Stable and regular reachability for relaxed hereditary differential systems

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STABLE AND REGULAR REACHABILITY FOR RELAXED HEREDITARY DIFFERENTIAL SYSTEMS*

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Abstract. We investigate reachability properties of relaxed nonlinear hereditary differential systems in the state space $W^{n,\infty}[-r, 0]$. Reachability of the relaxed system is equivalent to approximate reachability of the ordinary system. Local reachability for the relaxed system can be obtained under much weaker conditions than for the ordinary system.

The notion of stable reachability allows us to relate reachability of points in $\mathbb{R}^n$ and states in $W^{n,\infty}[-r, 0]$. Regular reachability is of primary importance in optimal control problems with function space end condition, because in case of regularity Lagrange multipliers in the dual of $W^{n,\infty}[-r, 0]$ can be identified with functions in $W^{n,\infty}[-r, 0]$. Regularly reachable final states and regular trajectories are characterized and it is shown that regularity is a generic property of trajectories. Examples are given where all trajectories reaching a certain final state are regular.

Introduction. In this paper, we study reachability properties of the following nonlinear hereditary differential system Σ:

(0.1) $$\dot{x}(t) = f(x_t, u(t), t) \qquad (t \in T := [t_0, t_1]),$$

(0.2) $$x_{t_0} = \varphi_0,$$

(0.3) $$u(t) \in \Omega(t) \qquad (t \in T),$$

where $0 \leqq r < \infty$, $t_1 - r > t_0$, $f : C^n[-r, 0] \times \mathbb{R}^m \times T \to \mathbb{R}^n$, $\Omega(t) \subset \mathbb{R}^m$ and $\varphi_0 \in C^n[-r, 0]$ are fixed and

$$x_t(s) := x(t + s), \qquad s \in [-r, 0].$$

r denotes the length of the system memory. The state of this system is given by the function segment x_t and the reachability theory depends essentially on the choice of the infinite dimensional state space Z.

Jacobs/Kao [15], Banks/Jacobs/Langenhop [2], [3], Colonius/Hinrichsen [11] dealt with complete exact reachability in the Sobolev space $Z = W^{n,2}[-r, 0]$ for unconstrained linear systems. This work was primarily motivated by fixed final state optimal control problems with unconstrained (resp. energy constrained) controls $u \in L_2^m(T)$. A natural choice of Z for problems with pointwise constrained controls in L_∞^m—as considered here—is $Z = W^{n,\infty}[-r, 0]$. (This compatibility requirement on the state and control function spaces is exposed very clearly by Kurcyusz/Olbrot [17].)

However, as is well known, there is a severe drawback of reachability theory in $W^{n,p}[-r, 0]$, $1 \leqq p \leqq \infty$: complete exact reachability for linear systems can only be guaranteed if the dimension m of the control space is not less than the dimension n of the phase space. Hence it appears reasonable to replace exact reachability by approximate reachability. This concept, in fact, has been studied for unconstrained linear systems by many authors (see, e.g., Delfour/Mitter [12], Manitius/Triggiani [20], Manitius [19] in $Z = \mathbb{R}^n \times L_2^n[-r, 0]$, Olbrot [23] in various Banach spaces). While this turned out successful for certain purposes (e.g., feedback stabilizability and the infinite time linear quadratic optimal control problem), it is inadequate for fixed final state optimal control problems. For these problems exact reachability of the linearized system is necessary in order to guarantee the existence of Lagrange multipliers (see

Kurcyusz [16]). Furthermore, with respect to the phase space $Z = W^{n,\infty}[-r, 0]$, Olbrot [23, Thm. 5.3] has shown that for linear unconstrained systems approximate reachability is equivalent to exact reachability; i.e., nothing is gained by allowing approximate reachability. In this paper we propose the following way out of this apparent dilemma: we study instead of the system (0.1), (0.2), $(\widetilde{0.3})$ the *relaxed system* in the sense of Warga [27]. That is, we consider instead of measurable control functions u satisfying $(\widetilde{0.3})$ relaxed (measure-valued) control functions

$$(0.3) \qquad\qquad v \in \mathscr{S}^{\#}$$

and replace u in (0.1) by v.

Exact reachability with relaxed controls is equivalent to approximate reachability with ordinary *pointwise constrained controls* (see Theorem 2.1 and Remark 2.1 below).

It is well known that for linear systems final states which are not exactly reachable can only be approximated by applying an *unbounded* sequence of control functions (see Kurcyusz [16]). We show, however, that neither this nor Olbrot's result on the equivalence of exact and approximate reachability in $W^{n,\infty}[-r, 0]$ remains true if the *control appears nonlinearly*. Here local reachability of the relaxed system is a much weaker property than local reachability of the ordinary system. In particular, the condition $m \ggeqq n$ on the dimensions m and n of the control and phase space, respectively, is not necessary for local reachability of the relaxed system. The consequences for optimal control problems are studied in the companion paper [10].

Since the controls are bounded we study local reachability. Now suppose we can reach a certain final state φ_1 with a trajectory x^0 corresponding to a control $v^0 \in \mathscr{S}^{\#}$. Then in order to reach φ in a neighborhood of φ_1 in $W^{n,\infty}[-r, 0]$, we first have to reach $\varphi(-r)$ at time $t_1 - r$ and then trace exactly the velocity function $\dot{\varphi}(t - t_1)$ on $[t_1 - r, t_1]$.

In order to achieve this we introduce the concept of *stable reachability*. This means that a complete neighborhood of $\varphi_1(-r) \in \mathbb{R}^n$ can be reached with arbitrarily small deviations from x^0. Then the hereditary effects influencing the velocity on $[t_1 - r, t_1]$ can be kept small, and in order to reach a complete neighborhood of φ_1 in $W^{n,\infty}[-r, 0]$ we only have to compensate small deviations from $\varphi_1 = x_{t_1}^0$.

If a complete neighborhood of $\dot{\varphi}_1(t - t_1) = \dot{x}^0(t) = f(x_t^0, v^0(t), t)$, $t \in [t_1 - r, t_1]$ in $L_\infty^n[t_1 - r, t_1]$ can be covered by altering the control v^0 only on $[t_1 - r, t_1]$, we say that φ_1 is reached *regularly* with x^0 (for exact definitions see Definitions 2.1, 2.2, 3.1). The notion of regular reachability is of primary importance for optimal control problems with function space end condition, because in case of regularity Lagrange multipliers in the dual of $W^{n,\infty}[-r, 0]$ can be identified with functions in $W^{n,\infty}[-r, 0]$ (see Colonius [9], [10] (this issue, pp. 695–712)). The analysis of the somewhat delicate relations among local, stable and regular reachability constitutes the main content of this paper. The results extend and sharpen those of Colonius [9]. After the preliminary § 1, we discuss in § 2 the relation between stable and local reachability. In particular, it turns out that both notions are equivalent for linear relaxed systems, while for nonlinear systems there may be final states which are locally but not stably reachable. In § 3, we show that regular reachability plus a stable reachability property in $\mathbb{R}^n$ implies stable reachability in $W^{n,\infty}[-r, 0]$. This, conversely, does not imply regular reachability, but a weaker property. For linear systems of the form (see (3.4) below)

$$\dot{x}(t) = L(t)x_t + B(t)u(t)$$

this property specializes to the well-known rank condition for $B(t)$ on $[t_1 - r, t_1]$ plus essential boundedness of the generalized inverse $B(t)^+$ on $[t_1 - r, t_1]$. This proves, in

the state space $W^{n,\infty}[-r, 0]$, a conjecture by Banks/Jacobs/Langenhop [3]. In §4, attention is focused on properties of the whole reachable set $\mathscr{R}$ and of the set $\mathscr{T}$ of trajectories. The subsets of regularly reachable states in $\mathscr{R}$ and of regular trajectories in $\mathscr{T}$ are described. It is shown that regularity is a generic property of trajectories. Examples are given where *all* trajectories reaching a certain final state are regular.

Notation and conventions. The Banach space of continuous functions on the compact set $A \subset \mathbb{R}^m$ with values in $\mathbb{R}^n$ is denoted by $C^n(A)$. For $1 \le p \le \infty$, $W^{n,p}[a, b]$ denotes the Sobolev space of absolutely continuous functions $x : [a, b] \to \mathbb{R}^n$ with derivative $\dot{x} \in L_p^n[a, b]$, that is, with p-integrable, respectively essentially bounded derivative. The norm in the Banach space $W^{n,p}[a, b]$ is given by $\|x\| := |(|x(a)|, \|\dot{x}\|_{L_p})|$, where $|\cdot|$ denotes the Euclidean norm in finite dimensional space. $W^{n,p}[a, b]$ is identified in the canonical way with $\mathbb{R}^n \times L_p^n[a, b]$. The topological dual of a Banach space X is denoted by X^*. The interior of a set A in a Banach space is denoted by int A and co A is its convex hull. For $\delta > 0$, $\text{int}_\delta A$ is the set of all points in int A having at least distance δ to the boundary ∂A of A.

$\mathscr{L}(X_1, X_2)$ is the space of bounded linear operators from a Banach space X_1 into a Banach space X_2. For a measurable subset $S \subset \mathbb{R}$, we write $(s \in S)$ instead of for Lebesgues almost all $s \in S$. For the sets $\Omega(t)$ of admissible control values we assume that $\Omega(t) \subset \Omega_0 (t \in T)$, where $\Omega_0 \subset \mathbb{R}^m$ is compact, $t \mapsto \Omega(t)$ is measurable, and $\Omega(t)$ is closed for almost all $t \in T$. The set of Radon probability measures on Ω_0 is denoted by $\text{rpm}(\Omega_0)$. The set of relaxed controls v defined on the fixed time interval $T := [t_0, t_1]$ with values in the set of Radon probability measures on Ω_0 having support contained in $\Omega(t)$ is denoted by $\mathscr{S}^*$. Then $\mathscr{S}^*$ can be identified with a subset of the dual space $\mathscr{N} := L_1(T, C(\Omega_0))^*$ and is considered in the weak* topology (for all this see Warga [27]).

