscr{A}_{Nh}^{(k+1)h} \subset_f \mathscr{A}_h^{kh}$.

(ii) *There exists a sequence* $C_t \in \mathbb{R}^{m \times m}$, $t = 0, 1, 2, \ldots$, *such that for all* t

$$\left[\sum \prod_{j=1}^{k+1} A_1 A_0^{i_j}\right] B_0 = \sum_{r=0}^{t+1} \left(\sum \prod_{j=1}^{k} A_1 A_0^{i_j}\right) B_0 C_{t+1-r},$$

where the sum at the left is taken over all $(i_1, \ldots, i_{k+1})$, $i_j \in \{0, 1, \ldots, t\}$, *with* $\sum_{j=1}^{k+1} i_j = t$, *and the inner sums at the right are taken over all* $(i_1, \ldots, i_k)$, $i_j \in \{0, 1, \ldots, r\}$, *with* $\sum_{j=1}^{k} i_j = r$. *In particular*, (ii) *is satisfied if*

(iii) *For all* $(i_1, \ldots, i_{k+1})$, $i_j \in \{0, 1, \ldots, n-1\}$,

$$\prod_{j=1}^{k+1} (A_1 A_0^{i_j}) \mathscr{B} \subset A_1^k \mathscr{B}, \qquad \textit{where} \quad \mathscr{B} := \operatorname{Im} B_0.$$

Proof. Condition (i) is equivalent to

$$\operatorname{Im}(A^{k+1}(s) B(s)) \subset \operatorname{Im}(A^k(s) B(s)) \qquad \text{over} \quad \mathbb{R}_p(s). \tag{4.1}$$

This holds iff there is an $m \times m$-matrix $C(s)$ with entries in $\mathbb{R}_p(s)$ such that

$$\begin{aligned}[(sI - A_0)^{-1} A_1]^{k+1} (sI - A_0)^{-1} B_0 \\ = [(sI - A_0)^{-1} A_1]^k (sI - A_0)^{-1} B_0 C(s),\end{aligned}$$

which is equivalent to

$$[A_1 (sI - A_0)^{-1}]^{k+1} B_0 = [A_1 (sI - A_0)^{-1}]^k B_0 C(s). \tag{4.2}$$

The left- and right-hand sides of (4.2) are equal iff the corresponding formal power series have the same coefficients, i.e.,

$$\left(A_1 \sum_{i=1}^{\infty} A_0^{i-1} s^{-i}\right)^{k+1} B_0 = \left(A_1 \sum_{i=1}^{\infty} A_0^{i-1} s^{-i}\right)^k B_0 \sum_{i=0}^{\infty} C_i s^{-i}, \tag{4.3}$$

where $C(s) = \sum_{i=0}^{\infty} C_i s^{-i}$, $C_i \in \mathbb{R}^{m \times m}$; without loss of generality $C_0 = 0$.
Thus (4.2) is equivalent to the condition

$$\left[\sum \prod_{j=1}^{k+1} (A_1 A_0^{i_j})\right] B_0 = \sum \prod_{j=1}^{k+1} (A_1 A_0^{i_j}) B_0 C_{i_{k+1}} \qquad \text{for all } t \geqslant 0, \tag{4.4}$$

where the sum at the left is taken over all $(i_1,\dots, i_{k+1})$, $i_j \geqslant 0$, with $\sum i_j = t$, and the sum at the right is taken over all $(i_1,\dots, i_{k+1})$, $i_j \geqslant 0$, with $\sum i_j = t+1$. The right-hand side may be rewritten as

$$\sum_{r=1}^{t+1} \left[\sum \prod_{j=1}^{k+1} (A_1 A_0^{i_j}) B_0 \right] C_r,$$

where the inner sums are taken over all $(i_1,\dots, i_k)$, $i_j \geqslant 0$, with $\sum i_j = t+1-r$. Then (4.4) and hence (i) are equivalent to (ii).

