FEEDBACK STABILIZATION OF ONE DIMENSIONAL SYSTEMS NEAR BIFURCATION POINTS

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1. Introduction.

Recently, much attention has been devoted to two areas in nonlinear control theory: The construction of smooth and non-smooth stabilizing feedbacks, and the control of systems around bifurcation points of the underlying, uncontrolled dynamics. In this note we announce some results on controllability in parametrized control systems of dimension 1 with compact control range, and we will use the results for the design of (not necessarily smooth) static feedback laws.

For one-dimensional control systems of the form $\dot{x} = X(x) + u(t)Y(x)$, with $u \in \mathcal{U} = \{u : \mathbb{R} \to U$, measurable$\}$ and $U \subset \mathbb{R}$ compact, connected, $0 \in \text{int } U$, the control sets (i.e. regions of complete controllability) are easily computed, see e.g.[CK]. For a parametrized family of control systems $\dot{x} = X_\alpha(x) + u(t)Y_\alpha(x)$ with $\alpha \in I$, an open interval in $\mathbb{R}$, the control sets need not depend continuously on α, even if X_α and Y_α are analytic in α. The bifurcation behavior of the control sets for these systems is actually characterized by two parameters: the original bifurcation parameter α, and a parameter $\rho \geq 0$, which determines the size of the control range via $U^\rho = \rho \cdot U$.

The basic question when dealing with static feedback stabilization of nonlinear systems is: When does (asymptotic) controllability imply stabilizability? The answer to this question depends heavily on the smoothness that one requires for the feedback law. Even for one-dimensional systems there may exist piecewise constant feedbacks, when no smooth feedback is available, see [SoSu] and the survey [So]. In this paper, we do not require smoothness of the feedbacks (they will be piecewise constant). This leads to the characterization that a one-dimensional control system is (asymptotically) stabilizable via static state feedback at a point x^0 iff x^0 is contained in some control set. Since nonlinear systems, even under the Lie algebra rank condition, need not be completely controllable nor asymptotically controllable to a point x^0 from all initial values, the problem of characterizing the domain for a feedback law has to be addressed. We show that this set coincides with the domain of attraction $\mathcal{A}(D)$ of the control set D with $x^0 \in D$. .

Next we study feedback stabilization near bifurcation points: Does there exist a feedback law that will delay or advance the original bifurcation point α_0? The answer here depends on whether the value of the bifurcation parameter is known (or estimated, which leads to concepts of adaptive control), or a common feedback law for a range of α-values is desired (robustness with respect to α). For points x^0, where the Lie algebra rank condition is satisfied (regular systems), robust feedback laws that stabilize at x^0 for a range of α-values do not exist in general. If, however, x^0 is a common fixed point for all $u \in U$ (singular systems), one obtains, for fixed control range, a precise estimate for the α-range where robust stabilization is possible, i.e. for advance or delay of the original bifurcation point α_0. Reference [WA] uses washout filters i.e. dynamic feedback to preserve the original bifurcation diagram even for regular systems.

2. Feedback stabilization of one-dimensional control systems.

In this paper we consider single-input one-dimensional systems of the form

(1) $$\dot{x} = X(x) + u(t)Y(x) \quad \text{in } \mathbb{R}^1,$$

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where $u \in \mathcal{U} = \{u : \mathbb{R} \to U, \text{ measurable}\}$ and $U \subset \mathbb{R}$ compact, connected with $0 \in \text{int } U$. We assume that X and Y are C^∞ vectorfields, and that $X + uY$ has at most finitely many zeros for all $u \in U$. Denote by $\varphi(t, x, u)$ the solution of (1) at time $t \in \mathbb{R}$ with initial value $x = \varphi(0, x, u)$ under the control action $u \in \mathcal{U}$. The positive (forward in time) orbit of a point $x \in \mathbb{R}$ is defined as $\mathcal{O}^+(x) := \{y \in \mathbb{R};$ there are $u \in \mathcal{U}$ and $t \geq 0$ with $\varphi(t, x, u) = y\}$, similarly for the negative orbit $\mathcal{O}^-(x)$. A control set $D \subset \mathbb{R}$ is a set satisfying (a) $cl\mathcal{O}^+(x) \supset D$ for all $x \in D$, (b) for all $x \in D$ there exists $u \in \mathcal{U}$ with $\varphi(t, x, u) \in D$ for all $t \geq 0$, and (c) maximality with respect to set inclusion. ("cl" denotes the closure of a set.)