Relaxed controls satisfy the following weak measurability requirement:

$$t \mapsto c(v(t)) := \int_{\Omega_0} c(\omega) v(t)(d\omega)$$

is measurable for each $c \in C(\Omega_0)$.

For f as in Lemma 1.1 below we define

$$f(x_t, v(t), t) := \int_{\Omega_0} f(x_t, \omega, t) v(t)(d\omega),$$

and identify an ordinary control u with the relaxed control $\delta_{u(\cdot)} \in \mathscr{S}^*$, where $\delta_{u(t)}$ denotes the point measure concentrated at $u(t) \in \Omega(t)$. T_1 denotes the final interval $[t_1 - r, t_1]$. Since the initial state φ_0 of the relaxed system Σ described by (0.1), (0.3) remains fixed, we tacitly assume that x satisfies $x_{t_0} = \varphi_0$, when we speak of "a trajectory x of Σ". A pair (x, v) is called a solution of Σ, if $v \in \mathscr{S}^*$ and x satisfies (0.1), (0.2).

1. Preliminaries. In this section we formulate conditions on f which will be needed in the sequel, and we investigate how the right-hand side of (0.1) depends on trajectories and controls. Then it is shown that the relaxation of the problem is equivalent to the convexification of the set of velocity vectors. Proofs are very condensed or omitted (see Colonius [8]).

LEMMA 1.1. *Assume that the following conditions on f are satisfied:*

(1.1) *The function $f : C^n[-r, 0] \times \mathbb{R}^m \times T \to \mathbb{R}^n$ is continuous in $(\varphi, \omega) \in C^n[-r, 0] \times \mathbb{R}^m$ and measurable in $t \in T$;*

(1.2) *There is $p : \mathbb{R}_+ \times T \to \mathbb{R}_+$ such that for all $x \in C^n[t_0 - r, t_1]$ and $\omega \in \Omega_0$*

$$|f(x_t, \omega, t)| \le p(\|x_t\|_\infty, t) \qquad (t \in T),$$

where $p(s, \cdot) \in L^1_\infty(T)$ for all $s \in \mathbb{R}_+$ and $p(\cdot, t)$ is monotonically increasing for almost all $t \in T$.

Let $((x^k, v^k)) \subset C^n[t_0 - r, t_1] \times \mathscr{S}^*$ be a sequence with $x^k \to x^0$ in $C^n[t_0 - r, t_1]$ and $v^k \to v^0$ weakly* in $\mathscr{S}^*$. Then $(f(x^k_t, v^k(t), t), t \in T) \to (f(x^0_t, v^0(t), t), t \in T)$ weakly* in $L^n_\infty(T)$, hence weakly in $L^n_2(T)$.

If $p(s, \cdot)$ in (1.2) is only known to be a L_2-function, the convergence property still holds in the weak L_2-topology.

Variants of this lemma implying weak L_2-convergence are well known, even for functions f allowing lags in the controls (Berkovitz [5], Warga [26], Bates [4]). Weak* convergence in L_∞ follows similarly.

LEMMA 1.2. *Assume that* (1.1) *and* (1.2) *hold and moreover the following condition is satisfied:*

(1.3) *The function f is continuously Fréchet differentiable in the first argument, the corresponding derivative $D_1 f(\varphi, \omega, t)$ is continuous in (φ, ω, t) and for all $\omega \in \Omega_0$*

$$\|D_1 f(\varphi, \omega, t)\| \leqq p(\|\varphi\|_\infty, t) \qquad (t \in T)$$

where p is as in (1.2).

Let $((x^k, v^k)) \subset C^n[t_0 - r, t_1] \times \mathscr{S}^*$ be a sequence with $x^k \to x^0$ in $C^n[t_0 - r, t_1]$ and $v^k \to v^0$ in the strong norm topology on $\mathscr{S}^*$ (see Warga [27, Thm. IV.1.9]). Then

$$\operatorname*{ess\,sup}_t |f(x^k_t, v^k(t), t) - f(x^0_t, v^0(t), t)| \to 0$$

and the Fréchet derivatives $D_1 f(\varphi, v^k(t), t)$ exist and have the form $(\psi \in C^n[-r, 0])$

$$D_1 f(\varphi, v^k(t), t)\psi = \int_{\Omega_0} D_1 f(\varphi, \omega, t) v^k(t)(d\omega);$$

furthermore

$$\operatorname*{ess\,sup}_t \|D_1 f(x^k_t, v^k(t), t) - D_1 f(x^0_t, v^0(t), t)\| \to 0.$$

Proof. Using (1.1)–(1.3) and the mean value theorem, we find

$$\|f(x^k_t, v^k(t), t) - f(x^0_t, v^0(t), t)\|_\infty$$

$$\leqq \operatorname*{ess\,sup}_{t,\omega} |f(x^k_t, \omega, t)| \|v^k - v^0\| + \operatorname*{ess\,sup}_{t,\omega} |f(x^k_t, \omega, t) - f(x^0_t, \omega, t)|$$

$$\leqq \operatorname*{ess\,sup}_t p(c_0, t) \|v^k - v^0\| + \operatorname*{ess\,sup}_t p(c_0, t) \|x^k - x^0\|_\infty$$

for a constant $c_0 > 0$. The right-hand side converges to 0 by assumption. The existence and the form of the derivatives $D_1 f(\varphi, v^k(t), t)$ follow by Warga [27, Thm. II.3.10]. For $\psi \in C^n[-r, 0]$ with $\|\psi\|_\infty \leqq 1$

$$\|D_1 f(x^k_t, v^k(t), t)\psi - D_1 f(x^0_t, v^0(t), t)\psi\|$$

$$\leqq \sup_{t,\omega} \|D_1 f(x^k_t, \omega, t) - D_1 f(x^0_t, \omega, t)\| \|v^k\| + \sup_{t,\omega} \|D_1 f(x^0_t, \omega, t)\| \|v^k - v^0\|.$$

This converges to 0, since $\Omega \times T$ is compact and $D_1 f$ is continuous.   □

LEMMA 1.3. *Let the conditions* (1.1) *and* (1.2) *be satisfied and assume that the trajectories of Σ are uniformly bounded. Then the set $\mathscr{T}$ of trajectories x of Σ is compact and sequentially compact in the weak* topology of $W^{n,\infty}(T)$ and the uniform norm topology.*

Proof. Sequential weak* compactness of $\mathscr{T}$ follows by sequential weak* compactness of $\mathscr{S}^*$, Warga [27, Thm. IV.3.11, and Lemma 1.1]. Since $W^{n,\infty}$ is the dual of the separable space $W^{n,1}$, weak* compactness of $\mathscr{T}$ is equivalent to weak* sequential

compactness (see Dunford/Schwartz [13, p. 437]. Finally, compactness in the topology of uniform convergence follows, because the embedding of $W^{n,\infty}(T)$ into $C^n(T)$ is compact. ☐

Remark 1.1. The trajectories of Σ are uniformly bounded, if (1.3) is satisfied. The following lemma allows us to characterize relaxed velocity vectors.

LEMMA 1.4. *For a measurable subset $S \subset T$, consider a measurable function $z : S \to \mathbb{R}^n$ and a function $\Phi : \Omega_0 \times S \to \mathbb{R}^n$ with $\Phi(\omega, \cdot)$ measurable for all $\omega \in \Omega_0$ and $\Phi(\cdot, t)$ continuous for a.a. $t \in S$. Then the following three conditions are equivalent:*

$$\text{(i)} \qquad z(t) \in \mathrm{co}\ \Phi(\Omega(t), t) \qquad (t \in S);$$

$$\text{(ii)} \qquad z(t) = \Phi(v(t), t) \qquad (t \in S)$$

for an element $v \in \mathscr{S}^*$;

$$\text{(iii)} \qquad z(t) = \sum_{i=0}^{n} \alpha_i(t)\Phi(u_i(t), t) \qquad (t \in S)$$

for some measurable $\alpha_i : T \to \mathbb{R}_+$ and $u_i : T \to \Omega_0$ with $\sum_{i=0}^{n} \alpha_i(t) = 1$ and $u_i(t) \in \Omega(t)$.

Proof. The proof follows by Warga [27, Thms. I.6.13, IV 3.13; compare also Thm. VI.3.2]. ☐

Remark 1.2. Consider

$$\{(f(x_t, v(t), t), t \in T) : v \in \mathscr{S}^*\}.$$

Then the lemma above implies that this set coincides with

$$\{z \in L^n_\infty(T) : z(t) \in \mathrm{co}\ f(x_t, \Omega(t), t)(t \in T)\}.$$

This shows that along a fixed trajectory x the set of relaxed velocity vectors coincides with the convex hull of the set of ordinary velocity vectors. Hence the relaxed system Σ is equivalent to the relaxed system considered by Oguztöreli [22, § 8.9].

LEMMA 1.5. *Assume that (1.1)–(1.3) hold and, additionally, that the following condition is satisfied:*

(1.4) *For each relaxed control $v \in \mathscr{S}^*$ there is a unique trajectory $x = S(v) \in C^n[t_0 - r, t_1]$ satisfying (0.1) and (0.2).*

Then $S(v)$ depends in a continuously Fréchet differentiable manner on $v \in \mathscr{S}^*$; for $v, v^0 \in \mathscr{S}^*$ and $x^0 := S(v^0)$, the trajectory $x(v) := DS(v^0)(v - v^0) \in C^n[t_0 - r, t_1]$ satisfies

$$\text{(1.5)} \qquad \dot{x}(t) = D_1 f(x_t^0, v^0(t), t)x_t + f(x_t^0, v(t) - v^0(t), t) \qquad (t \in T),$$

$$\text{(1.6)} \qquad x_{t_0} = 0.$$

Proof. The proof follows by the implicit function theorem and computation of derivatives according to Lemma 1.2. ☐

Remark 1.3. It is convenient to write the linearized system in the form (1.5), because (1.5) involves the relaxed control $v \in \mathscr{S}^*$; $v - v^0$ is *not* in the set $\mathscr{S}^*$ of relaxed controls.

