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A Note on the Existence of Lagrange Multipliers

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Abstract. This note extends a Lagrange multiplier theorem due to J. Zowe and S. Kurcyusz. Under some additional assumptions the required regularity condition can be weakened. The cone corresponding to the explicit constraint is replaced by its closure.

In [5], J. Zowe and S. Kurcyusz considered the following optimization problem

Minimize $f(x)$

subject to $x \in C \subset X$ and $g(x) \in K$,

where f is a real functional defined on a real Banach space X, g is a map from X into a real Banach space Y and K is a closed convex cone in Y with vertex at the origin.

For a set Q in a vector space and $q^0 \in Q$ define $Q(q^0)$ as the conical hull of Q:

$$Q(q^0) := \{\lambda(q - q^0) : \lambda \geqslant 0, q \in Q\}.$$

Zowe and Kurcyusz proved the following result [5, Theorem 3.1] (cp. also Robinson's paper [3]).

Theorem 1.1. *Let x^0 be an optimal solution of the considered problem and assume*:

(a) *f is differentiable and g is continuously differentiable in the sense of Fréchet with derivatives $f'(x^0)$ and $g'(x^0)$; the set C is closed and convex*;

(b) $g'(x^0)C(x^0) - K(g(x^0)) = Y$.

Then there exists a Lagrange multiplier $y^* \in Y^*$ satisfying

$$[f'(x^0) - y^* \cdot g'(x^0)]h \geqslant 0 \qquad \text{for all } h \in C(x^0). \tag{1.1}$$

$$\langle y^*, y \rangle \geqslant 0 \qquad \text{for all} \quad y \in K \quad \text{and} \quad \langle y^*, g(x^0) \rangle = 0. \tag{1.2}$$

M. Brokate [1] analysed the regularity condition (b) for problems with $C = X$ and showed that in some sense it cannot be improved.

This note is motivated by problems where condition (b) is not satisfied, because the cone $C(x^0)$ is "too small," and the analysis will be restricted to the case $K = \{0\}$. I will give conditions under which one can replace $C(x^0)$ by $\operatorname{cl} C(x^0)$ in the regularity condition. Furthermore, the differentiability assumption is weakened.

The following example illustrates that $C(x^0)$ need not be closed, even if C is.

Example 1.1. Let $Q \subset \mathbb{R}^m$ be compact and convex and define

$$C := \{u \in L_2([0,1], \mathbb{R}^m) : u(t) \in Q \text{ a.e.}\}.$$

Then C is closed and convex in $L_2([0,1], \mathbb{R}^m)$. However, for $u^0 \in C$,

$$C(u^0) := \{\lambda(u - u^0) : \lambda \geqslant 0, u(t) \in Q \text{ a.e.}\}$$

is a proper subset of

$$\operatorname{cl} C(u^0) = \{v \in L_2([0,1], \mathbb{R}^m) : v(t) \in \mathbb{R}_+ (Q - u^0(t)) \text{ a.e.}\}.$$

If $C(x^0)$ is "too small" it is a natural idea to restrict the image space Y of g. Then, however, one has also to restrict X in order to get a well-defined map g (which is not only defined on $C(x^0)$). Hence we make the following assumption:

$$\text{There exist Banach spaces } \tilde{X} \subset X \text{ and } \tilde{Y} \subset Y \text{ which are dense in } X \text{ and } Y \text{ respectively such that } C \subset \tilde{X} \text{ and } g(\tilde{X}) \subset \tilde{Y}. \tag{1.3}$$

Thus we are faced with the following situation (Fig. 1):

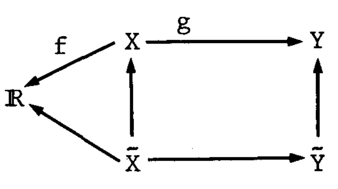

Fig. 1. Illustration of the problem treated in Thm. 1.2.

Now suppose that the regularity conditions (a) and (b) are satisfied on the "lower level", i.e.,

f considered as a map on $\tilde{X}$ with values in $\mathbb{R}$ is differentiable and g considered as a map on $\tilde{X}$ with values in $\tilde{Y}$ is continuously differentiable in the sense of Fréchet with derivatives $f'(x^0)$ and $g'(x^0)$; the set C is convex and closed in $\tilde{X}$; (1.4)

$$g'(x^0)C(x^0) = \tilde{Y}. \tag{1.5}$$

Then, under the assumptions (1.3)–(1.5) there exists by Thm. 1.1 a Lagrange multiplier $\tilde{y}^* \in \tilde{Y}^*$ satisfying

$$[f'(x^0) - \tilde{y}^* \cdot g'(x^0)]h \geqslant 0 \qquad \text{for all } h \in C(x^0). \tag{1.6}$$

Every element of Y^* can be considered as a continuous linear functional on $\tilde{Y}$, but the converse is not true. Thus the question arises, under what conditions $\tilde{y}^*$ can continuously be extended to a functional on Y. The following theorem is the main contribution of this note.