Two fundamentally different cases have to be considered:

(I) regular systems, where the Lie algebra rank condition is satisfied
   (H) $\dim \mathcal{LA}\{X + uY, u \in U\}(x) = 1$ for all $x \in \mathbb{R}$,

(II) singular systems, which contain common fixed points of X and Y.

Regular systems.

Under our assumptions, we consider all control sets of regular systems that have nonvoid interior. Then precise controllability holds in control sets, i.e. $\mathcal{O}^+(x) \supset \text{int } D$ for all $x \in D$. (If $X + uY$ can have infinitely many zeros for some $u \in U$, one point control sets may occur, but stabilization at these points is not possible.) The order between control sets is defined as

$$D \prec D' \text{ if there exists } x \in D \text{ with } cl\mathcal{O}^+(x) \cap D' \neq \phi.$$

For one-dimensional systems the control sets are the intervals of fixed points of $X + uY$, $u \in U$, where the lower boundary α belongs to D iff there exists $u \in U$ with $X(\alpha) + uY(\alpha) > 0$, and similarly for the upper boundary, see [CK] for a precise statement.

Singular systems.

If $x^0 \in \mathbb{R}$ is a point, where the Lie algebra rank condition is violated, then x^0 is a common fixed point of the vector fields X and Y, and hence of all (time varying) vector fields $X + u(t)Y$, $u \in \mathcal{U}$. In this case, $\{x^0\}$ is a one point (invariant) control set of (1). We extend the order between control sets to singular systems by defining for a one point control set $\{x^0\}$ and an arbitrary control set D with $\{x^0\} \cap clD = \phi$

$\{x^0\} \prec D$ if for all $\varepsilon > 0$ there exists $x \in \mathbb{R}$ with $|x - x^0| < \varepsilon$ and $cl\mathcal{O}^+(x) \cap D \neq \phi$.

Analogously, we define $D \prec \{x^0\}$.

Define the domain of attraction of a control set D by

$$\mathcal{A}(D) := \{y \in \mathbb{R}; cl\mathcal{O}^+(y) \cap D \neq \phi.$$

Next we specify our notion of feedback.

Definition. The control system (1) is locally feedback stabilizable at $x_0 \in \mathbb{R}$, if there exists an open neighborhood V of x_0 and a piecewise constant feedback function $F : V \to U$ such that x_0 is a fixed point of $\dot{x} = X(x) + F(x)Y(x)$ and this system, restricted to V, has a unique solution and is stable. Similarly, we define asymptotic and exponential local feedback stabilizability.

1. Theorem. *Assume that system (1) is regular. Then this system is locally feedback stabilizable at $x_0 \in \mathbb{R}$ iff there exists a control set $D \subset \mathbb{R}$ with int $D \neq \phi$ and $x_0 \in D$. In this case, the set of initial values from which the system can be stabilized at x_0, agrees with $\mathcal{A}(D)$.*

An analogous result holds for the singular case:

2. Theorem. *The system (1) is locally feedback stabilizable at $x_0 \in \mathbb{R}$, where x_0 is a common fixed point of X and Y, iff either*

(i) *$x_0 \notin clD$ for all control sets D of (1) with $\{x_0\} \neq D$, and $\{x_0\}$ is maximal with respect to $\prec$, or*

(ii) *$x_0 \in clD$ for some control set D, and $x_0 \in$ int $\mathcal{A}(\{x_0\})$.*

3. Bifurcation of control sets.

In order to understand control and stabilization of systems near bifurcation points, we have to generalize the results from the previous sections to control systems depending on a parameter. We will first consider the global bifurcation structure of one-dimensional control systems. In general, these systems are of the form

$$\dot{x} = f_\alpha(x, u) \text{ in } \mathbb{R}^1, \ \alpha \in I \subset \mathbb{R},$$

with $u \in \mathcal{U}^\rho = \{u : \mathbb{R} \to U^\rho, \text{ measurable}\}$ and $U^\rho = \rho \cdot U$ for $\rho \geq 0$, where $U \subset \mathbb{R}$ is compact and connected. For each α and ρ the control sets of the system can be computed, and their dependence on these parameters can be studied. Here we will only hint at some interesting effects that can occur around bifurcation points. It turns out that the structurally simpler case

$$(2_\alpha^\rho) \qquad \dot{x} = X(x) + f(\alpha + u(t))Y(x), \ \alpha \in I \subset \mathbb{R}, \ \rho \geq 0, \ u \in \mathcal{U}^\rho$$

covers all the interesting phenomena, and the results are easy to visualize. The general case can be treated by using the same ideas.