Remark 1.4. Assumption (1.4) requires existence and uniqueness for trajectories corresponding to *relaxed* controls. This problem can be reduced to existence and uniqueness theory of functional differential equations using the representation of relaxed velocities introduced by Gamkrelidze: by Lemma 1.4 for each relaxed trajectory x there exist measurable functions $\alpha_0, \alpha_1, \cdots, \alpha_n : T \to \mathbb{R}_+$ with $\sum \alpha_i(t) = 1$ and ordinary controls $u_0, u_1, \cdots, u_n$ with values in $\Omega(t)$ such that

$$\dot{x}(t) = \sum_{i=0}^{n} \alpha_i(t)f(x_t, u_i(t), t) \qquad (t \in T),$$

and conversely. Hence relaxed trajectories satisfy a functional differential equation; for results on the solution of these equations see Hale [14].

Remark 1.5. The set of trajectories of (1.5) coincides with the set of trajectories of the system

$$(1.6) \qquad\qquad \dot{x}(t) = D_1 f(x_t^0, v^0(t), t)x_t + u(t) \qquad (t \in T)$$

where $u : T \to \mathbb{R}^n$ are measurable functions with values $u(t)$ in

$$\mathrm{co}\, f(x_t^0, \Omega(t), t) - f(x_t^0, v^0(t), t) = \mathrm{co}\, f(x_t^0, \Omega(t), t) - \dot{x}^0(t).$$

This is a consequence of Lemma 1.4. Observe that (1.5) (and (1.6)) again is uniquely solvable by Hale [14, Chap. 6, Thm. 2.1].

2. Local and stable reachability. In this section we formally state the relation between the ordinary and the relaxed problem indicated in the introduction. Then we prove sufficiency of a criterion for stable reachability and show that local and stable reachability are equivalent for linear relaxed systems.

In the rest of this paper we assume that the conditions (1.1), (1.2) and (1.4) are satisfied.

THEOREM 2.1. *If the trajectories of* Σ *are uniformly bounded, the following assertions hold*:

(i) *Suppose* (x^0, v^0) *is a solution of* Σ. *Then there is a sequence* $((x^k, v^k))$ *of solutions of* Σ, *where* v^k *is an ordinary control, such that* $v^k \to v^0$ *weakly* in* $\mathscr{S}^*$ *and* $x^k \to x^0$ *in* $C^n[t_0 - r, t_1]$.

(ii) *If, conversely,* $((x^k, v^k))$ *is a sequence of solutions of* Σ *(where* v^k *is not necessarily an ordinary control) with* $x_{t_1}^k \to \varphi_1$ *in* $C^n[-r, 0]$, *then there exists a solution* (x^0, v^0) *of* Σ *such that for a subsequence* $v^k \to v^0$ *weakly* in* $\mathscr{S}^*$ *and* $x^k \to x^0$ *in* $C^n[t_0 - r, t_1]$; *in particular* $x_{t_1}^0 = \varphi_1$.

Proof. (i) Since the ordinary controls are dense in $\mathscr{S}^*$ (Warga [27, Thm. IV.3.10]) there is a sequence (v^k) of ordinary controls converging to v^0. Then a subsequence of the corresponding sequence of trajectories $x^k := S(v^k)$ converges by Lemma 1.3, and by Lemma 1.2 and assumption (1.4) its limit is x^0.

(ii) follows similarly. ☐

Remark 2.1. The assertions of the theorem remain true if uniform convergence of the trajectories (resp. final states) is replaced by weak* convergence in $W^{n,\infty}$.

The following definition presents an *abus de langage*; however, it is quite convenient.

DEFINITION 2.1. A function $\varphi_1 \in W^{n,\infty}[-r, 0]$ is called *locally reachable* at time t_1, if and only if there is a neighborhood N of φ_1 in $W^{n,\infty}[-r, 0]$ such that for each $\varphi \in N$ there is a trajectory x of Σ with $\varphi = x_{t_1}$. Analogously, we define local reachability of a point α in $\mathbb{R}^n$.

DEFINITION 2.2. A function $\varphi_1 \in W^{n,\infty}[-r, 0]$ is called *stably reachable* at time t_1 with a trajectory x^0 of Σ, if and only if for each neighborhood V of x^0 in $C^n[t_0 - r, t_1]$ there is a neighborhood N of φ_1 in $W^{n,\infty}[-r, 0]$ such that for each $\varphi \in N$ there is a trajectory $x \in V$ of Σ with $\varphi = x_t$. Analogously, we define stable reachability of a point α in $\mathbb{R}^n$.

Stable reachability of φ (resp., α) with x^0 means that one can reach a complete neighborhood of φ_1 (resp., α) with arbitrarily small deviations from x^0. Obviously this implies local reachability. However, the following example which is taken from Lee/Markus [18, p. 257] illustrates that in general the converse does not hold. It is a nonlinear system with $r = 0$, i.e., an ordinary differential system.

Example 2.1. Consider

$$\dot{x}^1(t) = x^2(t)u^1(t) - x^1(t)u^2(t),$$

$$\dot{x}^2(t) = -x^1(t)u^1(t) - x^2(t)u^2(t), \qquad t \in [0, \pi],$$

(2.1)

$$x^1(0) = 1, \qquad x^2(0) = 0,$$

$$(u^1(t), u^2(t)) \in \Omega := \{(\omega_1, \omega_2) \in \mathbb{R}^2 : |\omega_1| \leq 1, |\omega_2| \leq 1\}.$$

Since for fixed $x = (x^1, x^2)$ the velocity sets are convex, it suffices to consider ordinary controls (see Remark 1.2). In polar coordinates (r, φ), the system is described by

$$\dot{r}(t) = -r(t)u^2(t), \qquad \dot{\varphi}(t) = -u^1(t),$$

$$r(0) = 1, \qquad \varphi(0) = 0.$$

The reachable set at time $t_1 = \pi$ is an annulus around 0 with inner radius $e^{-\pi}$ and outer radius e^{π}. It is not difficult to see that all pairs (r, π) with $e^{-\pi} < r < e^{\pi}$ are locally reachable, but not stably reachable at time $t_1 = \pi$.

Remark 2.2. It seems that this is a general phenomenon: We conjecture that any point being for the first time in the interior of the reachable set without having—before that time—been at the boundary of the reachable set is not stably reachable.

Condition (2.2), below, is related to regular reachability (see § 3) and will play an important role in the rest of this paper. The following *stability lemma* shows that it remains valid under small perturbations of the trajectory x^0.

LEMMA 2.1. *Assume that* (1.3) *and the following condition are satisfied*:
(2.2) *For a measurable subset* $S \subset T$, *for* $x^0 \in C^n[t_0 - r, t_1]$, *and* $z \in L^n_\infty(S)$, *there is a neighborhood* V *of* $0 \in \mathbb{R}^n$ *such that*

$$V \subset z(t) + \operatorname{co} f(x^0_t, \Omega(t), t) \qquad (t \in S).$$

Then there are $\delta > 0$ *and a neighborhood* $\tilde{V}$ *of* $0 \in \mathbb{R}^n$ *such that for all* x *with* $\|x - x^0\|_\infty < \delta$

$$\tilde{V} \subset z(t) + \operatorname{co} f(x_t, \Omega(t), t) \qquad (t \in S).$$

Proof. Take V as an n-simplex with vertices $e_0, e_1, \cdots, e_n$. Then by Lemma 1.4 there are $v_i \in \mathscr{S}^*$ with

$$e_i = z(t) + f(x^0_t, v_i(t), t) \qquad (t \in S).$$

By Lemma 1.2, $f(x_t, v_i(t), t)$ is essentially uniformly close to $f(x^0_t, v_i(t), t)$ for small $\|x - x^0\|_\infty$. Thus there is $\delta > 0$ such that for all x with $\|x - x^0\|_\infty < \delta$ and for a.a. $t \in S$ the points $f(x_t, v_i(t), t)$, $i = 0, 1, \cdots, n$, are also vertices of n-simplices containing a fixed neighborhood $\tilde{V}$ of $0 \in \mathbb{R}^n$. This proves the lemma, since convex combinations of $f(x_t, v_i(t), t)$, $i = 0, 1, \cdots, n$ remain in $\operatorname{co} f(x_t, \Omega(t), t)$. $\quad\square$

Remark 2.3. Assumption (1.3) may be replaced by continuity of $f(\varphi, \omega, t)$ in (φ, ω, t) (compare the second part of Lemma 1.2).

THEOREM 2.2. *Suppose the trajectory* x^0 *of* Σ *reaches* $\varphi_1 \in W^{n,\infty}[-r, 0]$ *at time* t_1, *and there are* $\varepsilon > 0$ *and a neighborhood* V *of* $0 \in \mathbb{R}^n$ *such that*

(2.3) $$V \subset -\dot{x}^0(t) + \operatorname{co} f(x^0_t, \Omega(t), t) \qquad (t \in [t_1 - r - \varepsilon, t_1 - r]).$$

Then $\varphi_1(-r)$ *is stably reachable with* x^0 *at time* $t_1 - r$.

If the inclusion in (2.3) *holds also for a.a.* $t \in T_1$, *the function* φ_1 *is stably reachable with* x^0 *at time* t_1.

