On approximate and exact null controllability of delay systems *

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Approximate and exact null controllability of linear delay equations with a constant delay and scalar control are shown to be equivalent. The main tool is a recent result on equivalence between spectral controllability and finite spectrum assignability.

Keywords: Delay systems, Null controllability, Finite spectrum assignability.

1. Introduction

This note deals with null controllability of the following linear autonomous delay equation with *scalar* control:

$$(\Sigma) \quad \dot{x}(t) = A_0 x(t) + A_1 x(t-h) + bu(t), \quad t \geq 0,$$

where

$$x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R},$$

$$A_0, A_1 \in \mathbb{R}^{n\times n}, \quad b \in \mathbb{R}^{n\times 1},$$

and $h > 0$ is the length of the delay.

We show that approximate null controllability (in finite time) of (Σ) is equivalent to exact null controllability.

For two-dimensional systems, this was proved by Jacobs and Langenhop [2]. Marchenko [7] claimed the assertion for retarded systems with finitely many delays. However, his arguments seem to be incomplete as noticed in Salamon [8]. Salamon also gave a simple two-dimensional counterexample of a neutral system which is approximately, but not exactly null controllable.

The proof given here heavily relies on the ideas of finite spectrum assignability (Kamen [3], Manitius and Olbrot [6]). The main tool is a result by Watanabe, Ito, and Kaneko [9], which states that spectral controllability implies finite spectrum assignability.

* This research was performed during a visit to Mathematisches Institut der Universität Graz, Austria, supported by a grant from Deutsche Forschungsgemeinschaft.

2. Null controllability

We first give formal definitions of controllability notions and cite some known facts. Then the result on equivalence between approximate and exact null controllability is stated and proved.

Suppose that a control $u \in L_2(0, t_1; \mathbb{R})$ and an initial condition

$$x(0) = \varphi^0, \quad x(t) = \varphi^1(t) \quad \text{a.e. } t \in [-h, 0], \tag{2.1}$$

with

$$\varphi = (\varphi^0, \varphi^1) \in M^2 = \mathbb{R}^n \times L_2(-h, 0; \mathbb{R}^n)$$

for (Σ) are given. The corresponding solution of (Σ) is denoted by $x(\varphi, u, t)$ and the state $\hat{x}(t)$ at time t,

$$\hat{x}(t) = \hat{x}(\varphi, u, t) = (x(\varphi, u, t), x(\varphi, u)_t) \in M^2,$$

is given by the variation of constants formula

$$\hat{x}(t) = S(t)\varphi + \int_0^t S(t-s)(bu(s), 0)\, ds; \tag{2.2}$$

here

$$x(\varphi, u)_t(s) = x(\varphi, u, t+s), \quad s \in [-h, 0],$$

$S(t)$ is the absolutely continuous semigroup of operators associated with the homogeneous equation, and $(bu(s), 0) \in M^2$.

Let the attainable subspace $\mathscr{A}$ at time t_1 be defined by

$$\mathscr{A} := \left\{ \int_0^{t_1} S(t-s)(bu(s), 0)\, \mathrm{d}s : u \in L_2(0, t_1; \mathbb{R}) \right\} \subset M^2.$$

Definition 2.1. System (Σ) is exactly null controllable at time $t_1 > h$ if for every initial state $\varphi \in M^2$

$S(t_1)\varphi \in \mathscr{A}$.

Definition 2.2. System (Σ) is approximately null controllable at time $t_1 > h$ if for every initial state $\varphi \in M^2$

$S(t_1)\varphi \in \text{closure}_{M^2}\mathscr{A}$.

Definition 2.3. System (Σ) is spectrally controllable if for all λ in the spectrum of the infinitesimal generator A of $S(t)$, $t \geq 0$, the system

$$(\Sigma^\lambda) \quad \dot{x}^\lambda(t) = A^\lambda x^\lambda(t) + b^\lambda u(t)$$

obtained by spectral projection of (Σ) onto the generalized eigenspace corresponding to λ is controllable.

(For details on the spectrum of A and the spectral projection see Hale [1], Manitius [5].)

Spectral controllability can be characterized by the generalized Hautus condition

$$\operatorname{rank}(sI - A_0 - A_1 \exp(-sh), b) = n \quad \text{for all } s \in \mathbf{C}.$$

The following result is well known for much more general systems than (Σ) (see e.g. Salamon [8, IV, 1.11 Theorem]).

Theorem 2.4. *Suppose that system (Σ) is approximately null controllable at some time $t_1 > h$. Then it is spectrally controllable.*

Recently, Watanabe, Ito and Kaneko [9] proved the following result on finite spectrum assignability via algebraic methods.