We continue to work under the assumptions from Section 2., and distinguish again between regular and singular systems.

Regular systems.

Here we assume that the Lie algebra rank condition holds for (2_α^ρ) for all $\alpha \in I$, $\rho > 0$.

Singular systems.

Here we assume that there exists an open interval $I \subset \mathbb{R}$ such that the controlled system (2_α^ρ) has a common fixed point x^0 for all $\alpha \in I$.

Figures 1a. and 1b. show the typical bifurcation behavior of control sets for a system with pitchfork and transcritical bifurcation for two different control ranges ρ. Figure 2. illustrates the situation around a pitchfork bifurcation in a sigular system. (Analogous results can be obtained for other types of generic bifurcation scenarios.) In these figures the areas A,B, and C correspond to different types of control sets, F indicates sets of fixed points. For a given value α (and for the control range ρ as depicted in the figures) the corresponding part of A is an invariant control set (i.e. the control system cannot leave this set), the corresponding part of B is a (variant) control set with one bifurcation branch of the underlying system, while in C the (variant) control sets contain more than one bifurcation branch of the uncontrolled system.

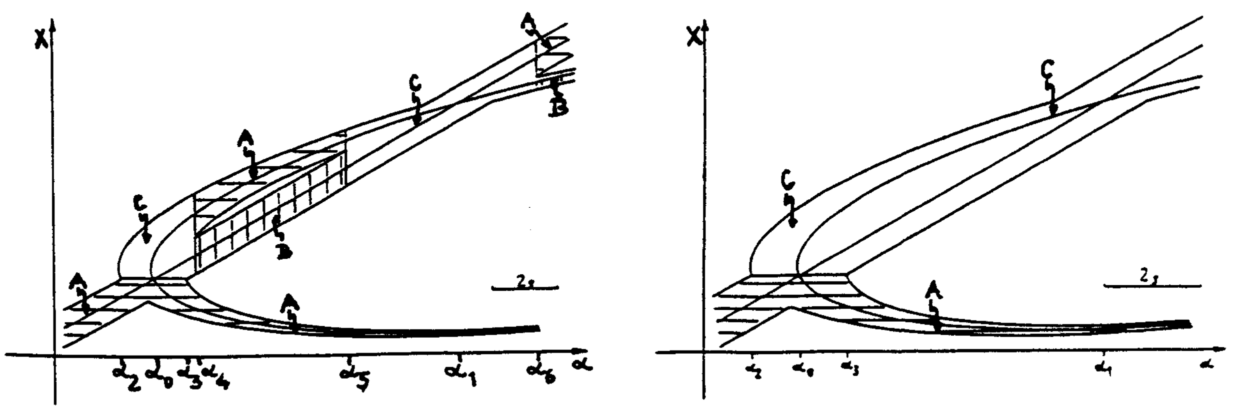

Figure 1. Control sets for a system with pitchfork bifurcation (x^0, α_0) and transcritical bifurcation (x^1, α_1) for two different control ranges ρ.

Comparison between the left Figure 3a and the right Figure 3.b. shows how controllability around bifurcation points is affected by the size ρ of the control range: While the lower region of invariant control sets varies continuously with ρ, the entire upper region in Figure 3.a. has merged into one variant control set in region C for Figure 3.b. This is due to the fact that with increasing ρ more of the global dynamics determines the controllability behavior, while the original bifurcation parameter α describes the situation locally.

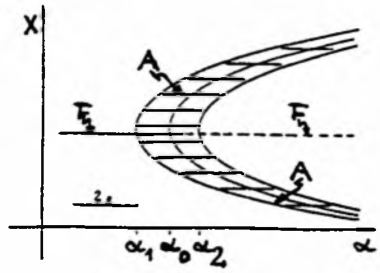

Figure 2. Control sets around a pitchfork bifurcation (x^0, α_0) in a singular system.
In this situation no variant control sets exist, but the fixed point x^0 becomes unstable at α_0. However, the control system is stabilizable at x^0 as long as $\rho \geq \alpha - \alpha_0$, see Theorem 3. Hence the bifurcation can be delayed to $\alpha_2 > \alpha_0$.