Proof. By the stability lemma, Lemma 2.1, there are $\delta_0 > 0$ and a neighborhood U of $\dot{x}^0|[t_1 - r - \varepsilon, t_1 - r] \in L^n_\infty$ such that for all x with $\|x - x^0\|_\infty < \delta_0$ and all $z \in U$

$$(2.4) \qquad z(t) \in \operatorname{co} f(x_t, \Omega(t), t) \qquad (t \in [t_1 - r - \varepsilon, t_1 - r]).$$

For all $\delta > 0$, the set N_δ defined by

$$N_\delta := \{\alpha \in \mathbb{R}^n : \alpha = x(t_1 - r) \text{ for some } x \in C^n[t_0 - r, t_1]$$

$$\text{with } x(t) = x^0(t) \text{ for } t \in [t_0 - r, t_1 - r - \varepsilon],$$

$$\|x - x^0\|_\infty < \delta \text{ and } \dot{x}|[t_1 - r - \varepsilon, t_1 - r] \in U\}$$

forms a neighborhood of $\varphi_1(-r)$ in $\mathbb{R}^n$. For $0 < \delta < \delta_0$ take $\alpha \in N_\delta$. Then there is $x \in C^n[t_0 - r, t_1]$ as in the definition of N_δ with

$$(2.5) \qquad \dot{x}(t) \in \operatorname{co} f(x_t, \Omega(t), t) \qquad (t \in [t_1 - r - \varepsilon, t_1 - r]).$$

By Lemma 1.4, x is a trajectory of Σ, reaching α by definition. Thus x^0 reaches $\varphi_1(-r)$ stably at time $t_1 - r$.

Stable reachability of φ_1 follows similarly, since for each $\delta > 0$ and each neighborhood U of $\dot{x}^0|[t_1 - r - \varepsilon, t_1]$ in L^n_∞ the set

$$\{\varphi \in W^{n,\infty}[-r, 0]: \varphi = x_{t_1} \text{ for some } x \in C^n[t_0 - r, t_1]$$

$$\text{with } x(t) = x^0(t) \text{ for } t \in [t_0 - r, t_1 - r - \varepsilon],$$

$$\|x - x^0\|_\infty < \delta, \text{ and } \dot{x}|[t_1 - r - \varepsilon, t_1] \in U\}$$

forms a neighborhood of φ_1 in $W^{n,\infty}[-r, 0]$.   □

Next we consider a class of systems Λ which are linear in x. As a *relaxed* system, Λ is linear.

$$\Lambda \qquad \begin{aligned} \dot{x}(t) &= L(t)x_t + b(v(t), t) \qquad (t \in T), \\ x_{t_0} &= \varphi_0, \end{aligned}$$

where $v \in \mathscr{S}^*$ and

(2.6) for $L: T \to \mathscr{L}(C^n[-r, 0], \mathbb{R}^n)$ the map $t \mapsto L(t)\varphi$ is measurable for all $\varphi \in C^n[-r, 0]$ and ess sup $\|L(t)\| < \infty$; the function $b: T \times \Omega_0 \to \mathbb{R}^n$ is measurable in the first, continuous in the second argument, and satisfies

$$\operatorname*{ess\,sup}_{t \in T} \sup_{\omega \in \Omega_0} |b(t, \omega)| < \infty;$$

furthermore, $\varphi_0 \in C^n[-r, 0]$ and $t_1 - r > t_0$.

For each $v \in \mathscr{S}^*$, the system Λ has a unique trajectory x with $x_t = \varphi_0$, Hale [14, p. 142]. The system Λ is a special case of Σ satisfying conditions (1.1)–(1.4).

Remark 2.4. Consider a function $b: T \times \Omega_0 \to \mathbb{R}^n$ with the same properties as in (2.6). Then the map $t \mapsto b(t, \cdot)$ may be considered as a map on T with values in $C^n(\Omega_0)$. Hence, since $C^n(\Omega_0)$ is separable, b may be identified with an element of $L_\infty(T, C^n(\Omega_0))$ where $\|b\|_\infty := \operatorname{ess\,sup}_{t \in T} \sup_{\omega \in \Omega_0} |b(t, \omega)|$ (cf. Warga [27, p. 122 and Theorem I.5.26]).

Remark 2.5. The set of trajectories of Λ coincides with the set of trajectories of the following system with ordinary controls:

$$\dot{x}(t) = L(t)x_t + u(t) \qquad (t \in T),$$

where the controls u are measurable functions defined on T with values $u(t)$ in co $b(\Omega(t), t)$. Hence the reachability theories for these two classes of systems coincide. However, we write down the proofs for the system Λ, since the corresponding optimal control theories are different and the extra expense of writing is minimal.

THEOREM 2.3. *For the system* Λ *a function* φ_1 *in* $W^{n,\infty}[-r, 0]$, *(respectively a point* α *in* $\mathbb{R}^n$*), is stably reachable iff it is locally reachable.*

Proof. One direction is trivial. Let N be a neighborhood of φ_1 in $W^{n,\infty}[-r, 0]$ such that each element of N is reachable at time t_1. Suppose that φ_1 is reached with the trajectory x^0 of Λ corresponding to $v^0 \in \mathscr{S}^*$. For $\varepsilon > 0$, define a neighborhood N^ε of φ_1 by

$$N^\varepsilon := \varphi_1 + \varepsilon(N - \varphi_1).$$

Let $\varphi^\varepsilon \in N^\varepsilon$. Then there is $\varphi \in N$ with $\varphi^\varepsilon = \varphi_1 + \varepsilon(\varphi - \varphi_1)$. The function φ is reachable, say with x corresponding to $v \in \mathscr{S}^*$. Then by linearity, φ^ε is reached with $x^\varepsilon := x^0 + \varepsilon(x - x^0)$ corresponding to

$$v^\varepsilon := v^0 + \varepsilon(v - v^0) \in \mathscr{S}^*.$$

For all $\delta > 0$ there is $\varepsilon > 0$ such that for all $\varphi^\varepsilon \in N^\varepsilon$ the trajectory x^ε satisfies $\|x^\varepsilon - x^0\|_\infty < \delta$, since the trajectories of Λ are uniformly bounded. Thus φ_1 is reached stably with x^0. The assertion for $\alpha \in \mathbb{R}^n$ follows similarly. □

Remark 2.6. Suppose that φ_1 is reached stably with x^0 and x^1 is any trajectory of Λ reaching φ_1. Then the theorem above shows that φ_1 is reached stably with x^1. Hence for linear systems it is not necessary to specify the trajectory reaching φ_1 stably, if φ_1 is stably reachable with some trajectory.

Remark 2.7. For scalar systems (i.e. $n = 1$) stable reachability of points in $\mathbb{R}$ follows under very mild assumptions. Suppose the trajectories of Σ are uniformly bounded and there are v^0, $\bar{v}$, $\underline{v} \in \mathscr{S}^*$ such that the map $v \mapsto x(v, t_1)$ has Gateaux derivatives at v^0 in the directions $\bar{v} - v^0$ and $\underline{v} - v^0$ satisfying

$$D(x(t_1, v^0); \bar{v} - v^0) > 0 \text{ and } D(x(t_1, v^0); \underline{v} - v^0) < 0.$$

Then $x(t_1, v^0)$ is stably reachable with $x(v^0)$, since the map $v \mapsto x(t_1, v)$ is continuous and $n = 1$.

3. Regular and stable reachability. In this section, we introduce the concept of regular reachability and investigate its relation to local and stable reachability.

DEFINITION 3.1. A function φ_1 in $W^{n,\infty}[-r, 0]$ is called *regularly reachable at time* t_1 *with a trajectory* x^0 *of* Σ, iff $\varphi_1(-r) = x^0(t_1 - r)$ and there exists a neighborhood V of $0 \in \mathbb{R}^n$ such that

(3.1) $V \subset -\dot{\varphi}_1(t - t_1) + \text{co } f(x_t^0, \Omega(t), t)$ $(t \in T_1)$.

A trajectory x^0 is called *regular*, iff $x_{t_1}^0$ is reached regularly at time t_1 with x^0; otherwise x^0 is called *irregular*. By Lemma 1.4 it follows—which is to be expected—that $\varphi_1 = x_{t_1}^0$, if x^0 reaches φ_1 regularly. A function $\varphi_1 \in W^{n,\infty}[-r, 0]$ is reached regularly with x^0 iff $\varphi_1(-r) = x^0(t_1 - r)$ and in L_∞^n

$$\dot{\varphi}_1 \in \text{int}\{z \in L_\infty^n[-r, 0]: z(t - t_1) \in \text{co } f(x_t^0, \Omega(t), t)(t \in T_1)\}$$

$$= \text{int}\{z \in L_\infty^n[-r, 0]: z(t - t_1) = f(x_t^0, v(t), t)(t \in T_1) \text{ for a } v \in \mathscr{S}^*\}.$$

Regular reachability means that not only $\dot{\varphi}_1(t - t_1)$, but a whole uniform neighborhood of $\dot{\varphi}_1(t - t_1)$ is contained in the set of relaxed velocity vectors if the system at time t is in the state x_t^0.

If the optimal trajectory x^0 is regular, the maximum principle constitutes a necessary optimality criterion (see Colonius [9], [10]).

The following theorem relates stable reachability of points in $\mathbb{R}^n$ and final states in $W^{n,\infty}[-r, 0]$ via the notion of regular reachability.

THEOREM 3.1. *Assume that* Σ *satisfies* (1.3) *and that for* $\varphi_1 \in W^{n,\infty}[-r, 0]$ *the following assumptions hold*:

(a) *the point* $\varphi_1(-r)$ *is reached stably with* x^0 *at* $t_1 - r$;

(b) *the function* φ_1 *is reached regularly with* x^0 *at* t_1.

Then φ_1 *is reached stably with* x^0 *at* t_1.