Theorem 1.2. *Suppose that conditions* (1.3)–(1.5) *are satisfied, and let* $\tilde{y}^*$ *be a Lagrange multiplier satisfying* (1.6). *Furthermore, assume*

$f'(x^0)$ *and* $g'(x^0)$ *can be extended to continuous linear maps on* X *with values in* $\mathbb{R}$ *and* Y, *respectively*; (1.7)

$$g'(x^0)\mathrm{cl}_X C(x^0) = Y. \tag{1.8}$$

Then $\tilde{y}^*$ *can continuously be extended to* Y.

The main tool in the proof of Thm. 1.2 is the following generalized open mapping theorem due to Zowe and Kurcyusz [5, Thm. 2.1] (cp. also [4]) which is specialized to our situation.

Let X_ρ and Y_ρ denote the balls around 0 with radius $\rho > 0$ in X and Y respectively.

Theorem 1.3. *Suppose* Q *is a closed convex subset of* X. *Let* $q^0 \in Q$ *and* T *be a continuous linear map on* X *with values in* Y. *Then the following statements are equivalent*:

(i) $Y = TQ(q^0)$

(ii) $Y_\rho \subset T((Q - q^0) \cap X_1)$ for some $\rho > 0$.

Proof of Theorem 1.2. We have to show that $\tilde{y}^*$ is bounded on a ball around the origin in Y. In order to apply Thm. 1.3, we make the following definitions:

$$Q := \mathrm{cl}_X C(x^0), \qquad q^0 := 0, \quad T := g'(x^0).$$

Then Q is a closed and convex cone and

$$Q(q^0) = \{\lambda(q - q^0) : \lambda \geqslant 0, q \in \mathrm{cl}_X C(x^0)\} = \mathrm{cl}_X C(x^0);$$

furthermore,

$$Q - q^0 = \mathrm{cl}_X C(x^0).$$

The assumptions of Thm. 1.3 are satisfied and (i) means

$$Y = g'(x^0)\mathrm{cl}_X C(x^0), \text{ i.e., } (1.8).$$

Hence there exists $\rho > 0$ such that

$$Y_\rho \subset g'(x^0)(\mathrm{cl}_X C(x^0) \cap X_1) \tag{1.9}$$

Now observe that for every $x \in \mathrm{cl}_X C(x^0) \cap X_1$ there is a sequence $(x_n) \subset C(x^0) \cap X_2$ converging to x.

Hence by (1.7) and (1.9) $\tilde{y}^*$ is bounded on Y_ρ if it is bounded on $g'(x^0)(C(x^0) \cap X_2)$. For $y \in g'(x^0)(C(x^0) \cap X_2) \subset \tilde{Y}$ there is $h \in C(x^0)$ with $\|h\|_X \leqslant 2$ and $y = g'(x^0)h$. By (1.6) it follows that

$$\begin{aligned} \langle \tilde{y}^*, y \rangle &= \langle \tilde{y}^*, g'(x^0)h \rangle \\ &\leqslant f'(x^0)h \\ &\leqslant 2\|f'(x^0)\| \end{aligned}$$

where $\|f'(x^0)\|$ is the norm of $f'(x^0)$ considered as a functional on X. The same arguments applied to $-y$ show that $\tilde{y}^*$ is bounded on Y_ρ.

This proves Thm. 1.2. □

Remark 1.1. Thm. 1.2 explains the general structure which underlies the proof in the second part of [2].

Remark 1.2. In Thm. 1.2, it is not necessary to assume that f and g are defined on all of X.

Let $\Lambda(x^0)$ denote the set of Lagrange multipliers $y^* \in Y^*$ satisfying

$$[f'(x^0) - y^* \cdot g'(x^0)]h \geqslant 0 \tag{1.10}$$

for all $h \in C(x^0)$.

Then the following result holds:

Proposition 1.1. *Let the assumptions of Thm. 1.2 be satisfied. Then $\Lambda(x^0)$ is a bounded set in Y^*.*

Proof. By continuity, the inequality (1.10) holds on $\mathrm{cl}_X C(x^0)$. Furthermore, Thm. 1.3 implies that for some $\rho > 0$ and for all $y \in Y$ with $\|y\|_Y \leqslant \rho$ there is $h \in \mathrm{cl}_X C(x^0)$ with $\|h\| \leqslant 1$ and $y = g'(x^0)h$. Hence, for any $y^* \in \Lambda(x^0)$,

$$\begin{aligned} \langle y^*, y \rangle &= \langle y^*, g'(x^0)h \rangle \\ &\leqslant f'(x^0)h \\ &\leqslant \|f'(x^0)\| \end{aligned}$$

The same argument applied to $-y$ shows that $\|y^*\|$ is bounded. □

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References

1. Brokate M (1980) A regularity condition for optimization in Banach spaces: Counterexamples. Appl Math Optim 6:189–192
2. Colonius F (1981) A penalty function proof of a Lagrange multiplier theorem in Banach spaces with application to linear delay systems. Appl Math Optim 7:309–334
3. Robinson SM (1976) Stability theory for systems of inequalities in nonlinear programming, part II: Differentiable nonlinear systems. SIAM J Numer Anal 13:497–513
4. Robinson SM (1976a) Regularity and stability for convex multivalued functions. Mathematics Oper Res 1:130–143
5. Zowe J, Kurcyusz S (1979) Regularity and stability for the mathematical programming problem in Banach spaces. Appl Math Optim 5:46–62