4. Feedback stabilization near bifurcation points.

In a family of one-dimensional differential equations $\dot{x} = X_\alpha(x), \alpha \in I$, bifurcation means change of stability behavior for a branch a fixed points. If the differential equations are controlled (via bounded inputs), the question arises, whether there exist admissible feedbacks that stabilize the system around a bifurcation point. This question can be made more precise in various ways:

a) The stabilizing feedback can depend on the bifurcation parameter α, i.e. α is known, and
b) the stabilizing feedback should work for a wide range of the parameter α, i.e. one looks for robust stabilization with respect to α, while the stabilizing feedback does not depend on α.

The answer to a) is obtained through a combination of the techniques used above. In the regular case, let (x^0, α_0) be a bifurcation point of the uncontrolled equation such that the bifurcation branches emanating from x^0 one continuously increasing and decreasing, respectively, around x^0. Then x^0 lies in the interior of a control set of $(2^\rho_{\alpha_0})$ for all $\rho > 0$, and hence the system can be stabilized at x^0 via bounded feedback with values in U^ρ. For $\alpha \neq \alpha_0$ there exists $\rho^*(\alpha)$ (which may be ∞), such that for $\rho > \rho^*(\alpha)$ the corresponding control system is stabilizable at x^0, while for $\rho < \rho^*(\alpha)$ this may not be possible. Here ρ^* is defined by $\rho^*(\alpha) = \inf\{\rho > 0; x^0 \notin \text{int } D^\rho_\alpha\}$, where D^ρ_α is the control set of (2^ρ_α) containing x^0. (If no such control set exists for some $\alpha \in I$ and all $\rho > 0$, then $\rho^*(\alpha) = 0$.) The set of initial values, from which the system can be stabilized at the fixed point x^0 is given by the domain of attraction as defined in Section 2.

The solution to Problem b) is treated here in the context of static state feedback, i.e. we do not assume that α can be estimated (which would lead to adaptive control) nor do we introduce feedback dynamics (leading e.g. to washout filters, see [WA] for an example). Then feedback stabilization under uncertainty in α takes different forms for regular and for singular systems. For regular systems, one cannot guarantee that one feedback law will stabilize the system at the same point x^* for various values of α. However, if for given a control range $\rho > 0$ one has a connected family D^ρ_α of control sets (i.e. there exists a continuous function $f : I \to \bigcup_{\alpha \in I} D^\rho_\alpha$ with $f(\alpha) \in \text{int } D^\rho_\alpha$) such that $\text{int} \bigcap_{\alpha \in I} D^\rho_\alpha \neq \phi$, then there exists a nonempty, open set B of initial values and a common feedback law $u : B \to U^\rho$ such that for each $\alpha \in I$ the sytem (2^ρ_α) is stabilized at some point in $D = \bigcup_{\alpha \in I} D^\rho_\alpha$. In particular, if $\alpha : \mathbb{R} \to I$ is time varying, then there is $T > 0$ such that the solution of $\dot{x} = X(x) + f(\alpha(t) + u(x))Y(x)$ lies in D for all $t \geq T$. This result follows directly from the feedback construction used to prove Theorem 1. Note that the set B of initial values can be strictly contained in $\bigcap_{\alpha \in I} \mathcal{A}(D^\rho_\alpha)$: This is e.g. the case if for each $\alpha \in I$ the domain of attraction $\mathcal{A}(D^\rho_\alpha)$ contains a (with respect to $\prec$ smallest) control set $\hat{D}^\rho_\alpha$, such that $\hat{D}^\rho_\alpha$ forms a connected family, but $\bigcap_{\alpha \in I} \hat{D}^\rho_\alpha = \phi$.

For singular systems we are interested in a common feedback law $u(x)$ such that the system is stabilized at the common fixed point x^0 for all $\alpha \in I$. We consider again the situation of a pitchfork bifurcation.

3. Theorem. *Consider a system with a pitchfork bifurcation. Let $\rho > 0$ be given, and let (x^0, α_0) be the bifurcation point. Then there exists a common feedback $u(x)$ such that the systems (2^ρ_α) are (asymptotically) stable at x^0 for all $\alpha \in I$ with $\alpha_0 \in$ int I, iff $\alpha \leq \alpha_0 + \rho$ for all $\alpha \in I$, i.e. iff $\rho \geq \alpha - \alpha_0 = \rho(\alpha)$.*

For a proof just notice that if $\rho \geq \alpha - \alpha_0$, then there exists $u \in U^\rho$ such that x^0 is asymptotically stable for the system (2^ρ_α) using the constant feedback u. Vice versa, if $\rho < \alpha - \alpha_0$ then we have three control sets, of which $\{x^0\}$ is the smallest one with respect $\prec$. Hence the system is not stabilizable at x^0 for such an α according to Theorem 2.

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