Proof. By assumption (b) and the stability lemma 2.1, there are $\delta > 0$ and a neighborhood N of $\varphi_1 \in W^{n,\infty}[-r, 0]$ such that for all $\varphi \in N$ and all x with $\|x - x^0\|_\infty < \delta$

$$\dot{\varphi}(t - t_1) \in \operatorname{co} f(x_t, \Omega(t), t) \qquad (t \in T_1).$$

By assumption (a), there is a neighborhood N_1 of $\varphi_1 \in W^{n,\infty}[-r, 0]$, such that for all $\varphi \in N_1$ there is $x^\varphi \in C^n[t_0 - r, t_1 - r]$ with $\|x^\varphi - x^0\|_\infty < \delta$ and

$$\dot{x}^\varphi(t) = f(x_t^\varphi, v^\varphi(t), t) \qquad (t \in [t_0, t_1 - r])$$

for a relaxed control $v^\varphi \in \mathcal{S}^*$, and

$$x_{t_0}^\varphi = \varphi_0, \qquad x^\varphi(t_1 - r) = \varphi(-r).$$

Without loss of generality, we may assume that $N_1 \subset N$. The function x^φ can be extended to an absolutely continuous function on $[t_0 - r, t_1]$ by

$$x_{t_1}^\varphi := \varphi.$$

It only remains to prove that x^φ is a trajectory of Σ. We have to show that there is $v \in \mathcal{S}^*$ such that

$$\dot{x}^\varphi(t) = \dot{\varphi}(t - t_1) = f(x_t^\varphi, v(t), t) \qquad (t \in T_1)$$

or, equivalently,

$$\dot{\varphi}(t - t_1) \in \operatorname{co} f(x_t^\varphi, \Omega(t), t) \qquad (t \in T_1).$$

This follows by $N_1 \subset N$ and $\|x^\varphi - x^0\|_\infty < \delta$. □

COROLLARY 3.1. *Assume that* $\varphi_1 \in W^{n,\infty}[-r, 0]$ *is regularly reachable with* x^0. *Then for all* y *in a neighborhood of* $\dot{\varphi}_1 \in L_\infty^n[-r, 0]$ *there is a trajectory* x *of* Σ *with*

$$\dot{x}_{t_1} = y \text{ and } x(t) = x^0(t) \text{ for } t \in [t_0, t_1 - r].$$

Proof. First reach $\varphi_1(-r)$ with x^0. Then use regularity as above in order to steer the system to

$$\left(\varphi_1(-r) + \int_{-r}^{s} y(\tau) \, d\tau, s \in [-r, 0] \right) \in W^{n,\infty}[-r, 0].$$

□

Theorem 3.1 shows that, up to a finite dimensional reachability condition at $t_1 - r$, regular reachability implies stable reachability. We shall prove a partial converse of this for a special class of systems.

Define the class of systems Σ_a, where the control term appears addititively, by:

$$\dot{x}(t) = a(x_t, t) + b(v(t), t) \qquad (t \in T),$$

$$x_{t_0} = \varphi_0,$$

where $v \in \mathcal{S}^*$ and, in addition to the requirements of (2.6), (3.2) the map $a : C^n[-r, 0] \times T \to \mathbb{R}^n$ is continuous in the first and measurable in the second argument, and $|a(\varphi, t)|$ is essentially bounded for bounded arguments.

Σ_a is a special case of Σ satisfying (1.1) and (1.2). Furthermore, we assume unique solvability, i.e., (1.4).

Remark 3.1. As in Remark 2.5, the set of trajectories of Σ_a coincides with the set of trajectories of

$$\dot{x}(t) = a(x_t, t) + u(t),$$

where the controls u take values $u(t)$ in co $b(\Omega(t), t)$.

We shall make use of the following generalization of a classical result due to A. Denjoy.

THEOREM 3.2. *Let X be a complete separable metric space, S a compact metric space, $h : [a, b] \times S \to X$ such that $h(\cdot, s)$ is measurable and $h(t, \cdot)$ is continuous for all $(t, s) \in [a, b] \times S$. Then for a subset T' of $[a, b]$ with $\lambda(T') = \lambda([a, b])$ and for all $t \in T'$ there is a set $E \subset [a, b]$ such that t is a point of density of E and $h | E \times S$ is continuous in t.*

Proof. By a strong version of Lusin's theorem (see Warga [27, Thm. I.5.26(2)]), there is for all $\varepsilon > 0$ a closed subset F_ε of $[a, b]$ such that $\lambda([a, b] \backslash F_\varepsilon) \leqq \varepsilon$ and $h | F_\varepsilon \times S$ is continuous. This implies the theorem above in exactly the same way as the usual Lusin's theorem implies Natanson [21, Satz 2, p. 296]. $\square$

THEOREM 3.3. *Assume that the trajectories of Σ_a are uniformly bounded, and that there is a neighborhood U of $z^0 \in L^n_\infty(T_1)$ such that for all $z \in U$ there is a trajectory x with $z = (\dot{x})_{t_1}$. Then*

$$\text{int}\{z \in L^n_\infty(T_1): z(t) \in \text{co } b(\Omega(t), t)(t \in T_1)\} \neq \varnothing.$$

Proof. There is a trajectory x^0 with $z^0 = (\dot{x}^0)_{t_1}$. Define for $t \in T_1$

$$V_t := -z^0(t) + a(x^0_t, t) + \text{co } b(\Omega(t), t).$$

Then V_t is convex and $0 \in V_t$. Due to the assumptions on Σ_a, all sets V_t are contained in a compact set $C \subset \mathbb{R}^n$. Consider $\Phi : t \mapsto V_t$ as a map from T_1 into the set of closed subsets of C. Then Φ is well defined and measurable by Warga [27, Thm. I.7.6]. Thus by Warga [27, Thm. I.7.8], there is a countable set $\{\xi_i : i \in \mathbb{N}\}$ of measurable selections of Φ such that

$$\{\xi_i(t): i \in \mathbb{N}\} \text{ is dense in } \Phi(t) = V_t \text{ for a.a. } t \in T_1.$$

For $y \in \mathscr{E} := \{y \in \mathbb{R}^n: |y| = 1\}$ and $t \in T_1$ let

$$\bar{h}(y, t) := \sup\{y\xi_i(t): i \in \mathbb{N}\}, \qquad \underline{h}(y, t) := \inf\{y\xi_i(t): i \in \mathbb{N}\}.$$

Then $\bar{h}$ and $\underline{h}$ are continuous in y, measurable in t and satisfy

$$\bar{h}(y, t) = \max\{y\xi: \xi \in V_t\}, \qquad \underline{h}(y, t) = \min\{y\xi: \xi \in V_t\}.$$

We shall prove that there are a constant $\alpha > 0$ and a set $T' \subset T_1$ with $\lambda(T') = \lambda(T_1)$ such that for all $t \in T'$ and all $y \in \mathscr{E}$

(3.3) $$\bar{h}(y, t) \geqq \underline{h}(y, t) + \alpha.$$

Then, by Warga [27, Thm. I.7.10], there exists a measurable function $\xi : T_1 \to C$ with $\xi(t) \in V_t$, $t \in T'$ such that for all $y \in \mathscr{E}$ and all $t \in T'$

$$\min\{y\xi: \xi \in V_t\} + \frac{\alpha}{2} \leqq y\xi(t) \leqq \max\{y\xi: \xi \in V_t\} - \frac{\alpha}{2}.$$

Then $\xi \in \text{int}\{z \in L^n_\infty(T_1): z(t) \in V_t(t \in T_1)\}$, and the theorem follows.

The set $\mathscr{E}$ and the set $\mathscr{T}$ of trajectories of Σ_a are compact metric spaces, since $\mathscr{T}$ is compact in $C^n[t_0-r, t_1]$ by Lemma 1.3. Consider a as a map: $\mathscr{T} \times T_1 \to \mathbb{R}^n$, $(x, t) \mapsto a(x_t, t)$. Then a is continuous in x and measurable in t.

Now Theorem 3.2 implies that there is $T' \subset T_1$ with $\lambda(T') = \lambda(T)$ such that for all $t' \in T'$, all $y \in \mathscr{E}$, and all $x \in \mathscr{T}$ $\bar{h}(y, t)$, $\underline{h}(y, t)$, and $a(x_t, t)$ are approximately continuous in $t = t'$. There exists a constant $\alpha > 0$ with the following property: For any $t' \in T'$ and any $y \in \mathscr{E}$, there is a step function z such that $z^0 + z \in U$ and that $yz(t)$ has a jump of length greater than 2α in $t = t'$.

For any such z there exists a solution (x, v) of Σ_a with

$$z(t) = -z^0(t) + a(x_t, t) + b(v(t), t) \qquad (t \in T_1).$$

Hence

$$yz(t) = y[a(x_t, t) - a(x_t^0, t)] + y[-z^0(t) + a(x_t^0, t) + b(v(t), t)] \qquad (t \in T_1).$$

The left-hand side has a jump of length greater than 2α in $t = t'$ and $y[a(x_t, t) - a(x_t^0, t)]$ is approximately continuous in $t = t'$. Hence (3.3) follows. $\square$

Remark 3.2. The proof above owes a lot to the proof of Schwarzkopf [24, Thm. 1].

Remark 3.3. Let $\varphi_1 \in W^{n,\infty}[-r, 0]$. Then the assumptions of Theorem 3.3 are satisfied for $z^0(t) = \dot{\varphi}_1(t - t_1)$, $t \in T_1$, if φ_1 is reached locally (resp. stably) with x^0 at t_1, and the trajectories of Σ_a are uniformly bounded. Then it also follows that $\varphi_1(-r)$ is reached locally (resp. stably) with x^0 at $t_1 - r$. If one could prove that V_t contains a neighborhood of $0 \in \mathbb{R}^n$ which is independent of $t \in T_1$, the converse of Theorem 3.1 would hold. However, in the following we give an example of a linear system where a certain final state φ_1 is locally reachable (i.e., by Theorem 2.3 also stably reachable), but reached irregularly with a certain trajectory x^0. Hence in general, we cannot show more than Theorem 3.3. Intuitively, this may be explained as follows:

In the proof of Theorem 3.1, we reach φ_1 with x^0. In order to reach all φ in a neighborhood of $\varphi_1 \in W^{n,\infty}[-r, 0]$, we reach first $\varphi(-r)$; by stable reachability of $\varphi_1(-r)$, we can achieve this with small deviations from $x^0|[t_0, t_1-r]$. The hereditary effects influencing the possible velocity vectors of the system on T_1 are compensated using regular reachability of φ_1: By the stability lemma 2.1, regularity allows to choose controls in such a way that *arbitrary* small deviations from x^0 are compensated and all velocity vectors $\dot{\varphi}(t - t_1)$ in a uniform neighborhood of $\dot{\varphi}_1(t - t_1)$ are reached.