Theorem 2.5. *Suppose that system (Σ) is spectrally controllable, and let an arbitrary set Λ of n complex numbers containing with λ also its complex conjugate be given.*

Then there exists a feedback control u^{F} of the form

$$u^{\mathrm{F}}(t) = \sum_{i=0}^{N} d_i x(t - ih) + \int_{-Nh}^{0} \zeta(\theta) x(t+\theta)\, \mathrm{d}\theta \tag{2.3}$$

where N is a positive integer, $d_i \in \mathbb{R}^{1\times n}$, and $\zeta \in L_2(-Nh, 0; \mathbb{R}^{1\times n})$, such that the spectrum of the feedback system coincides with Λ.

Observe that the state space of the feedback system

$$(\Sigma_{\mathrm{F}}) \quad \dot{x}(t) = A_0 x(t) + A_1 x(t-h) + b\left[\sum_{i=0}^{N} d_i x(t-ih) + \int_{-Nh}^{0} \zeta(\theta) x(t+\theta)\mathrm{d}\theta\right]$$

is $\mathbb{R}^n \times L_2(-Nh, 0; \mathbb{R}^n)$.

Hence the semigroup of operators $S_{\mathrm{F}}(t)$, $t \geq 0$, corresponding to (Σ_{F}) operates on this space.

Theorem 2.6. *Under the assumptions of Theorem 2.5, define*

$$V := \operatorname{Im} S_F(nNh) \subset \mathbb{R}^n \times L_2(-Nh, 0; \mathbb{R}^n).$$

Then V is a finite-dimensional vector space and contained in the sum of the generalized eigenspaces corresponding to $\lambda \in \Lambda$.

Proof. By Kappel [4, Theorem 11.7], the sum of the generalized eigenspaces corresponding to $\lambda \in \Lambda$ is an n-dimensional (complex) vector space. Now (see Manitius [5]) V is contained in the closure of this space. By finite dimensionality, the assertion follows.

The next theorem presents the contribution of this note.

Theorem 2.7. *If system (Σ) is approximately null controllable at time $t_1 > h$, then it is exactly null controllable at every time $t_2 > (n-1)Nh + h$.*

Proof. Suppose that (Σ) is approximately null controllable at time t_1. Then by Theorem 2.4 it is spectrally controllable. Since spectral controllability is invariant under the feedback (2.3), also system (Σ_F) is spectrally controllable. Thus, by Theorem 2.6, the elements of Im $S_F(nNh)$ are controllable. This proves that (Σ_F) is exactly null controllable at $t > nNh$.

Now let an initial state

$$\varphi \in \mathbb{R}^n \times L_2(-h, 0; \mathbb{R}^n)$$

for (Σ) be given. Extend φ to an element

$$\bar{\varphi} \in \mathbb{R}^n \times L_2(-Nh, 0; \mathbb{R}^n)$$

by setting $\bar{\varphi}(t) := 0$ for $t \in [-Nh, -h)$. For a control function u, let $\hat{x}_F(\bar{\varphi}, u, t)$ be the corresponding state of (Σ_F) at time t. Then the $\mathbb{R}^n$-component $x_F(\bar{\varphi}, u, t)$ of $\hat{x}(\hat{\varphi}, u, t)$ is for $t \geq 0$ a solution $x(\varphi, u', t)$ of (Σ) corresponding to the initial state φ and a certain control function u'. Then, by the first part of the proof, there exists a control u such that

$$x(\varphi, u', t) = x_F(\bar{\varphi}, u, t) = 0$$

for $t \geq t_3 > (n-1)Nh$.

Taking into account that the state space of (Σ) is $\mathbb{R}^n \times L_2(-h, 0; \mathbb{R}^n)$, this proves that φ is null controllable in every time $t_2 > (n-1)Nh + h$.

Remark. It would be highly desirable to have an upper bound for N.

3. Comments

The present note deals with the rather special class of delay systems with a constant delay and scalar control. This is due to the fact that the crucial tool in the proof — spectral controllability implies finite spectrum assignability — has been stated only for these systems (Watanabe, Ito and Kaneko [9] allow the slightly more general case of finitely many *commensurate* delays in state and control). However, I must admit that the proof in [9] — even for this relatively simple class of systems — appears very complicated. I hope that the present note will be another motivation to give a more convincing proof of this important result.

Acknowledgements

I thank Andrzej Manitius for bringing [9] to my attention and an anonymous referee for several valuable comments.

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