Now suppose that (iii) holds. Then the Cayley–Hamilton Theorem applied over $\mathbb{R}$ shows that for all $t = 0, 1, 2,\dots$

$$\left[\sum \prod_{j=1}^{k+1} A_1 A_0^{i_j} \right] . \mathscr{B} \subset \sum \left[\prod_{j=1}^{k+1} (A_1 A_0^{i_j}) . \mathscr{B} \right] \subset A_1^k . \mathscr{B},$$

where the sums are taken over all $(i_1,\dots, i_{k+1})$, $i_j \geqslant 0$, with $\sum i_j = t$. Hence also

$$\left[\sum \prod_{j=1}^{k} A_1 A_0^{i_j} \right] . \mathscr{B} + \sum_{r=1}^{t+1} \left[\sum \prod_{j=1}^{k} A_1 A_0^{i_j} \right] . \mathscr{B} \subset A_1^k . \mathscr{B}, \tag{4.5}$$

where the sum in the first term is taken over all $(i_1,\dots, i_{k+1})$, $i_j \geqslant 0$, with $\sum i_j = t$, and the inner sums in the second term are taken over all $(i_1,\dots, i_k)$, $i_j \geqslant 0$, with $\sum i_j = r$. Now define recursively a sequence $C_t \in \mathbb{R}^{m \times m}$, $t = 0, 1, 2,\dots$, as follows: Take $C_0 := 0$, and suppose that $C_0,\dots, C_t \in \mathbb{R}^{m \times m}$ have been defined. Then by (4.5) each column of

$$\left[\sum \prod_{j=1}^{k+1} A_1 A_0^{i_j} \right] B_0 - \sum_{r=1}^{t+1} \left[\sum \prod_{j=1}^{k} A_1 A_0^{i_j} \right] B_0 C_{t+1-r}$$

is in

$$\left[\sum \prod_{j=1}^{k+1} A_1 A_0^{i_j} \right] . \mathscr{B} + \sum_{r=1}^{t+1} \left[\sum \prod_{j=1}^{k} A_1 A_0^{i_j} \right] . \mathscr{B} \subset A_1^k . \mathscr{B},$$

where the sums are taken as in (4.5). Hence there is a matrix $C_{t+1} \in \mathbb{R}^{m \times m}$ such that (ii) holds.

Remark 4.1. Consider the case $k = 0$. Then (ii) reduces to: There exists a sequence $C_t \in \mathbb{R}^{m \times m}$, $t = 0, 1, 2,\dots$, such that

$$A_1 A_0^t B_0 = \sum_{r=0}^{t+1} B_0 C_{t+1-r} \qquad \text{for all } t.$$

Invoking the Cayley–Hamilton Theorem over $\mathbb{R}$, this is easily seen to be equivalent to

$$A_1 A_0^t . \mathscr{B} \subset \mathscr{B} \qquad \text{for all } t = 0, 1,\dots, n-1.$$

Kurcyusz and Olbrot showed in [7] (see also [8]) that this condition is equivalent to closedness of $\mathscr{A}_T = \mathscr{A}_T^0$ in $W^{1,p}([-h, 0], \mathbb{R}^n)$. Using Theorem 3.2 and the fact that $\mathscr{A}_h$ is always closed in $W^{1,p}([-h, 0], \mathbb{R}^n)$, one sees that this characterization follows also from Theorem 4.1. In order to characterize closedness of $\mathscr{A}_h^{kh}$, we prepare the following Lemma.

LEMMA 4.1. *Suppose that L is an operator as considered in* (2.2) *with* $L_0 = \cdots = L_{k-1} = 0$. *Then the following two conditions are equivalent*:

(i) $q_\infty(L) = q_k(L)$;

(ii) $$\operatorname{Im} \begin{pmatrix} L_k \\ L_{k+1} \\ \vdots \\ L_{k+j} \end{pmatrix} \cap \begin{pmatrix} 0 \\ \mathbb{R}^n \\ \vdots \\ \mathbb{R}^n \end{pmatrix} \subset \operatorname{Im} \begin{pmatrix} 0 \\ L_k \\ \vdots \\ L_{k+j-1} \end{pmatrix} + \cdots + \operatorname{Im} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ L_k \end{pmatrix}$$

for all $j \in \mathbb{N}$.