Now one can try to reach $\varphi(-r)$ in such a way that the hereditary effects influencing the system behavior on T_1 prepare the reaching of φ. Then it is not necessary to compensate *arbitrary* small deviations from x^0 on $[t_0, t_1-r]$ and we may get local reachability without regularity. This is performed in the following scalar example. Theorem 3.3 shows the limitations of this manipulation: One can achieve a shift of the set of velocity vectors, but V_t must contain interior points.

Example 3.1. Consider

$$\dot{x}(t) = x(t-2) + u(t) \qquad (t \in T := [0, 6]),$$

$$x_0 = 0,$$

$$\Omega := [-3, 1].$$

We claim that

$$\varphi_1(t) := \begin{cases} t+2, & t \in [-2, -1], \\ 1, & t \in [-1, 2] \end{cases}$$

is locally and irregularly reachable. Define an (ordinary) control function u^1 by

$$u^1(t) := \begin{cases} 0 & \text{for } t \in [0, 4], \\ 1 & \text{for } t \in (4, 5], \\ 0 & \text{for } t \in (5, 6]. \end{cases}$$

The corresponding trajectory x^1 is given by

$$x^1(t) := \begin{cases} 0 & \text{for } t \in [-2, 4], \\ t - 4 & \text{for } t \in (4, 5), \\ 1 & \text{for } t \in [5, 6]. \end{cases}$$

Clearly x^1 is irregular and satisfies $x_6^1 = \varphi_1$. Now we shall show that the assumptions of Theorem 2.2 are satisfied for a certain trajectory x^0. This will imply that φ_1 is locally reachable.

Define a control function u^0 by

$$u^0(t) = \begin{cases} 1 & \text{for } t \in [0, 2), \\ -2 & \text{for } t \in [2, 4), \\ -\frac{1}{2}(t-4)^2 + 2(t-2) - 5 & \text{for } t \in [4, 5), \\ -\frac{1}{2}(t-4)^2 + 2(t-2) - 6 & \text{for } t \in [5, 6]. \end{cases}$$

The corresponding trajectory x^0 is given by

$$x^0(t) = \begin{cases} 0 & \text{for } t \in [-2, 0], \\ t & \text{for } t \in [0, 2], \\ \frac{1}{2}(t-2)^2 - 2t + 6 & \text{for } t \in [2, 4], \\ t - 4 & \text{for } t \in [4, 5], \\ 1 & \text{for } t \in [5, 6]. \end{cases}$$

Clearly $x_6^0 = \varphi_1$.

An elementary analysis shows that $u^0(t)$ is uniformly in the interior of Ω on $[2, 6]$; i.e., the assumptions of Theorem 2.2 are satisfied.   □

Assume that (1.3) holds. Then the system Σ can be linearized, and the linearized system Σ_{lin} has the form (1.5), (1.6) as indicated in Lemma 1.5.

Regular reachability has the remarkable property, that it is invariant under linearization. More precisely, we have

PROPOSITION 3.1. *Suppose that (1.3) holds, and $\varphi_1 = x_{t_1}^0$, where x^0 is the trajectory of Σ corresponding to $v^0 \in \mathcal{S}^*$. Then x^0 reaches φ_1 regularly iff the zero trajectory of Σ_{lin} reaches $0 \in W^{n,\infty}[-r, 0]$ regularly.*

Proof. Write down the definitions!   □

This property motivates interest in regular reachability of $0 \in W^{n,\infty}[-r, 0]$ with the zero trajectory. Consider the following linear system:

(3.4) $$\dot{x}(t) = L(t)x_t + B(t)v(t) \qquad (t \in T),$$

(3.5) $$x_{t_0} = 0,$$

where we assume that the assumptions in (2.6) are satisfied and $B \in L_\infty(T, \mathbb{R}^{n \times m})$.

COROLLARY 3.2. *Let $0 \in \text{int}_\delta \text{co } \Omega(t)(t \in T_1)$ for a $\delta > 0$. Then the following two conditions on the system (3.4), (3.5) are equivalent:*

 (i) *The trajectory $x^0 = 0$ reaches $0 \in W^{n,\infty}[-r, 0]$ regularly at time t_1;*

 (ii) *Rank $B(t) = n(t \in T_1)$ and $B(t)^+$ is essentially bounded on T_1.*

Furthermore, $0 \in W^{n,\infty}[-r, 0]$ *is locally reachable at time* t_1, *iff* (ii) *holds and* $0 \in \mathbb{R}^n$ *is locally reachable at time* $t_1 - r$.

Proof. Define the multiplication operator $\tilde{B} : L^m_\infty(T_1) \to L^n_\infty(T_1)$ by

$$(\tilde{B}u)(t) = B(t)u(t) \qquad (t \in T_1).$$

By Kurcyusz/Olbrot [17, Lemma 3], $\tilde{B}$ has a closed image iff $B(t)^+$ is essentially bounded on T_1. Hence $B(t)^+$ is essentially bounded and rank $B(t) = n$ $(t \in T_1)$ iff the linear map B is surjective, hence open. Because of $0 \in \text{int}_\delta \text{ co } \Omega(t)$, this in turn is equivalent to the existence of a neighborhood V of $0 \in \mathbb{R}^n$ such that

$$V \subset V_t = \text{co } B(t)\Omega(t) = B(t) \text{ co } \Omega(t) \qquad (t \in T_1).$$

This proves the equivalence of (i) and (ii). If (i) holds and $0 \in \mathbb{R}^n$ is locally reachable at $t_1 - r$, then $0 \in W^{n,\infty}[-r, 0]$ is locally reachable by Theorem 2.3 and Theorem 3.1.

The converse follows by Theorem 3.3 and the assumption

$$0 \in \text{int}_\delta \text{ co } \Omega(t)(t \in T_1). \qquad \square$$

Remark 3.4. The last equivalence in the corollary above is the analogue of results in the theory of *unconstrained* linear hereditary systems. Here the following is known (for controls $u \in L^m_p(T)$, $1 \leqq p \leqq \infty$):

Suppose that $B(t)^+$ is essentially bounded on T_1. Then each element $\varphi \in W^{n,p}[-r, 0]$ is reachable at time t_1, iff each element $\alpha \in \mathbb{R}^n$ is reachable at time $t_1 - r$ and rank $B(t) = n$ $(t \in T_1)$ (see Jacobs/Kao [15], Banks/Jacobs/Langenhop [2], [3], Kurcyusz/Olbrot [17], Colonius/Hinrichsen [11]). In fact, by a category argument, Corollary 3.2 implies the result above for $p = \infty$. Furthermore, complete reachability in $W^{n,\infty}$ implies even that $B(t)^+$ is essentially bounded on T_1. This proves a conjecture by Banks/Jacobs/Langenhop [3] (see also Kurcyusz/Olbrot [17, p. 48] and Colonius/Hinrichsen [11, p. 878]). In turn the cited result implies the last assertion of Corollary 3.2, provided that $B(t)^+$ is essentially bounded on T_1 as shown in Colonius [9, Remark 10].

Remark 3.5. If $B(t)$ depends continuously on t and has a constant rank, then by Kurcyusz/Olbrot [17, Lemma 4] the generalized inverse $B(t)^+$ is bounded.

Remark 3.6. Suppose there is $\varepsilon > 0$ such that for a.a. $t \in [t_1 - r - \varepsilon, t_1 - r]$ one has $0 \in \text{int}_\delta \text{ co } \Omega(t)$ for a $\delta > 0$, and the generalized inverse $B(t)^+$ is essentially bounded and has rank n. Then by Theorem 2.2 $0 \in \mathbb{R}^n$ is locally reachable at time $t_1 - r$.

4. Analysis of the reachable set $\mathcal{R}$ of final states and of the set $\mathcal{T}$ of trajectories. In §3, we analyzed reachability properties of a specific final state φ_1 and of a specific trajectory x^0. Now we change our point of view by analyzing properties of the whole set $\mathcal{T}$ of trajectories:

$$\mathcal{T} := \{x \in C^n[t_0 - r, t_1] : x \text{ is a trajectory of } \Sigma\}$$

and of the whole reachable set $\mathcal{R}$ of final states

$$\mathcal{R} := \{\varphi \in W^{n,\infty}[-r, 0] : \text{there is } x \in \mathcal{T} \text{ with } \varphi = x_{t_1}\}.$$

Since the initial state φ_0 remains fixed throughout, we may identify $x \in \mathcal{T}$ with $x|T$ and consider $\mathcal{T}$ as a subset of $W^{n,\infty}(T)$. The set $\mathcal{R}$ will always be considered in $W^{n,\infty}$-topology.

Observe that by definition $\varphi \in \text{int } \mathcal{R}$ iff φ is locally reachable.