In particular, (ii) *holds if* $\operatorname{Im} L_{k+j} \subset \operatorname{Im} L_k$ *for all* $j \in \mathbb{N}$.

Proof. First, we observe that (i) holds iff for all $j \in \mathbb{N}$

$$q_{k+j}(L) = q_k(L),$$

i.e.,

$$\operatorname{rank} \mathscr{L}_{k+j} = (j+1) \operatorname{rank} \mathscr{L}_k = (j+1) \operatorname{rank} L_k. \tag{4.6}$$

Thus we have to prove the equivalence of (ii) and (4.6). Certainly, $\operatorname{rank} \mathscr{L}_{k+j}$-$\operatorname{rank} \mathscr{L}_{k+j-1} \geqslant \operatorname{rank} L_k$. We show first that (ii) implies the converse of this inequality. Let $\{\ell_k^i, i \in I\}$ be a maximal set of linearly independent columns in L_k. Consider the corresponding set of columns in $\mathscr{L}_{k+j}$ (we omit the zeros corresponding to $L_0, \ldots, L_{k-1}$):

$$M := \left\{ \begin{pmatrix} \ell_k^i \\ \ell_{k+1}^i \\ \vdots \\ \ell_{k+j}^i \end{pmatrix}, i \in I \right\} \cup \left\{ \begin{pmatrix} 0 \\ \ell_k^i \\ \vdots \\ \ell_{k+j-1}^i \end{pmatrix}, i \in I \right\} \cup \cdots \cup \left\{ \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \ell_k^i \end{pmatrix}, i \in I \right\},$$

where ℓ_{k+r}^i is a column in L_{k+r}.

Let $(\ell_k, \ldots, \ell_{k+j})^T$ be one of the first m columns of $\mathscr{L}_{k+j}$.

By assumption, there are $\alpha_{i0} \in \mathbb{R}$ such that

$$\ell_k = \sum_{i \in I} \alpha_{i0} \ell_k^i.$$

Then

$$\begin{pmatrix} \ell_k - \sum \alpha_{i0} \ell_k^i \\ \ell_{k+1} - \sum \alpha_{i0} \ell_{k+1}^i \\ \vdots \\ \ell_{k+j} - \sum \alpha_{i0} \ell_{k+j}^i \end{pmatrix} \in \operatorname{Im} \begin{pmatrix} L_k \\ L_{k+1} \\ \vdots \\ L_{k+j} \end{pmatrix} \cap \begin{pmatrix} 0 \\ \mathbb{R}^n \\ \vdots \\ \mathbb{R}^n \end{pmatrix}.$$

Use of (ii) shows that this implies rank $\mathscr{L}_{k+j}$-rank $\mathscr{L}_{k+j-1} \leqslant$ rank L_k. Hence (4.6) holds. Conversely, assume that (ii) is not satisfied, i.e., there is for some $j \geqslant 1$ an element $(0, \ell_{k+1},..., \ell_{k+j-1})^T$ in

$$\operatorname{Im} \begin{pmatrix} L_k \\ L_{k+1} \\ \vdots \\ L_{k+j-1} \end{pmatrix} \cap \begin{pmatrix} 0 \\ \mathbb{R}^n \\ \vdots \\ \mathbb{R}^n \end{pmatrix},$$

which is not in

$$\operatorname{Im} \begin{pmatrix} 0 \\ L_k \\ \vdots \\ L_{k+j-1} \end{pmatrix} + \cdots + \operatorname{Im} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ L_k \end{pmatrix}.$$

We claim that $(0, \ell_{k+1},..., \ell_{k+j-1})^T$ is not linearly dependent on M, defined as above. This will imply that (4.6) does not hold.

Suppose that there are $\alpha_{il} \in \mathbb{R}$ with

$$\begin{pmatrix} 0 \\ \ell_{k+1} \\ \vdots \\ \ell_{k+j-1} \end{pmatrix} = \sum_{i \in I} \alpha_{i0} \begin{pmatrix} \ell_k^i \\ \ell_{k+1}^i \\ \vdots \\ \ell_{k+j}^i \end{pmatrix} + \cdots + \sum_{i \in I} \alpha_{ij-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \ell_k^i \end{pmatrix}.$$

By linear independence of $\{\ell_k^i\}$ it follows that $\sum_{i \in I} |\alpha_{i0}| = 0$, and $(0, \ell_{k+1},..., \ell_{k+j-1})^T$ is in

$$\operatorname{Im} \begin{pmatrix} 0 \\ L_k \\ \vdots \\ L_{k+j-1} \end{pmatrix} + \cdots + \operatorname{Im} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ L_k \end{pmatrix},$$

contradicting the definition of $(0, \ell_k,..., \ell_{k+j-1})^T$.

It remains to prove that $\operatorname{Im} L_{k+i} \subset \operatorname{Im} L_k$, $i \geqslant 1$, implies (ii). We have to show

$$\left\{ \begin{pmatrix} L_{k+1}x \\ \vdots \\ L_{k+j}x \end{pmatrix}, L_k x = 0, x \in \mathbb{R}^n \right\} \subset \operatorname{Im} \begin{pmatrix} L_k \\ \vdots \\ L_{k+j-1} \end{pmatrix} + \cdots + \operatorname{Im} \begin{pmatrix} 0 \\ \vdots \\ L_k \end{pmatrix}.$$

There exist $x_1, x_2, x_3 \in \mathbb{R}^n$ with

$$L_{k+1}x = L_k x_1, \qquad -L_{k+1}x_1 = L_k x_2, L_{k+2}x_2 = L_k x_3.$$

Then

$$\begin{pmatrix} L_{k+1}x \\ L_{k+2}x \\ \vdots \\ L_{k+j}x \end{pmatrix} = \begin{pmatrix} L_k x_1 \\ L_{k+1}x_1 \\ \vdots \\ L_{k+j-1}x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ -L_{k+1}x_1 \\ \vdots \\ -L_{k+j-1}x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ L_{k+2}x \\ \vdots \\ L_{k+j}x \end{pmatrix}$$

$$= \begin{pmatrix} L_k x_1 \\ L_{k+1}x_1 \\ L_{k+2}x_1 \\ \vdots \\ L_{k+j-1}x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ L_k(x_2+x_3) \\ L_{k+1}(x_2+x_3) \\ \vdots \\ L_{k+j-2}(x_2+x_3) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -L_{k+2}x_1 + L_{k+3}x \\ \vdots \\ -L_{k+j-1}x_1 + L_{k+j}x \end{pmatrix}$$

Repeated application of these arguments proves the assertion.

THEOREM 4.2. *The small attainability subspace* $\mathscr{A}_h^{kh}$ *is closed in* $W^{k+1,p}([-h,0], \mathbb{R}^n)$ *if the following condition is satisfied*:

$$\prod_{j=1}^{k} (A_1 A_0^{i_j}) \mathscr{B} \subset A_1^k \mathscr{B} \qquad \textit{for all } i_1, \ldots, i_k \in \{0, 1, \ldots, n-1\}.$$

Proof. First observe that $\mathscr{A}_h^0$ is always closed in $W^{1,p}([-h,0], \mathbb{R}^n)$. Now let $k \geqslant 1$. We know that $\mathscr{A}_h^{kh}$ is closed iff $\operatorname{Im} L_1^{(k)} = \operatorname{Im}(A_v^k B_v)$ is closed.