LEMMA 4.1. *Consider the system* Σ_a *of* §3, *and assume that* x^0, $x^1 \in \mathcal{T}$ *correspond to* v^0, $v^1 \in \mathcal{S}^*$ *and that* x^1 *is regular. Then the trajectories* $x^\gamma \in \mathcal{T}$ *corresponding to* $v^\gamma := (1 - \gamma)v^0 + \gamma v^1 \in \mathcal{S}^*$ *where* $0 < \gamma \leqq 1$ *are regular.*

Proof. By regularity of x^1 there is $\delta > 0$ such that

$$\dot{x}^1(t) - a(x_t^1, t) = b(v^1(t), t) \in \text{int}_\delta \, \text{co} \, b(\Omega(t), t) \qquad (t \in T_1).$$

Since $b(v^0(t), t) \in \text{co} \, b(\Omega(t), t)$ $(t \in T_1)$ and $\text{co} \, b(\Omega(t), t)$ is convex, this implies for $0 < \gamma \leqq 1$

$$\dot{x}^\gamma(t) - a(x_t^\gamma, t) = b(v^\gamma(t), t) = (1 - \gamma)b(v^0(t), t) + \gamma b(v^1(t), t)$$

$$\in \text{int}_{\gamma\delta} \, \text{co} \, b(\Omega(t), t) \qquad (t \in T_1).$$

This shows regularity of x^γ.   □

Remark 4.1. Using the same construction as in the proof above, one can see that for Σ_a the set $\mathscr{T}$ is path connected, i.e., any two trajectories x^0, $x^1 \in \mathscr{T}$ can be connected by a continuous path $(x^\gamma, \gamma \in [0, 1])$. One only has to prove that the map $v^\gamma \to x^\gamma : \mathscr{S}^* \to W^{n,\infty}(T)$ is continuous with respect to norm topology on $W^{n,\infty}(T)$ and strong norm topology on $\mathscr{S}^*$. This follows by Lemma 1.3 and Lemma 1.2.

Define the finite dimensional reachable set $\mathscr{R}^f$ at time $t_1 - r$ as

$$\mathscr{R}^f := \{\alpha \in \mathbb{R}^n : \text{there is } \varphi \in \mathscr{R} \text{ with } \alpha = \varphi(-r)\},$$

and let

$$\mathscr{R}_\alpha := \{\varphi \in \mathscr{R} : \varphi(-r) = \alpha\} \quad \text{for } \alpha \in \mathscr{R}^f.$$

Then

$$\mathscr{R} = \bigcup_{\alpha \in \mathscr{R}^f} \mathscr{R}\alpha$$

is a decomposition of the reachable set parametrized by the elements of the finite dimensional reachable set at time $t_1 - r$. We identify $\mathscr{R}_\alpha$ with a subset of $L_\infty^n[-r, 0]$, and consider $\mathscr{R}_\alpha$ in the induced topology. Then

$$\text{int} \, \mathscr{R}_\alpha = \{\varphi_1 \in \mathscr{R}_\alpha : \text{there is a } \delta > 0 \text{ such that all } \varphi \in W^{n,\infty}[-r, 0]$$

$$\text{with } \varphi(-r) = \alpha \text{ and } \|\dot{\varphi}_1 - \dot{\varphi}_\infty\| < \delta \text{ are in } \mathscr{R}_\alpha\}.$$

Corollary 3.1 shows that each regularly reachable φ_1 lies in $\text{int} \, \mathscr{R}_{\varphi_1(-r)}$.

THEOREM 4.1. *Assume that Σ_a satisfies (1.3). Then*

   (i) *there exists a regularly reachable final state φ_1 in $\mathscr{R}_\alpha$ for each $\alpha \in \mathscr{R}^f$ iff*

(4.1) $$\text{int}_\delta \, \text{co} \, b(\Omega(t), t) \neq \varnothing \qquad (t \in T_1) \quad \text{for a } \delta > 0.$$

   (ii) *Suppose (4.1) holds. Then the set of regularly reachable final states is open and dense in $\mathscr{R}_\alpha$ for each $\alpha \in \mathscr{R}^f$.*

   (iii) *Suppose that φ_1 is reached regularly and stably with a certain trajectory x^0. Then there is a neighborhood N of φ_1 in $W^{n,\infty}[-r, 0]$ such that all $\varphi \in N$ are regularly reachable.*

Proof. (i) One direction is trivial. Suppose (4.1) holds and let $z^0 \in L_\infty^n(T_1)$ with

$$z^0(t) \in \text{int}_\delta \, \text{co} \, b(\Omega(t), t) \qquad (t \in T_1).$$

For any trajectory x^0 define x^1 by

$$x^1(t) := x^0(t), \quad t \in [t_0 - r, t_1 - r], \quad \dot{x}^1(t) = a(x_t^1, t) + z^0(t) \qquad (t \in T_1).$$

Then x^1 is a regular trajectory and $\varphi_1 := x_{t_1}^1 \in \mathscr{R}_{x^0(t_1 - r)}$.

   (ii) Let $\varphi_1 \in \mathscr{R}$ be reached with the trajectory x^0 corresponding to $v^0 \in \mathscr{S}^*$. By (i) there exists $\varphi_2 \in \mathscr{R}_{\varphi_1(-r)}$ reached with a regular trajectory x^2 corresponding to

$v^2 \in \mathscr{S}^*$. Define x^1 as the trajectory corresponding to the control $v^1 \in \mathscr{S}^*$ which coincides with v^0 on $[t_0, t_1 - r)$ and with v^2 on T_1. Then $x^1(t_1 - r) = \varphi_1(-r)$ and Lemma 4.1 together with Lemma 1.2 implies that in each neighborhood of φ_1 in $\mathscr{R}_{\varphi_1(-r)}$ there is a regularly reachable final state.

The set of regularly reachable final states is open in $\mathscr{R}_\alpha$ for $\alpha \in \mathscr{R}^f$ by the stability lemma 2.1.

(iii) The assumptions imply that there is $\delta > 0$ such that

$$\dot{\varphi}_1(t - t_1) - a(x_t^0, t) \in \text{int}_{3\delta} \text{ co } b(\Omega(t), t) \qquad (t \in T_1).$$

Furthermore there is a neighborhood N of φ_1 in $W^{n,\infty}[-r, 0]$, such that all $\varphi \in N$ are reached with trajectories x^φ satisfying

$$\operatorname*{ess\ sup}_{t \in T_1} |a(x_t^\varphi, t) - a(x_t^0, t)| < \delta.$$

If we choose N so small that $\|\varphi - \varphi_1\| < \delta$ for all $\varphi \in N$, we find

$$\dot{\varphi}(t - t_1) - a(x_t^\varphi, t) \in \text{int}_\delta \text{ co } b(\Omega(t), t) \qquad (t \in T_1). \qquad \square$$

COROLLARY 4.1. *Suppose that* Σ_a *satisfies* (1.3). *Then in* L_∞^n $\text{int}\{\dot{\varphi} \in L_\infty^n(T_1): \varphi \in \mathscr{R}\} \neq \varnothing$ *iff* (4.1) *holds.*

Proof. If condition (4.1) is satisfied, then by Theorem 4.1 there exists a regularly reachable final state φ_1. By Corollary 3.1 we find $\dot{\varphi}_1 \in \text{int}\{\dot{\varphi} \in L_\infty^n(T_1): \varphi \in \mathscr{R}\}$. The converse follows by Theorem 3.3. $\square$

Remark 4.2. If (4.1) holds a.e. on $[t_1 - r - \varepsilon, t_1]$ for an $\varepsilon > 0$, then by Theorem 2.2 it follows that int $\mathscr{R} \neq \varnothing$.

Remark 4.3. If the function b does not depend explicitly on t and $\Omega(t) = \Omega_0$ $(t \in T)$, then the condition (4.1) (nonempty interior in L_∞-norm) reduces to int co $b(\Omega_0) \neq \varnothing$. Furthermore, by the remark above and Corollary 4.1, this condition is equivalent to int $\mathscr{R} \neq \varnothing$.

Remark 4.4. By Remark 2.4, functions b satisfying the requirements in (2.6) may be considered as elements in $L_\infty(T, C^n(\Omega_0))$. Now suppose that for a.a. $t \in T$ the set $\Omega(t)$ contains at least $n + 1$ points. Then the set of functions b satisfying condition (4.1) is open and dense in $L_\infty(T, C^n(\Omega_0))$, and in this sense condition (4.1) is *generic*. Openness is easily seen. Density can be proved using a construction similar to that in the proof of Lemma 4.1: There exist $n + 1$ measurable functions $\omega_i: T \to \Omega_0$, $i = 0, 1, \cdots, n$ with $\omega_i(t) \in \Omega(t)$ and $\omega_i(t) \neq \omega_j(t)$ for a.a. $t \in T$ and $i \neq j$. Then for a.a. $t \in T$ there exists $b_1(t, \cdot) \in C^n(\Omega_0)$ mapping $\omega_0(t), \cdots, \omega_n(t)$ into $n + 1$ points in $\mathbb{R}^n$ being in general position, i.e., having the property $\text{int}_\delta \text{ co}\{b_1(t, \omega_0(t)), \cdots, b_1(t, \omega_n(t))\} \neq \varnothing$ for some $\delta > 0$. We may assume that δ is independent of t and that $b_1 \in L_\infty(T, C^n(\Omega_0))$. Let b_0 be an arbitrary element of $L_\infty(T, C^n(\Omega_0))$, and define

$$b^\gamma(t, \omega) := \gamma b_1(t, \omega) + (1 - \gamma) b_0(t, \omega) \quad \text{for } t \in T, \quad \omega \in \Omega_0.$$

Then $b^\gamma \to b_0$ in $L_\infty(T, C^n(\Omega_0))$ and

$$\text{int}_{\gamma\delta} \text{ co }\{b\gamma(t, \omega_0(t)), \cdots, b^\gamma(t, \omega_n(t))\} \neq \varnothing,$$

i.e., the functions $b\gamma$ satisfy condition (4.1).

Remark 4.5. Theorem 4.1 (ii) shows that regular reachability is *generic* in $\mathscr{R}_\alpha$. Part (iii) shows regular reachability is preserved under small perturbations of φ_1, provided that it is also stable.