Define the multiplication operator $\tilde{A}_1$ on $W^{k,p}([-h,0], \mathbb{R}^n)$ by

$$(\tilde{A}_1 \psi)(t) := A_1 \psi(t), \qquad t \in [-h, 0].$$

Then A_v can be written as the composition of $\tilde{A}_1$ and an isomorphism between $W^{k,p}([-h,0], \mathbb{R}^n)$ and the closed subspace of all elements

$\varphi \in W^{k+1}([-h, 0], \mathbb{R}^n)$ with $\varphi(-h) = 0$. Hence $\operatorname{Im}(A_v^k B_v)$ is closed in $W^{k+1,p}([-h, 0], \mathbb{R}^n)$ iff the operator

$$L := \tilde{A}_1 A_v^{k-1} B_v$$

has a closed image. By Theorem 2.1 this is equivalent to

$$q_k(L) = q_\infty(L) := \sup q_i(L).$$

For the Laplace transform $L(s)$ of L we obtain

$$\begin{aligned} L(s) &= A_1 A^{k-1}(s)\, B(s) \\ &= \sum_{t=k}^{\infty} s^{-t} \left[\sum \prod_{j=1}^{k} (A_1 A_0^{i_j - 1})\, B_0 \right], \end{aligned}$$

where the sum is taken over all $(i_1, ..., i_k)$, $i_j \geqslant 1$, with $\sum i_j = t$. Thus the coefficient L_{k+t} of s^{-k+t} is given by

$$L_{k+t} = \sum \prod_{j=1}^{k} (A_1 A_0^{i_j})\, B_0, \qquad t \geqslant 0, \tag{4.7}$$

where the sum is taken over all $(i_1, ..., i_k)$, $i_j \geqslant 0$, with $\sum i_j = t$. By definition of q_i

$$q_i(L) = 0 \qquad \text{for} \quad i = 0, 1, ..., k-1,$$
$$q_k(L) = \operatorname{rank} L_k = \operatorname{rank}(A_1^k B_0).$$

Lemma 4.1 and the Cayley–Hamilton Theorem applied over $\mathbb{R}$ imply that

$$q_{k+j}(L) = q_k(L) \qquad \text{for all } j \geqslant 1,$$

if the condition in the theorem is satisfied.

Remark 4.2. Using the explicit formula (4.7) for L_{k+t}, one can give a complete characterization for closedness of $\mathscr{A}_h^{kh}$ in the style of Lemma 4.1. We omit this, since the criterion is very technical.

Remark 4.3. Theorem 4.2 corrects an erroneous statement in [2, Theorem 3.1].

Now we will analyse *how typical* the closedness property of the small attainability subspace $\mathscr{A}_{Nh}^h$ is. By Theorem 4.1 and Theorem 4.2, $\mathscr{A}_{Nh}^h$ is closed if for all $i, i_1, i_2 \in \{0, 1, ..., n-1\}$,

$$A_1 A_0^i \mathscr{B} \subset A_1 \mathscr{B} \tag{4.8}$$

and

$$A_1 A_0^{i_1} A_1 A_0^{i_2} \mathscr{B} \subset A_1 \mathscr{B}. \tag{4.9}$$

These conditions are certainly satisfied if

$$\operatorname{Im} A_1 = \operatorname{Im}(A_1 B_0), \tag{4.10}$$

being equivalent to

$$\operatorname{rank} A_1 = \operatorname{rank}(A_1 B_0). \tag{4.11}$$

If $m \geqslant n$, this condition is not very restrictive in the space of all system matrices $(A_0, A_1, B_0) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$. In the following we will be concerned with the more important case $m < n$.

First observe that (4.8) implies (4.10) if the pair (A_0, B_0) is controllable, hence (4.10) is only a slight strengthening of (4.8), (4.9).

Since always rank $(A_1 B_0) \leqslant m$, condition (4.10) can only be satisfied if $\operatorname{rank} A_1 \leqslant m$. Hence it cannot be "generic" in the space of admissible triples (A_0, A_1, B_0). We "tighten" the space of admissible triples by considering only

$$V := \{(A_0, A_1, B_0) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}, \operatorname{rank} A_1 \leqslant m\}$$

and study the subset of triples leading to a closed set $\mathscr{A}^h_{Nh}$. We need some elementary notions from algebraic geometry to formulate a precise genericity statement.

An (affine) variety (over $\mathbb{R}$) is defined as the set of common zeros of a finite collection of polynomials (cp. [9, 10]). A variety is called irreducible if it is not the proper union of two varieties. Each variety can be written as the union of a finite number of irreducible varieties, called its components. The dimension of an irreducible variety X is the (finite) transcendence degree of its field $\mathbb{R}(X)$ of rational functions. The dimension of a variety is the maximal dimension of its components. Identifying $\mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ with $\mathbb{R}^{2n^2+nm}$, the set V is seen to be a variety. We need the following lemma.

LEMMA 4.2. *The variety* $X := \{A_1 \in \mathbb{R}^{n\times n}: \operatorname{rank} A_1 \leqslant m\}$ *is irreducible.*

Proof. Define a map $\tau: \mathbb{R}^{mn} \times (\mathbb{R}^m)^{n-m} \to X$ in the following way: The elements of $\mathbb{R}^{mn}$ form the first m rows of an $n \times n$-matrix. The elements of the $n-m$ copies of $\mathbb{R}^m$ are used as coefficients in linear combinations of these m rows; these linear combinations form the last $n-m$ rows. Then the rank of the resulting matrix is not greater than m, and τ is continuous in the Zariski topology. Thus by [9, p. 7] the closure of $\operatorname{Im} \tau$ is an irreducible variety, since affine space is irreducible, and irreducibility is preserved under continuous mapping and taking the closure. Furthermore, one can show by elementary arguments that $\operatorname{Im} \tau$ is dense in X; hence X is irreducible.

It follows from the lemma that also

$$V = \mathbb{R}^{n\times n} \times X \times \mathbb{R}^{n\times m}$$

is an irreducible variety. Now we can formulate the genericity statement.

THEOREM 4.3. *Let $m < n$. Then the set of all triples $(A_0, A_1, B_0) \in V$ for which the small attainability subspace $\mathscr{A}^h_{Nh}$ is not closed in $W^{2,2}([-h, 0], \mathbb{R}^n)$ is contained in a proper subvariety W of V with* $\dim W < \dim V$.

Proof. Define

$$W := \{(A_0, A_1, B_0) \in V: \operatorname{rank}(A_1 B_0) < m\}.$$

Then W contains all triples $(A_0, A_1, B_0) \in V$ for which $\mathscr{A}^h_{Nh}$ is not closed: Suppose that in V $\operatorname{rank}(A_1 B_0) \geqslant m$. Then $\operatorname{rank}(A_1 B_0) = \operatorname{rank} A_1$ and $\mathscr{A}^h_{Nh}$ is closed.

Now W is the set of common zeros of finitely many polynomials p defined by minors of $A_1 B_0$. Hence W is a variety. Clearly, $W \neq V$. Hence there is a polynomial p defining W such that V is not contained in the set of zeros of p. Since V is irreducible this implies that each irreducible component of W and hence W itself has dimension strictly lower than V has.

Remark 4.4. Condition (4.10) does *not* specify a variety.

Remark 4.5. Define an operator $F: W^{1,p}([-h, 0], \mathbb{R}^n) \to \mathbb{R}^n \times W^{1,p}([-h, 0], \mathbb{R}^n)$ as the restriction of the usual F-operator on M^p (see [5]):

$$F\varphi := (\varphi(0), A_1 \varphi).$$

Then the space $\mathscr{A}^h_T$ is closely related to the following *F-attainability subspace* $F\mathscr{A}_T$ defined by

$$F\mathscr{A}_T := \{F\varphi \in \mathbb{R}^n \times W^{1,p}([-h, 0], \mathbb{R}^n): \varphi \in \mathscr{A}_T\}.$$

The space $F\mathscr{A}_T$ is isomorphic to $\mathscr{A}^h_T$ (under the obvious isomorphism), and Theorem 4.3 generically characterizes closedness of $F\mathscr{A}_T$ in $\mathbb{R}^n \times W^{1,p}([-h, 0], \mathbb{R}^n)$.

ACKNOWLEDGMENTS

I thank Dieter Praetzel-Wolters for a helpful discussion on the proof of Lemma 4.1. I also thank the referee for criticism that helped to eliminate some ambiguity in the genericity arguments.

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