For linear systems, we can completely characterize regular reachability by strengthening the properties (ii) and (iii) in Theorem 4.1.

COROLLARY 4.2. *Suppose that for the linear system* Λ *condition* (4.1) *is satisfied. Then* $\varphi_1 \in W^{n,\infty}[-r, 0]$ *is regularly reachable iff* $\varphi_1 \in \operatorname{int} \mathcal{R}_{\varphi_1(-r)}$. *Furthermore, each element of* $\operatorname{int} \mathcal{R}$ *is regularly reachable.*

Proof. In view of Corollary 3.1 we have to show that each $\varphi_1 \in \operatorname{int} \mathcal{R}_{\varphi_1(-r)}$ is regularly reachable. By Theorem 4.1(ii) there is $\varphi = W^{n,\infty}[-r, 0]$ such that $\varphi_1 + \varphi \in \mathcal{R}_{\varphi_1(-r)}$ is regularly reachable, say with x^0, and $\varphi_1 - \varphi \in \mathcal{R}_{\varphi_1(-r)}$ is reachable, say with x^1. By linearity and Lemma 4.1,

$$\varphi_1 = \tfrac{1}{2}(\varphi_1 + \varphi) + \tfrac{1}{2}(\varphi_1 - \varphi)$$

is reached regularly with $\tfrac{1}{2}x^0 + \tfrac{1}{2}x^1$. Furthermore for each $\varphi \in \operatorname{int} \mathcal{R}$ we have $\varphi \in \operatorname{int} \mathcal{R}_{\varphi(-r)}$. This implies the last assertion. $\quad\square$

Remark 4.6. It follows from Corollary 4.2 that each final state, which is reachable only with irregular trajectories lies in $\partial \mathcal{R}$ and clearly

$$\bigcup_{\alpha \in \mathcal{R}^f} \partial \mathcal{R}_\alpha \subset \partial \mathcal{R}.$$

However, there are regularly reachable final states in $\partial \mathcal{R} \setminus \bigcup_{\alpha \in \mathcal{R}^f} \partial \mathcal{R}_\alpha$: For instance, reach $\alpha \in \partial \mathcal{R}^f$ with a trajectory x^0 satisfying $\dot{x}^0(t) - L(t)x_t^0 \in \operatorname{int}_\delta \operatorname{co} b(\Omega(t), t)(t \in T_1)$ for a $\delta > 0$. Then $x_{t_1}^0 \in \partial \mathcal{R} \setminus \partial \mathcal{R}_\alpha$.

Regular reachability of a final function φ_1 means that there exists a certain trajectory reaching φ_1 regularly. Now in an optimization problem, we are interested in a specific *unknown* trajectory, the optimal one, out of all trajectories reaching φ_1. Thus regular reachability of φ_1 is only *the minimal degree of well-posedness* we have to require in an optimal control problem. In the following, we go a step further by asking: What can be said about regularity of *all* trajectories reaching a given final function φ_1?

Consider the system Σ_a. Here the regularity condition for x^0 is satisfied iff there is a neighborhood V of $0 \in \mathbb{R}^n$ such that

$$V \subset -\dot{\varphi}_1(t - t_1) + a(x_t^0, t) + \operatorname{co} b(\Omega(t), t) \qquad (t \in T_1).$$

Now suppose that we can specify a uniform bound c_0 for $|a(x_t^0, t|, \ t \in T_1$. If $\operatorname{co} b(\Omega(t), t)$ can be made big enough compared to c_0, all trajectories reaching φ_1 are regular. We give two examples where this idea applies.

Example 4.1.

$$\dot{x}(t) = a(x_t, t) + v(t) \qquad (t \in T),$$

$$x_{t_0} = \varphi_0, \qquad x_{t_1} = \varphi_1,$$

where a is as in the definition of Σ_a, and additionally, satisfies

$$|a(x_t, t)| \leqq 1 \qquad (t \in T_1)$$

for all trajectories x reaching φ_1. x is regular, iff there is a neighborhood V of $0 \in \mathbb{R}^n$ such that

$$V + \dot{\varphi}_1(t - t_1) - a(x_t, t) \subset \operatorname{int}_\delta \operatorname{co} \Omega(t) \qquad (t \in T_1).$$

for a $\delta > 0$.

If, e.g., $\varphi_1 = 0$, take Ω as a n-simplex with vertices $\omega_0, \omega_1, \cdots, \omega_n$ containing the unit ball of $\mathbb{R}^n$ in its interior. Then, obviously, all trajectories reaching φ_1 are regular. Observe that by Corollary 4.2 $\mathcal{R}_{\varphi_1(-r)}$ is open.

Example 4.2.

$$\dot{x}(t) = A_1 x(t-1) + v(t) \qquad (t \in T := [0, k+1]),$$

$$x_0 = \varphi_0, \qquad x_{k+1} = \varphi_1,$$

where φ_0, $\varphi_1 \in C^n[-1, 0]$, $k \in \mathbb{N}$, $A_1 \in M_{nn}$, $\Omega := \{\omega \in \mathbb{R}^n : |\omega| \leq q\}$ and $q > 0$. Then for $t \in [k, k+1]$

$$|A_1 x(t-1)| \leq \|A_1\| \|x_k\|_\infty \leq \|A_1\| \{(1 + \|A_1\|) \|x_{k-1}\|_\infty + q\}$$

$$\leq \|A_1\| \left\{ (1 + \|A_1\|)^k \|\varphi_0\|_\infty + q \sum_{j=0}^{k} (1 + \|A_1\|)^j \right\}.$$

All trajectories reaching φ_1 are regular, if

$$(4.2) \qquad \|\dot{\varphi}_1\|_\infty + \|A_1\| \left\{ (1 + \|A_1\|)^k \|\varphi_0\|_\infty + q \sum_{j=0}^{k} (1 + \|A_1\|)^j \right\} < q.$$

If, e.g., $\varphi_0 = \varphi_1 = 0$ and

$$(4.3) \qquad \|A_1\| \sum_{j=0}^{k} (1 + \|A_1\|)^j < 1,$$

condition (4.2) is satisfied for all $q > 0$.

This simple example illustrates that the regularity requirement for *all* trajectories reaching a given final function is very restrictive: Observe that (4.3) implies $\|A_1\| < 1/(k+1)$. However, if Ω may change in time, regularity can be guaranteed if $\Omega(t)$ is big enough for $t \in T_1$.

The following theorem characterizes regular trajectories by a reachability property, and claims that regularity is a *generic* property of trajectories.

THEOREM 4.2. $x^0 \in \mathcal{T}$ *is a regular trajectory of* Σ_a *iff in* $L_\infty^n(T_1)$

$$(4.4) \qquad (\dot{x}^0)_{t_1} \in \text{int}\{(\dot{x})_{t_1} : x \in \mathcal{T} \text{ and } x(t) = x^0(t) \text{ for } t \in [t_0 - r, t_1 - r]\}.$$

If φ_1 *is a regularly reachable final state of* Λ *the set of regular trajectories is open and dense in* $\{x^0 \in \mathcal{T} : x_{t_1}^0 = \varphi_1\}$.

Proof. One direction follows by Corollary 3.1. Conversely, assume that $x^0 \in \mathcal{T}$ satisfies (4.4). By Corollary 4.1 and the construction in the proof of Theorem 4.1(ii), there is in each neighborhood of $x^0 \in W^{n,\infty}(T)$ a regular trajectory x^1 coinciding with x^0 on $[t_0 - r, t_1 - r]$. By (4.4), x^1 may be chosen such that $x^2 := x^0 - (x^1 - x^0) \in \mathcal{T}$. Then by Lemma 4.1

$$x^0 = \frac{x^1}{2} + \frac{x^2}{2}$$

is regular.

The final assertion of the theorem follows again by the construction in the proof of Theorem 4.1(ii) and linearity.   □

Remark 4.7. The results above illustrate that regularity occurs also for higher dimensional systems with scalar control. In Example 4.1, take the vertices $\omega_0, \omega_1, \cdots, \omega_n$ as images of the points i/n, and consider $\tilde{\Omega} := \{0, 1/n, \cdots, 1\} \subset [0, 1]$ as new control set with control actions ω_i. More generally, condition (4.1) is, e.g.,

satisfied for the scalar control

$$b(\omega) = \begin{pmatrix} \omega \\ \omega^2 \\ \vdots \\ \omega^n \end{pmatrix}, \quad \text{where } \Omega(t) = \Omega_0 := [0, 1] \qquad (t \in T).$$

Then int co $b(\Omega_0) \neq \varnothing$, and hence it follows for the linear system Λ that int $\mathcal{R} \neq \varnothing$; each element of int $\mathcal{R}$ is regularly reachable and the regular trajectories reaching φ_1 form an open, dense subset of the set of all trajectories reaching φ_1.

Remark 4.8. Let $B \subset C^n[t_0 - r, t_1]$ and assume that there is a neighborhood V of $0 \in \mathbb{R}^n$ such that for all $x^0 \in B$

$$V \subset -\dot{\varphi}_1(t - t_1) + \text{co } f(x_t^0, \Omega, t) \qquad (t \in T_1)$$

(compare Definition 3.1). Then it appears natural to apply a fixed point theorem in order to prove local reachability of φ_1. In fact, if B is defined by certain growth conditions on f, and a finite dimensional condition for the reachability of $\varphi_1(-r)$ at time $t_1 - r$ is added, one can prove such a result using Kakutani's fixed point theorem (Colonius [8, Thms. 3.1, 3.3]). For Example 4.1, this yields that $\varphi_1 = 0$ is locally reachable [8, Beispiel 4.1]. In the context of control theory, this classical argument was used, e.g., by Tarnove [25]. Angell [1] and Chukwu [6], [7] applied it to hereditary differential systems in order to achieve reachability of a fixed final state.

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